AN INTRODUCTION TO PARTITIONS AND THEIR CRANKS

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Abstract. The crank function on integer partitions provides a combinatoric explanation for Ramanujan's Congruences on the partition function. This paper aims to, in a self-contained manner, provide the reader with a solid understanding of the Andrews and Garvan paper "Dyson's Crank of a Partition", which formally defined the crank and proved its fulfillment of Freeman Dyson's much earlier conjecture of its existence and properties. This includes the historical context of said paper, the concepts and tools necessary to understand its assertions, and the proofs which it contains.

Contents

1. Partitions, Compositions, and Ferrers/Young Diagrams

Definition 1.1. A *partition*, or *integer partition*, of a natural number n is an unordered collection of natural numbers, called *parts*, such that the sum of the collection is n. Different *parts* in the same partition need not have different values.

Definition 1.2. $p(n)$, called the *partition function*, is defined as the number of unique integer partitions of n.

Convention 1.3. The parts of a partition are usually written in decreasing order.

Example 1.4. The partitions of 4:

$$4$$
$$3+1$$
$$2+2$$
$$2+1+1$$
$$1+1+1+1$$
$$p(4) = 5$$

Convention 1.5. $p(0) = 1$. That is, the number of partitions of zero is equal to one. Additionally, the number of partitions of zero which possess any given property it is sensible to assign to a partition is equal to one. This may be informally interpreted as the existence of the "null partition", the partition with no parts.

Definition 1.6. A *composition*, or *integer composition*, of a natural number n is an *ordered* collection of natural numbers, called *parts*, such that the sum of the collection is n. Different *parts* in the same composition need not have different values.

Example 1.7. Some, but not all, compositions of 4:

$$3 + 1$$
$$1 + 3$$
$$2 + 1 + 1$$
$$1 + 2 + 1$$
$$1 + 1 + 2$$

Remark 1.8. The properties of compositions are generally less complex than those of partitions. Many nontrivial results in the theory of partitions, such as a formula for $p(n)$, have readily provable analogues regarding compositions. An interesting exercise left to the reader is to find and prove a formula for the number of integer compositions of n. There are many ways to approach this problem, but as a hint, one might count the ways to tile a one-by-n line of squares using tiles of varying sizes. As an interesting base case, try solving the problem first with only tiles of size one and two.

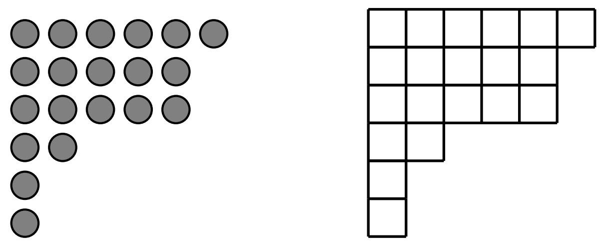
The remainder of this paper will focus on partitions as opposed to compositions.

Two useful, practically equivalent tools in the study of partitions are Ferrers diagrams and Young diagrams. These diagrams, examples of which can be seen on the next page, provide a simple way to visualize partitions. In the study of partitions, these diagrams are often used to aid intuition until results can be made formal by more symbolic methods. Because of this, we present them casually and without formal definitions, though such things do exist for their usage in other fields.

Both Ferrers and Young Diagrams organize themselves using the same basic scheme. Each row represents a single part of the partition and uses a number of graphical units equal to the size of the part to visually represent it. These rows are left justified and presented in the standard order of decreasing part size. The resultant diagram has a number of graphical units equal to n, the number being partitioned.

In Ferrers diagrams, the basic graphical unit is the dot. This choice helps in seeing the diagram as something to be manipulated, where dots can be freely moved around to transform one partition of n into another.

In Young diagrams, the basic graphical unit is the box, where the boxes share adjacent edges. This representation does not lend itself to mental manipulation to the same extent as a Ferrers diagrams' dots, but has its own benefit in the fact that the boxes may be written in. Though we do not discuss it further here, there are some applications of partitions for which it may be useful to assign a number to each unit in a partition. In such cases the boxes provide a convenient place to record this additional information.



A Ferrers Diagram (Left) & Young Diagram (Right) of $\{6, 5, 5, 2, 1, 1\}$

Definition 1.9. Two partitions are called *conjugate* if, in their Ferrers or Young diagrams, the k^{th} row of one has the same number of elements as the k^{th} column of the other for all natural numbers k. This can be visualized as a reflection of the partition's Ferrers/Young diagrams about the diagonal. More formally, we can construct the *conjugate partition* B of a partition A by letting the k^{th} part of B be equal to the number of parts in A which are at least k.

Theorem 1.10. *Conjugation is a well-defined, bijective function from the set of partitions of n to itself.*

Notation 1.11. Let ρ^{-1} denote the conjugate partition of ρ. Note that this is not standard notation.

Definition 1.12. A partition is called *self-conjugate* if $\rho^{-1} = \rho$.

Definition 1.13. We define the following functions on integer partition ρ.

$$\text{parts}(\rho) = (\text{the number of parts in } \rho)$$

$$\text{total}(\rho) = (\text{the sum of all parts in } \rho)$$

$$\max(\rho) = (\text{the largest part in } \rho)$$

Example 1.14. Let ρ be the partition $\{6, 5, 5, 2, 1, 1\}$, the same as is depicted by our Ferrers/Young diagram examples. Then we have the following.

$$\text{parts}(\rho) = 6$$

$$\text{total}(\rho) = 20$$

$$\max(\rho) = 6$$

This example, alongside the diagram examples which depict the same partition, brings attention to the facts: that $\max(\rho)$ is equal to the maximum width of a Ferrers/Young diagram; that $\text{total}(\rho)$ is equal to the number of geometric units in a Ferrers/Young diagram, so then also is the natural number being partitioned; and that $\text{parts}(\rho)$ is equal to the maximum height of a Ferrers/Young diagram.

Theorem 1.15. *For any partition ρ,*

$$\max(\rho) = \text{parts}(\rho^{-1}) \tag{1.16}$$

and

$$\text{total}(\rho) = \text{total}(\rho^{-1}). \tag{1.17}$$

Formal proofs of both (1.16) and (1.17) follow quickly from the definition of a conjugate partition.

One simple visualization of Theorem 1.15's results uses the Ferrers/Young diagrams of the partition in question, ρ. The width and the height of the diagrams are equal to $\max(\rho)$ and $\text{parts}(\rho)$, respectively. Since conjugation may be thought of as the reflection of these diagrams across the diagonal, (1.16) is a formalization of the fact that such a reflection would swap the height and width of the diagrams. Likewise, (1.17) is a formalization of the fact that such a reflection does not alter the total number of dots contained in the diagrams, which represents the number being partitioned.

2. Ramanujan's Congruences and Dyson's Crank Conjecture

Ramanujan, alongside Hardy, studied in great depth the behavior of the partition function $p(n)$. Their discoveries include the asymptotic behavior of $p(n)$ as n approaches infinity as well as the existence of and proofs for three surprising congruences on $p(n)$. These congruences are the starting point from which the story this paper follows began.

Theorem 2.1. *Ramanujan's Congruences*

The following three congruences hold for all nonnegative integers k:

$$p(5k+4) \equiv 0 \pmod{5}$$

$$p(7k+5) \equiv 0 \pmod{7}$$

$$p(11k+6) \equiv 0 \pmod{11}$$

Ramanujan proved these congruences directly around 1919, with the finalized collection of proofs [9] being published posthumously; the third proof required a different and more complex method than the first two and so was published some time after them. Today, we know that these are the only congruences of this form where $a > b$ in the $p(ak+b)$ expression. However, the existence of *these* three congruences posed a further question. If we know that the number of partitions of $(5k+4)$ is divisible by five, can we find a natural way to divide the actual partitions of numbers $(5k+4)$ into five classes of equal size? If so, can this be done with the other congruences, too? In short, we might wonder if there exists some combinatorial interpretation of the congruences.

This question was approached in 1944 by Freeman Dyson, at the time an undergrad at Trinity College, Cambridge, when he published the paper "Some Guesses in The Theory of Partitions" [2], the source of the rest of the definitions, theorems, and conjectures included in this section. This paper contained a number of conjectures regarding both proven and unproven congruences on values of the partition function. The most important of the conjectures focused on Ramanujan's Congruences. The first two such conjectures, called the Rank Conjectures, followed from the paper's introduction of a method of assigning to partitions a novel integer invariant called the rank.

Definition 2.2. The *rank* of a partition ρ is defined as

$$r(\rho) = \max(\rho) - \text{parts}(\rho).$$

We denote by $N(m,n)$ the number of partitions of n with rank m and denote by $N(m,q,n)$ the number of partitions of n with rank congruent to m modulo q.

Though Dyson's paper did not prove its main conjectures, it did prove a few basic identities which the rank obeys.

Theorem 2.3. *Basic Rank Identities*

$$N(m,q,n) = \sum_{k=-\infty}^{+\infty} N(m+kq,n) \tag{2.4}$$

$$N(m,n) = N(-m,n) \tag{2.5}$$

$$N(m,q,n) = N(-m,q,n) = N(q-m,q,n) \tag{2.6}$$

Let us take a moment to explain what these identities show. (2.4) is just a formalized definition of $N(m,n)$. (2.5) states that the partitions of n are distributed symmetrically about zero by their ranks. This may be seen through the fact that for any partition ρ with rank m, ρ^{-1} has rank $-m$. That is,

$$r(\rho) = \max(\rho) - \text{parts}(\rho) = \text{parts}(\rho^{-1}) - \max(\rho^{-1}) = -r(\rho^{-1}).$$

As conjugation is a bijection, then, each partition of n with rank m comes in a pair symmetric about zero. As a result, the second identity holds. (2.6) does the fairly trivial job of extending the second identity to prove that the distribution remains symmetric about zero when the ranks are taken modulo some q.

Dyson's two Rank Conjectures were primarily concerned with the possibility that the classification of partitions according their rank modulo five or seven offered a combinatoric explanation of Ramanujan's first two congruences.

The following is a the formal statement of this proposition.

Conjecture 2.7. *Dyson's Rank Conjectures*

$$\begin{aligned} N(0,5,5k+4) = N(1,5,5k+4) &= N(2,5,5k+4) \\ &= N(3,5,5k+4) = N(4,5,5k+4) \end{aligned} \tag{2.8}$$

$$\begin{aligned} N(0,7,7k+5) = N(1,7,7k+5) &= N(2,7,7k+5) = N(3,7,7k+5) \\ &= N(4,7,7k+5) = N(5,7,7k+5) = N(6,7,7k+5) \end{aligned} \tag{2.9}$$

The rest of Dyson's Rank Conjectures are more assorted and less structured.

Conjecture 2.10. *Dyson's Other Rank Conjectures*

$$N(1,5,5k+1) = N(2,5,5k+1)$$
$$N(0,5,5k+2) = N(2,5,5k+2)$$
$$N(0,5,5k+4) = N(1,5,5k+4) = N(2,5,5k+4)$$
$$N(2,7,7k) = N(3,7,7k)$$

$$N(1,7,7k+1) = N(2,7,7k+1) = N(3,7,7k+1)$$
$$N(0,7,7k+2) = N(3,7,7k+2)$$
$$N(0,7,7k+3) = N(1,7,7k+3) = N(2,7,7k+3) = N(3,7,7k+3)$$
$$N(0,7,7k+4) = N(1,7,7k+4) = N(3,7,7k+4)$$
$$N(0,7,7k+5) = N(1,7,7k+5) = N(2,7,7k+5) = N(3,7,7k+5)$$
$$N(0,7,7k+6) = N(1,7,7k+6) = N(2,7,7k+6) = N(3,7,7k+6)$$

One might note the complete absence of a rank conjecture pertaining to Ramanujan's $p(11k + 6)$ congruence. This conjecture is missing because *the corresponding assertion is not true for this congruence.* Even for low choices of k, the rank fails to equally distribute partitions into 11 equally-sized rank congruence classes. In reaction to this, Dyson made a much vaguer and more ambitious conjecture which predicted the existence of another invariant on integer partitions which is more "recondite," or difficult to find, than the rank. We present this conjecture abridged but with the original wording minimally changed.

Conjecture 2.11. *Dyson's Crank Conjecture*
That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; [we] shall call this hypothetical coefficient the "'crank" of the partition, and denote by $M(m, q, n)$ the number of partitions of n whose crank is congruent to m modulo q; that

$$M(m, q, n) = M(q - m, q, n);$$

that

$$M(0, 11, 11k + 6) = M(1, 11, 11k + 6) = M(2, 11, 11k + 6)$$
$$= M(3, 11, 11k + 6) = M(4, 11, 11k + 6);$$

[and] that numerous other relations exist analogous to [the rank conjectures and the conjectures in (2.10).]

Remark 2.12. Dyson notes that this may be the first major occasion in mathematics where a concept is named before it is defined.

The remainder of this paper deals with the discovery, forty years later, of a working definition of a crank function and how the partition invariants which it produces were proven to provide a unified, combinatoric explanation of all three of Ramanujan's congruences, just as Dyson conjectured.

3. Generating Functions, q-Series Notation, and $p(n)$

This section will introduce and build familiarity with the primary set of tools which will be used to prove that the crank function, as will be defined in section five, operates in accordance with Dyson's Crank Conjecture. Included are: an introduction to generating functions, used to algebraically represent and manipulate sequences; an instructional derivation of the generating function for the sequence produced by $p(n)$; and a short account of q-series notation, a shorthand for an expression which naturally emerges in partition-related generating functions.

Definition 3.1. A *generating function* is a unique polynomial of arbitrary degree where the coefficient of the k^{th} power of the function's variable is defined to be the index k term from a given sequence. The sequence $\{a_0, a_1, a_2, a_3, \dots\}$, then, has the following generating function:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots.$$

This method of embedding a sequence within a function is primarily of use in that it allows intuitive and well-defined algebraic manipulation of sequences for the purposes of analysis.

Generating functions are often able to be shortened into closed form expressions. More often than not, this form visibly encodes a relationship between terms which fully describes the sequence. For concision, referencing generating functions is usually done using its closed form expression, when one is known. When a closed form is unknown or abnormally bulky for a given function, an infinite sum or infinite product form is also preferable to the expanded polynomial.

Remark 3.2. For the most part, a generating function is not meant to be evaluated at a specific value. Instead of acting as a substitute value, the variable of the polynomial works as a record-keeping device, keeping the different values in the sequence in separate monomials, each unable to interfere with the others' values.

Using this format, sequence manipulation can be done intuitively, using ordinary algebra on the functions. For example, multiplying a generating function by its variable to the first power produces the generating function of the encoded sequence after being shifted one index to the right.

The following example establishes perhaps the most well-known generating function: the geometric series formula.

Example 3.3. Consider the sequence of all ones: $A_k = 1, 1, 1, 1, 1, \dots$

By definition, the generating function $A(x)$ of A_k has polynomial form

$$A(x) = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + \cdots = 1 + x + x^2 + x^3 + x^4 + \dots \tag{3.4}$$

Notice that shifting the elements of the sequence once to the right, without replacement of the index zero term, is equivalent to subtracting one from the index zero element. We can write these operations, and so also this equivalence, algebraically and in terms of the generating function $A(x)$.

$$\begin{aligned} xA(x) = x(1 + x + x^2 + x^3 + x^4 + \dots) &= x + x^2 + x^3 + x^4 + x^5 + \dots \\ &= (1 + x + x^2 + x^3 + x^4 + x^5 + \dots) - 1 = A(x) - 1 \end{aligned} \tag{3.5}$$

This equivalence visibly encodes the two defining characteristics of A_k. As each monomial in $(A(x) - 1)$ of degree greater than zero is equal to x times the monomial of one degree less, this is a constant sequence. As the degree zero term on the left is clearly zero, the right side implies that the index zero monomial of $A(x)$ equals one. This equality, then, is a complete description of the sequence, so it is not surprising that we can find a new, closed form expression for the generating function.

$$xA(x) = A(x) - 1 \tag{3.6}$$

$$1 = A(x)(1 - x) \tag{3.7}$$

$$\frac{1}{1 - x} = A(x) = 1 + x + x^2 + x^3 + x^4 + \dots \tag{3.8}$$

While this example does produce a generating function with a useful formula, it is important to remember that the defining characteristic of the function $A(x) = \frac{1}{1-x}$ as a generating function is its unique encoding of the elements of the sequence $A_k = 1, 1, 1, 1, 1, \ldots$.

Additionally, we note that in our use of algebraic operations on generating functions we make the assumption that their polynomial forms actually converge to a specific, manipulable, algebraic value. For $A(x)$, this is only true when $|x| < 1$. Our primary use of generating functions, however, is as record-keeping devices as opposed to evaluated functions. Correspondingly, so long as the function exists at *some* x, we need not think about the function's full domain when converting between its expanded and closed forms. Such considerations are only necessary when checking the compatibility of transformations which only hold on certain domains.

Remark 3.9. There exist other types of generating functions, most notably *exponential generating functions*. The only difference between ordinary and exponential generating functions is that the coefficient of the k^{th} term in an exponential generating function is the k^{th} value in the sequence multiplied by $\frac{1}{k!}$, rather than just the k^{th} value. That is, the exponential generating function on x of the sequence $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ is the following.

$$a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \frac{a_4 x^4}{4!} + \frac{a_5 x^5}{5!} + \ldots$$

This attribute of exponential generating functions, behaving conveniently under differentiation and integration, often makes them more useful in the analyses of sequences for which those operations can be interpreted meaningfully as shifting operations. Further types of generating functions include Lambert series and Bell series.

Ordinary generating functions, from here on just called generating functions, are extremely useful in the study of partitions. The sequences represented by "partition generating functions" generally follow a standard format: the index n term in the sequence is the number of partitions of n which have some given property or set of properties. In other words, the generating functions have monomials with powers n and respective coefficients counting some subset of the partitions of n. Some commonly explored properties of partitions include consisting of distinct parts, consisting of only odd parts, being self-conjugate, having a certain rank, and having a certain rank modulo another number.

Convention 3.10. Following from Convention 1.5, that the number of partitions of zero with any given property is considered equal to one, we have that the zero index term of any partition-counting sequence will be equal to one. By extension, the constant term in any partition-counting generating function is equal to one.

We will now take as an instructional example a derivation of the generating function for $p(n)$. As a reminder, $p(n)$ is the number of unique partitions of some positive integer n. The central line of this argument comes from Jennifer French's "Vector Partitions" [6].

Theorem 3.11. *The Generating Function of $p(n)$:*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=0}^{\infty} \frac{1}{1-q^k} \tag{3.12}$$

Proof. We will take a gradual, constructive approach to determining the generating function of $p(n)$ so as to serve the secondary purpose of preparing the reader for interpreting more complex generating functions later in the paper.

Consider the generating function counting the partitions of n for which all of their parts are equal to one. It is immediately obvious that there is exactly one such partition for any n, meaning that the sequence represented is $1, 1, 1, 1, \ldots$ and that, as seen previously, the generating function in question must be

$$\sum_{n=0}^{\infty} p_1(n)q^n = 1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}. \tag{3.13}$$

Now consider the generating function counting the partitions of n for which all of their parts are equal to some natural number k. The sequence is straightforward: if n is divisible by k, it can be partitioned in one way; if n is not divisible by k, it cannot be partitioned at all, so the value at index n is zero, making the q^n term disappear. That is,

$$\begin{aligned}\sum_{n=0}^{\infty} p_k(n)q^n &= 1 + q^k + q^{2k} + q^{3k} + q^{4k} + \ldots \\ &= (q^k)^0 + (q^k)^1 + (q^k)^3 + (q^k)^4 + \cdots = \frac{1}{1-x^k}.\end{aligned} \tag{3.14}$$

Now we consider how we might go about combining two such generating functions to arrive at the generating function for partitions of n which are allowed parts of two sizes. The answer is that we simply multiply the generating functions. To see why this works, consider the product of two such generating functions:

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\{1,2\}}(n)q^n &= \left[\sum_{n=0}^{\infty} p_1(n)q^n\right]\left[\sum_{n=0}^{\infty} p_2(n)q^n\right] = \frac{1}{1-q}\cdot\frac{1}{1-q^2} \\ &= (1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + \ldots)(1 + q^2 + q^4 + q^6 + \ldots)\end{aligned} \tag{3.15}$$

To see how this counts partitions using parts equal to one or two, consider how multiplying these polynomials together determines the coefficients for the q^k term in $p_{\{1,2\}}$. Take q^6 as an example. After multiplying the polynomials but before combining terms based on the power of their variable, every term is the result of the product of one term each from the polynomials being multiplied. Since all of the coefficients in $p_1(q)$ and $p_2(q)$ are equal to one, this means that the coefficient of q^6 *after* combining terms will be exactly the number of ways in which one term from each sequence, when multiplied together, produce a term of degree six.

Multiplying, we have the following four products which produce a q^6 term: q^0 from $p_1(q)$ and q^6 from $p_2(q)$; q^2 from $p_1(q)$ and q^4 from $p_2(q)$; q^4 from $p_1(q)$ and q^2 from $p_2(q)$; and q^6 from $p_1(q)$ and q^0 from $p_2(q)$.

Remembering that the coefficients of each of these factors is one because there is only one way to partition a multiple of n using n, we find that the coefficient of the q^6 term of the final product is the number of ways to partition six using ones and twos. A closer look at the individual sub-products we just discussed shows us how.

Each sub-product deals with a different way in which the "work" of reaching a total of six can be distributed between ones and twos. Though hidden behind polynomials, we have counted the partition $\{1,1,1,1,1,1\}$ as $(q^6)(q^0)$, $\{1,1,1,1,2\}$ as $(q^4)(q^2)$, $\{1,1,2,2\}$ as $(q^2)(q^4)$, and $\{2,2,2\}$ as $(q^0)(q^6)$. Since these are the only products of degree six and their coefficients are all equal to one, we have that the term $4q^6$ in the polynomial $p_{\{1,2\}}$ is counting the ways to partition six using only ones and twos.

As the same argument provides the corresponding counting property to all coefficients of the polynomial $p_{\{1,2\}}$, we have that generating function for the number of ways to partition n using ones and twos is equal to the following.

$$\begin{aligned}(3.16)\quad \sum_{n=0}^{\infty} p_{\{1,2\}}(n)q^n &= \left[\sum_{n=0}^{\infty} p_1(n)q^n\right]\left[\sum_{n=0}^{\infty} p_2(n)q^n\right] = \frac{1}{(1-q)(1-q^2)} \\ &= (1+q+q^2+q^3+q^4+\dots)(1+q^2+q^4+\dots) \\ &= 1+q+2q^2+2q^3+3q^4+3q^5+4q^6+\dots\end{aligned}$$

More generally, the generating function for the number of ways to partition n using naturals j,k where $j \neq k$ is equal to

$$\begin{aligned}(3.17)\quad \sum_{n=0}^{\infty} p_{\{j,k\}}(n)q^n &= \frac{1}{(1-q^j)(1-q^k)} \\ &= (1+q^j+q^{2j}+q^{3j}+\dots)(1+q^k+q^{2k}+q^{3k}+\dots).\end{aligned}$$

In fact, this polynomial multiplication method of distributing the task of "reaching" the partition's total, n, between different part values can be taken even further. The product of generating functions

$$(3.18)\qquad p_1(q)p_2(q) = (1+q+q^2+\dots)(1+q^2+q^4+\dots)$$

generates monomials with coefficient one, each of which corresponds to a way to allocate "space" in a partition between ones and twos. For exactly the same reasons, the product of generating functions

$$(3.19)\qquad p_1(q)p_2(q)p_3(q) = (1+q+q^2+\dots)(1+q^2+\dots)(1+q^3+\dots)$$

generates monomials with coefficient one, each of which corresponds to a way to allocate "space" in a partition between ones, twos, *and* threes.

In the same way, where S is some finite subset of the natural numbers, we can straightforwardly extend our result to

$$(3.20)\qquad \sum_{n=0}^{\infty} p_S(n)q^n = \prod_{s\in S}(1+q^s+q^{2s}+\dots) = \prod_{s\in S}\frac{1}{1-q^s}.$$

For some natural n, consider $p(n)$, the number of partitions of n with no restrictions on what parts are permitted. Since no partition of n can have a part greater than n, we know that the first n coefficients of $p(n)$ are exactly the coefficients of $p_S(n)$, where S is the set of all naturals less than or equal to n.

Therefore, we also know that as N goes to positive infinity, the coefficients of

$$\prod_{k=1}^{N} \frac{1}{1-q^k} \tag{3.21}$$

go to the corresponding coefficients in $p(n)$. Using this, we can finally state the desired result.

$$\sum_{n=0}^{\infty} p(n)q^n = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \frac{1}{1-q^k} \tag{3.22}$$

□

Before we wade into more complex waters, we take advantage of the prevalence of a certain form of series found within many partition generating functions:

$$\prod_{k=0}^{B} (1-aq^k),$$

where a is some expression and B is either some non-negative integer or ∞.

These series tend not to admit nontrivial decompositions, so we will largely treat them as an elementary expression hidden behind the following notation.

Notation 3.23. q-Series Notation

$$(a;q)_d = (a)_d = \prod_{k=0}^{d-1} (1-aq^k) \tag{3.24}$$

$$(q)_d = (q;q)_d \tag{3.25}$$

$$(a;q)_\infty = (a)_\infty = \prod_{k=0}^{\infty} (1-aq^k) \tag{3.26}$$

$$(q)_\infty = (q;q)_\infty \tag{3.27}$$

Remark 3.28. Though widely used in the study of partitions and therefore a necessary inclusion in any description of q-series notation, the shortened forms $(a)_d$, $(q)_d$, $(a)_\infty$, or $(q)_\infty$ will not be used in this paper; for readers new to the notation, the shortening is unlikely to be worth the potential loss of clarity. Additionally, we draw special attention to the fact that $(a;q)_d$ does not include $(1-aq^d)$ as a factor, instead cutting off after $(1-aq^{d-1})$.

The generating function of $p(n)$ is not quite complex enough to justify always shortening it with q-series notation. However, as we are now quite familiar with this function, we will use it for our example of the notation's implementation.

Example 3.29. q-Series Form of the Generating Function of $p(n)$

$$\sum_{n\geq 0} p(n)q^n = \prod_{k=0}^{\infty} \frac{1}{1-q^k} = \frac{1}{(1;q)_\infty} = (1;q)_\infty^{-1}$$

4. Vector Partitions and the Function $N_V(m,n)$

This section will provide an introduction to and discussion on an important variation on the partition called the vector partition. We will be able to define a crank function which operates according to Dyson's Crank Conjecture, though only on these vector partitions. Section five will rigorously extend this result to an integer crank function which does the same. We do not, however, include in this paper the proof that the crank on vector partitions works according to Dyson's conjecture, nor do we derive the generating function which counts these vector partitions according to it. For clarity, any reference to a "partition" which does not explicitly use the term "vector partition" is a reference to an integer partition. All results from this section come from the second chapter of Garvan's PhD thesis, "Dyson's Crank for Vector Partitions" [3].

Vector partitions are ordered triples of integer partitions. Importantly, the constituent partitions are not required to be partitions of the same natural number, nor do they need to fulfill any other restriction other than those we explicitly provide in the following definition.

Definition 4.1. Let the set

$$V = \{(\rho_1, \rho_2, \rho_3) \mid \rho_1 \text{ a partition with distinct parts; } \rho_2, \rho_3 \text{ partitions}\} \tag{4.2}$$

be called the set of all *vector partitions*. Each vector partition $v = (\rho_1, \rho_2, \rho_3) \in V$ has a *sum* defined as

$$s(v) = \text{total}(\rho_1) + \text{total}(\rho_2) + \text{total}(\rho_3), \tag{4.3}$$

a *weight* defined as

$$\omega(v) = (-1)^{\text{parts}(\rho_1)}, \tag{4.4}$$

and a *crank* defined as

$$r(v) = \text{parts}(\rho_2) - \text{parts}(\rho_3). \tag{4.5}$$

Remark 4.6. Unlike for integer partitions, which can be organized intuitively according to the number they partition, the choice of a "size invariant" with which to organize the set of vector partitions V is non-obvious. For our purposes, the most convenient such "size invariant" on V is the sum $s(v)$. This is the number which *would* be partitioned if one were to amalgamate the three constituent partitions in v into one large integer partition.

Now that we have our vector partitions, we move on to counting them. We want to count the number of vector partitions with sum n and crank m, as we will eventually see that this count is the same as that of integer partitions according to total n and crank m. Back to vector partitions, though, we are counting those $v \in V$ with $s(v) = n$ and $r(v) = m$.

Definition 4.7. We define $N_V(m,n)$ to be the *weighted* count of vector partitions of n with crank m. More formally, where

$$R = \{v \in V \mid s(v) = n,\ r(v) = m\} \tag{4.8}$$

we have

$$N_V(m,n) = \sum_{v \in R} \omega(v). \tag{4.9}$$

Following parallel to the path taken in Ramanujan's and Dyson's analysis of integer partitions, we now introduce congruences into our study of vector partitions.

Definition 4.10. We define $N_V(m,t,n)$ to be the number of vector partitions with sum n such that the crank is congruent to m mod t. More formally,

$$N_V(m,t,n) = \sum_{k=-\infty}^{+\infty} N_V(m+kt,n).$$

Remark 4.11. The iteration of k all the way from $-\infty$ to $+\infty$ in Definition 4.10 is technically unnecessary, as for any given sum n the possible cranks are bounded by n and $-n$. These extreme values would themselves only occur in the disallowed cases where (ρ_1,ρ_2,ρ_3) equals either $(\emptyset,\emptyset,n)$ or $(\emptyset,n,\emptyset)$, respectively. Despite the fact that the distribution of the crank for a given sum n is bounded, in Definition 4.10 does not bound the allowed values of its index, k. The lack of bounds was chosen first because its use does not impact the resulting sum and second because it shows more clearly that we are counting all vector partitions with sum n.

Notation 4.12. As we continue, we will frequently refer to monomials of the form Az^mq^n, so we explicitly provide the following terminology. A will be referred to as the coefficient of the monomial, as per usual. Collectively, m and n will be called the *degrees* of the monomial. Individually, we call m the z-degree and n the q-degree of the monomial.

Though we do not derive it in this paper, the following is the generating function of $N_V(m,n)$.

Theorem 4.13. *The Generating Function of $N_V(m,n)$:*

$$\sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} N_V(m,n)z^mq^n = \prod_{k=1}^{\infty}\frac{1-q^k}{(1-zq^k)(1-z^{-1}q^k)} \tag{4.14}$$

Though this generating function is a function on two variables, the basic functionality remains that of any generating function: when expanded into polynomial form, the coefficient of the term with degrees m and n is equal to the weighted count of vector partitions with crank m and sum n. This result is not particularly difficult for an experienced reader to derive and so is generally taken for granted in the papers which use it. A careful reading of this paper through the end should sufficiently equip the reader to derive (4.14) themselves. Doing so is left as a somewhat challenging exercise.

At face value, (4.14) doesn't reveal much; as m and n vary, the method by which vector partitions are being counted is not at all obvious.

However, consider what happens when we evaluate for $z=1$. This does not change any coefficients or q-degrees, so we are still counting the weights of vector partitions, grouped by their sums. However, we are no longer separating them based on their cranks. This means that evaluating at $z=1$ provides a generating function which counts the vector partitions $v \in V$, each adding its respective weight $\omega(v)$ to the coefficient of the monomial with q-degree equal to $s(n)$.

Carrying out this evaluation yields the following generating function.

$$\sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} N_V(m,n)q^n = \prod_{k=1}^{\infty}\frac{1}{1-q^k} \tag{4.15}$$

Incidentally, we have just shown a remarkable parallel between vector and integer partitions: (4.15) is equal to the generating function with $p(n)$. That is, for all natural numbers n, the count of the integer partitions of n is equal to the weighted count of the vector partitions of n. The following is this result expressed formally.

Theorem 4.16. *For all nonnegative integers n,*

$$\sum_{m=-\infty}^{+\infty} N_V(m,n) = p(n).$$

This parallel between vector and integer partitions is of no small importance to Dyson's conjectures in light of the main result of Garvan's thesis.

Theorem 4.17. *The Vector Crank Theorem*

We have the following three statements.

$$N_V(0,5,5n+4) = N_V(1,5,5n+4) = \cdots = N_V(4,5,5n+4) = \frac{\sum_{m=-\infty}^{+\infty} N_V(m,5n+4)}{5} = \frac{p(5n+4)}{5} \tag{4.18}$$

$$N_V(0,7,7n+5) = N_V(1,7,7n+5) = \cdots = N_V(6,7,7n+5) = \frac{\sum_{m=-\infty}^{+\infty} N_V(m,7n+5)}{7} = \frac{p(7n+5)}{7} \tag{4.19}$$

$$N_V(0,11,11n+6) = N_V(1,11,11n+6) = \cdots = N_V(10,11,11n+6) = \frac{\sum_{m=-\infty}^{+\infty} N_V(m,11n+6)}{11} = \frac{p(11n+6)}{11} \tag{4.20}$$

That is, the vector partition analogues of all three of Ramanujan's congruences not only hold but can be combinatorially explained by the given crank function on vector partitions. More concisely, the vector partition analogue of Dyson's Crank Conjecture holds.

5. The Crank Function on Integer Partitions

This section will present the definition of the crank on integer partitions and show that it inherits the results which the Vector Crank Theorem proved for $N_V(m,n)$ in the previous section. Unless otherwise noted, results in this section come from the Andrews/Garvan paper "Dyson's Crank of a Partition" [5].

We begin with the "recondite", or difficult to find, definition of the integer crank function which Dyson conjectured the existence of over forty years prior to its actual discovery:

Definition 5.1. The Crank of an Integer Partition

Let ρ be an integer partition. Let $\omega(\rho)$ be the number of parts in ρ equal to one and let $\mu(\rho)$ be the number of parts in ρ greater than $\omega(\rho)$. The *crank* of ρ is defined as follows.

$$c(\rho) = \begin{cases} \max(\rho) & \text{if } \omega(\rho) = 0 \\ \mu(\rho) - \omega(\rho) & \text{if } \omega(\rho) \neq 0 \end{cases}$$

To the reader this might seem at first like an extraordinarily arbitrary choice of function and feel not at all like a fundamental way to classify partitions. The remainder of this last section will demonstrate that this definition emerges naturally and in its entirety from the conversion of the previous section's vector crank to a form applicable to integer partitions.

Before diving into this derivation, however, we prove a useful lemma for transforming q-series.

Lemma 5.2. *The q-Analogue of the Binomial Series* [4]
For real numbers a, q, t such that $q, t \in (-1, 1)$,

$$\frac{(at;q)_\infty}{(t;q)_\infty} = \prod_{k=0}^{\infty} \frac{(1-atq^k)}{(1-tq^k)} = 1 + \sum_{k=1}^{\infty} \frac{(1-a)(1-aq)\dots(1-aq^{k-1})t^k}{(1-q)(1-q^2)\dots(1-q^k)} = 1 + \sum_{k=1}^{\infty} \frac{(a)_k t^k}{(q;q)_k} \tag{5.3}$$

Proof. Consider the generating function on variable t equal to the first or second expression in (5.3). Let $A = A(a,q)$ be the sequence for fixed a and q which this generating function encodes. That is, consider

$$F(t) = \prod_{k=0}^{\infty} \frac{(1-atq^k)}{(1-tq^k)} = \sum_{k=0}^{\infty} A_k t^k. \tag{5.4}$$

First, we need to show that such a polynomial expansion of $F(t)$ exists on some interval so that we can meaningfully manipulate it as an algebraic term.

The sequence of partial products in the infinite product form of $F(t)$ uniformly converges. Though we don't go into detail here, this leverages the fact that for any sequence of bounded, complex functions $f_k(t)$ for which $\sum_{k=0}^{\infty} |f_k(t)|$ converges uniformly, the sequence of partial products of $\prod_{k=0}^{\infty}(1 + f_k(t))$ also converges uniformly on the same domain. We can show this for on $[-1+\epsilon, 1-\epsilon]$ for fixed a real, $q \in (-1,1)$, and $\epsilon > 0$, meaning that $F(t)$ is uniformly convergent on $(-1,1)$. By Morera's Theorem, then, $F(t)$ is analytic on $(-1,1)$. By definition, this gives the existence of our right-side polynomial on this domain.

We can now move on with the proof.

Our first observation about (5.4) is that evaluating the infinite product form of $F(t)$ gives us $F(0) = 1$.

Further analysis requires finding a nontrivial relation of $F(t)$ to itself. To do this, we extract the $k = 0$ expression from the infinite product form of (5.4) to get

$$F(t) = \frac{(1-at)}{(1-t)} \prod_{k=1}^{\infty} \frac{(1-atq^k)}{(1-tq^k)}. \tag{5.5}$$

Rearranging,

$$(1-t)F(t) = (1-at) \prod_{k=1}^{\infty} \frac{(1-atq^k)}{(1-tq^k)}. \tag{5.6}$$

By redefining the indexing term k to be one less, this in turn gives that

$$(1-t)F(t) = (1-at)\prod_{k=0}^{\infty}\frac{(1-atq^{k+1})}{(1-tq^{k+1})} \tag{5.7}$$
$$= (1-at)\prod_{k=0}^{\infty}\frac{(1-a(tq)q^{k})}{(1-(tq)q^{k})} = (1-at)F(tq).$$

Importantly, notice how we pulled q^1 out of a greater power of q and into the argument of an encompassing function. This allowed us to convert a large expression into a more familiar form. This strategy will be used extensively in the proof of this paper's main theorem.

Condensing the last chain of equations, we get our relation of $F(t)$ to itself:

$$(1-t)F(t) = (1-at)F(tq). \tag{5.8}$$

After distributing, this is equivalent to

$$F(t) - tF(t) = F(tq) - atF(tq). \tag{5.9}$$

Now we convert this statement about a generating function into a statement about the terms of the sequence it encodes.

The left side of this equation describes the generating function of A minus the generating function of A shifted one term to the right. For a specific index k of the sequence, then, this left side of (5.9) expresses the generating function of the values which can be term-wise given by $A_k - A_{k-1}$.

The right side of this equation describes something slightly more complex. We take it one term at a time.

Consider $F(tq)$. In the polynomial form of F we are discussing, the input to the function is raised to the power of the index as a record-keeping strategy. When (tq) is fed in instead of just t, this has the effect of multiplying each monomial in the polynomial expansion by q^k, where k is the degree, the index, of that specific monomial. As q is a constant, this effect changes only the coefficient of the monomial. In short, $F(tq)$ is the generating function of the sequence defined at index k by $q^k A_k$.

Now consider $atF(tq)$. We know that $F(tq)$ is the generating function of the sequence defined by $q^k A_k$. Multiplying by a is simply multiplying by a constant and multiplying by t shifts the sequence once to the right. This means that $atF(tq)$ is the generating function of the sequence $aq^k A_k$, just shifted once to the right.

We can now see the following term-wise description of the right side of (5.9): $q^k A_k - q^{k-1}A_{k-1}$.

Putting all of this together, we have the full term-wise version of (5.9): for any positive index k of a term in A,

$$A_k - A_{k-1} = q^k A_k - q^{k-1}A_{k-1}. \tag{5.10}$$

After rearranging to isolate A_k,

$$A_k = \frac{(1-aq^{k-1})}{(1-q^k)}A_{k-1}. \tag{5.11}$$

As we know that $F(0) = 1$, meaning also that $A_0 = 1$, we have by induction the following formula for the function A_k:

$$A_k = \prod_{d=0}^{k-1} \frac{(1 - aq^d)}{(1 - qq^d)} = \frac{\prod_{d=0}^{k-1}(1 - aq^d)}{\prod_{d=0}^{k-1}(1 - qq^d)} = \frac{(a;q)_k}{(q;q)_k} \tag{5.12}$$

We note that this formula does not work for the index zero term, which corresponds to $F(0) = 1$, since (5.12) is defined inductively and treats this as the base case. To compensate, we simply define $A_0 = 1$ when using this new formula. In conjunction with (5.12), this lets us construct A's generating function $F(t)$ as the following power series, which is what we wanted to show.

$$F(t) = 1 + \sum_{k=1}^{\infty} \frac{(a;q)_k t^k}{(q;q)_k} \tag{5.13}$$

□

We are finally ready to state and prove the primary result of the Andrews/Garvan paper: the Integer Crank Theorem.

Theorem 5.14. *The number of integer partitions of $n > 1$ with crank m is equal to $N_V(m, n)$.*

Proof. We will approach this proof in two parts. First, we will rephrase our known generating function for $N_V(m, n)$ in more familiar terms. Second, we will show that our rephrased generating function, which sorts vector partitions according to their sum and crank, is also the generating function of the set of integer partitions sorted according to their total, when greater than one, and their crank. This equivalence of generating functions then implies our result.

As promised, we begin with our generating function of $N_V(m, n)$. Unless otherwise noted and until we explicitly finish the first part of this proof, all of the following expressions are restatements of this same function.

$$\sum_{m=-\infty}^{+\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \prod_{k=1}^{\infty} \frac{1 - q^k}{(1 - zq^k)(1 - z^{-1}q^k)} \tag{5.15}$$

Our first intention is to make the function more concise using q-series notation. We start by distributing the infinite product onto the three expressions within it. We also extract the first term of the numerator's product from its infinite product.

$$\frac{(1 - q)\prod_{k=2}^{\infty}(1 - q^k)}{\prod_{k=1}^{\infty}(1 - zq^k)\prod_{k=1}^{\infty}(1 - z^{-1}q^k)} \tag{5.16}$$

In order to rephrase the infinite products in such a way that they range from zero to infinity, as required by q-series notation, we factor the powers of q as we did in the proof of Lemma 5.2.

$$\frac{(1 - q)\prod_{k=0}^{\infty}(1 - (q^2)q^k)}{\prod_{k=0}^{\infty}(1 - (zq)q^k)\prod_{k=0}^{\infty}(1 - (z^{-1}q)q^k)} \tag{5.17}$$

As a result, we are able to convert $N_V(m, n)$'s generating function almost entirely into q-series.

$$\frac{(1 - q)(q^2;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \tag{5.18}$$

Next, we want to apply Lemma 5.2. Before we do so, we need to resolve two concerns.

First, Lemma 5.2 only holds when $q, t \in (-1, 1)$. This is not an issue, as we do not intend to evaluate this function at any values. As was the case in Lemma 5.2, the well-definition of a function on *some* region is enough to allow meaningful algebraic manipulation of the function. So long as we do not use another lemma or collection of lemmas which disallow these values, we encounter no issues.

Second, in order to make use of Lemma 5.2, we need to find in (5.18) some expression of the form $\frac{(at;q)_\infty}{(t;q)_\infty}$ for some adequate t, q. To accomplish this, we separate our function into two parts.

$$\frac{1-q}{(zq;q)_\infty} \cdot \frac{(q^2;q)_\infty}{(z^{-1}q;q)_\infty} \tag{5.19}$$

Let $a = zq$ and let $t = z^{-1}q$. With these set, Lemma 5.2 gives that

$$\begin{aligned} \frac{1-q}{(zq;q)_\infty} \cdot \frac{(q^2;q)_\infty}{(z^{-1}q;q)_\infty} &= \frac{1-q}{(zq;q)_\infty} \cdot \frac{(at;q)_\infty}{(t;q)_\infty} \\ &= \frac{1-q}{(zq;q)_\infty}\left[1 + \sum_{k=1}^{\infty} \frac{(a;q)_k t^k}{(q;q)_k}\right] \\ &= \frac{1-q}{(zq;q)_\infty}\left[1 + \sum_{k=1}^{\infty} \frac{(zq;q)_k (z^{-1}q)^k}{(q;q)_k}\right]. \end{aligned} \tag{5.20}$$

Distributing into the brackets and preparatorily grouping some terms, we get the following.

$$\frac{1-q}{(zq;q)_\infty} + \sum_{k=1}^{\infty}\left(\frac{1-q}{(q;q)_k} \cdot \frac{(zq;q)_k}{(zq)_\infty} \cdot \frac{(z^{-1}q)^k}{1}\right) \tag{5.21}$$

A number of non-obvious cancellations occur here. For clarity, we will deal with them one by one according to the groupings seen in (5.21) before recombining them and continuing.

Factoring in and out different powers of q, the first grouping simplifies into a single q-series.

$$\begin{aligned} \frac{1-q}{(q;q)_k} &= \frac{1-q}{(1-qq^0)(1-qq^1)\dots(1-qq^{k-1})} \\ &= \frac{1-q}{(1-q)(1-q^2)\dots(1-q^k)} \\ &= \frac{1}{(1-(q^2)q^0)(1-(q^2)q^1)\dots(1-(q^2)q^{k-2})} = \frac{1}{(q^2;q)_{k-1}}. \end{aligned} \tag{5.22}$$

The same toolset simplifies the second grouping just as much.

$$\begin{aligned} \frac{(zq;q)_k}{(zq;q)_\infty} &= \frac{(1-(zq)q^0)\dots(1-(zq)q^{k-1})}{(1-(zq)q^0)\dots(1-(zq)q^{k-1})(1-(zq)q^k)\dots} \\ &= \frac{(1-zq)(1-zq^2)\dots(1-zq^k)}{(1-zq)(1-zq^2)\dots(1-zq^k)(1-zq^{k+1})\dots} \\ &= \frac{1}{(1-(zq^{k+1})q^0)(1-(zq^{k+1})q^1)\dots} = \frac{1}{(zq^{k+1})_\infty}. \end{aligned} \tag{5.23}$$

Lastly and straightforwardly,

$$\frac{(z^{-1}q)^k}{1} = \frac{z^{-k}q^k}{1}. \tag{5.24}$$

Substituting (5.22), (5.23), and (5.24) into (5.21), we have that the generating function of $N_V(m, n)$ is equal to

$$\frac{1-q}{(zq;q)_\infty} + \sum_{k=1}^{\infty} \frac{z^{-k}q^k}{(q^2;q)_{k-1}(zq^{k+1})_\infty}. \tag{5.25}$$

With (5.25) we conclude our rephrasing of the function into an interpretable format, so we now move on to the second part of this proof: showing that this is also the generating function for the number of integer partitions of $n > 1$ with integer crank equal to m. We will be primarily calling upon the insights gained in the derivation and proof of Theorem 3.11, the generating function of $p(n)$.

Each of the two summands in (5.25) counts a disjoint subset of the integer partitions where, between them, every partition is counted exactly once. We will investigate these how these summands operate as generating functions one at a time, beginning with the second:

$$\sum_{k=1}^{\infty} \frac{z^{-k}q^k}{(q^2;q)_{k-1}(zq^{k+1})_\infty}. \tag{5.26}$$

As the index k of this infinite sum is always a natural number, we can safely expand (5.26) to get the following.

$$\sum_{k=1}^{\infty} \frac{z^{-k}q^k}{(1-q^2)(1-q^3)\dots(1-q^k)(1-zq^{k+1})(1-zq^{k+2})\dots} \tag{5.27}$$

For now, we focus on the denominator of this expression and do so outside of the context of the infinite sum or the numerator. Remembering our proof of the generating function of $p(n)$, we know that the inclusion of the term $(1-q^k)$ in the denominator of a partition generating function counts partitions which are "permitted" to have parts equal to k. We can see, then, that this function is some variation on counting all partitions with no parts equal to one.

There exists a difference, however, in that all factors that "permit" a part value x greater than k appear not as $(1-q^x)$, but instead as $(1-zq^x)$. To see what this contributes to the generating function's representative monomial for any given integer partition, we remember that the crux of most closed form partition generating functions is the following expansion.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \tag{5.28}$$

If we substitute aq^k for x, we find that

$$\frac{1}{1-aq^k} = 1 + (aq^k)^1 + (aq^k)^2 + (aq^k)^3 + \dots. \tag{5.29}$$

Our derivation of the generating function of $p(n)$ showed that we can consider each possible integer partition to be represented by a unique product of one term each from the polynomial expansions of this sort when discussing a partition generating function. The chosen term $(q^k)^x$ from the expansion of $\frac{1}{1-q^k}$ corresponds to the presence of exactly x parts equal to k within the partition.

We can see, then, that if we change $\frac{1}{1-q^k}$ to $\frac{1}{1-aq^k}$ and by extension each $(q^k)^x$ in its expansion to $(zq^k)^x$, a power of the variable z will appear in each partition's unique, representative product. The power on that z variable will increase by one for each part in the partition equal to k.

Armed with an understanding as to how the z's in (5.27) operate, we can see that, *still setting the numerator and infinite sum aside*, (5.27) counts all partitions that have no parts equal to one. Further, all partitions of n_0 with m_0 parts greater than k are counted by representative products of z-degree m_0 and q-degree n_0. The reader is advised to make sure they understand the reasons why before continuing.

We now restate (5.27) and consider how the infinite sum and the numerator change things.

$$\sum_{k=1}^{\infty} \frac{z^{-k}q^k}{(1-q^2)(1-q^3)\dots(1-q^k)(1-zq^{k+1})(1-zq^{k+2})\dots}$$

Neither the infinite sum nor the numerator fundamentally changes how the factors in the denominators expand, so the core of what we are counting is still the same set of things. Any effects of these new considerations necessarily lie in changes to how how each partition is interpreted and, by extension, the degrees of its representative product.

Instead of counting partitions *without* parts equal to one, consider (5.27) to be counting partitions according to the exact, nonzero number of parts equal to one they have: the index of the infinite sum, k.

For a fixed k, there is a natural bijection between the partitions of n with no ones and the partitions of $(n+k)$ with exactly k ones. Therefore, the denominator we have been discussing is a generating function which, for each fixed k, counts these as well. We have established that this denominator produces for each partition ρ with no ones a monomial with coefficient one and variables $z^{m_0}q^{n_0}$. Remember that m_0 is the number of parts in ρ greater than k, which we have since defined as the number of parts equal to one. Additionally, n_0 is number partitioned by this oneless partition ρ.

Consider the impact on these values by multiplying them by the numerator. The monomial which had been representing ρ now has z-degree $(m_0 - k)$ and q-degree (n_0+k). In other words, it has been mapped across the bijection between partitions with no ones and k ones to the partition ρ_k, the partition with the same parts as ρ but with the addition of k ones. As $(n_0 + k)$ is the number which is partitioned after the inclusion of the k ones, each new partition is properly mapped to the appropriate q-degree, representing the natural number it partitions. Remembering that m_0 is equal to the number of parts in ρ greater than k and noticing that the appending of more ones will never change this value, we can also see that $(m_0 - k) = \mu(\rho_k) - \omega(\rho_k)$, meaning that the z-degree is properly mapped to $c(\rho_k)$ as presented by Definition 5.1.

Altogether we now see that (5.27) is a generating function which counts all partitions with at least one part equalling one. In the full polynomial expansion of (5.27), and so also that of the more condensed (5.26), the coefficient of the monomial of z-degree m and q-degree n is equal to the number of such partitions of natural numbers n with crank m.

Having completed our analysis of the second summand in (5.25), we begin our analysis of the first.

Here is the section of (5.25) in question.

$$\frac{1-q}{(zq;q)_\infty} = \frac{1-q}{(1-(zq)q^0)(1-(zq)q^1)\dots} = \frac{1-q}{(1-zq)(1-zq^2)\dots} \tag{5.30}$$

The presence of the $(1-q)$ in the numerator lends this function some resistance to the methods we used to interpret both $p(n)$ and the other half of (5.25). To compensate, we reorient our analysis in two ways.

First, we realize that just as the crank function $c(\rho)$ was equivalent to $\mu(\rho)-\omega(\rho)$ when counting all partitions ρ for which $\omega(\rho) \neq 0$, $c(\rho)$ is equivalent to $\max(\rho)$ when counting all partitions ρ for which $\omega(\rho) = 0$. Our original goal is then equivalent to showing that (5.30) is a generating function which counts partitions ρ with $\omega(\rho) = 0$, giving the product corresponding to each a z-degree equalling $\max(\rho)$.

Second, we use what we know about conjugate partitions to count these partitions indirectly. Theorem 1.10 gave that conjugation is a bijection on partitions of n. Theorem 1.15 gave that $\max(\rho) = \text{parts}(\rho^{-1})$ for all integer partitions ρ. Combining these, we know that for any n, there exists a bijection between the set of partitions ρ for which $\max(\rho) = m$ and the set of partitions ρ^{-1} for which $\text{parts}(\rho) = m$.

From our discussions so far, we can tell that

$$\frac{1}{(1-zq)(1-zq^2)(1-zq^3)\dots} \tag{5.31}$$

is a generating function for which the coefficient of the monomial with z-degree m and q-degree n is the number of partitions ρ of n such that $\text{parts}(\rho) = m$. As a bijection preserves cardinality, we know that it is also a generating function for which the coefficient of the monomial with these same degrees is the number of partitions ρ of n such that $\max(\rho) = m$.

Next we investigate the other chunk of the numerator of (5.30).

$$\frac{q}{(1-zq)(1-zq^2)(1-zq^3)\dots} \tag{5.32}$$

(5.31) counts the partitions ρ of n according to $\max(\rho)$, but does *not* impose the condition that the counted partitions must have no parts equal to one. Our goal is then to show that (5.32) counts all partitions ρ of n with at least one part equal to one according to $\max(\rho)$, since subtracting such a function from (5.31) would then result in an interpretation of (5.30) which corresponds perfectly to the $\omega(\rho) = 0$ case of our crank definition.

In our analysis of (5.26) we found that the effect of a factor of q^k in the numerator of a partition generating function was that rather than counting the partitions generated by the denominator, the function counted the partitions which result from appending exactly k ones to each generated partition.

Since the addition of a one to any given partition does not change the value of its greatest part, this means that the coefficient of the monomial with z-degree m and q-degree n in the polynomial expansion of (5.32) is equal to the number of partitions ρ of n such that $\max(n) = m$, but $\omega(\rho) \geq 1$. This is exactly the interpretation we wanted.

The interpretation fails, however, when $n = 1$. This is because the representative product for the partition consisting of a single one will be the product of the q term in the denominator multiplied by the constant term, equalling one, of each of the polynomial expansions from the denominator. This will correctly set the partition's q-degree to its total, one, but will incorrectly set the partition's z-degree to zero. The crank of this partition, according to Definition 5.1, should be negative one. This interpretation of (5.30), then, is only correct for indexes of the generating function which are greater than one. In fact, this restriction goes so far as to be reflected in our formal statement of this theorem.

Combining our results, we have that

$$\frac{1}{(1-zq)(1-zq^2)\dots} - \frac{q}{(1-zq)(1-zq^2)\dots} = \frac{1-q}{(zq;q)_\infty} \tag{5.33}$$

is the generating function such that the coefficient of the monomial with z-degree m and q-degree n in its polynomial expansion counts the number of partitions ρ of $n > 1$ with $\omega(\rho) = 0$ and $c(\rho) = \max(\rho) = m$.

We showed earlier that (5.26) is the generating function where the coefficient of the monomial in its polynomial expansion with z-degree m and q-degree n counts the number of partitions ρ of n such that $\omega(\rho) \neq 0$ and $c(\rho) = \mu(\rho) - \omega(\rho) = m$.

Now we have also shown that (5.30) is the generating function where the coefficient on the monomial of the same degrees is equal to the number of partitions ρ of $n > 1$ such that $\omega(\rho) = 0$ and $c(\rho) = \max(\rho) = m$.

Directly summing these gives us the generating function on *all* integer partitions such that the coefficient of the monomial with z-degree m and q-degree n is equal to the number of partitions ρ of $n > 1$ such that $c(\rho) = m$.

Formally, where $M(m, n)$ is the number of integer partitions of $n > 1$ with crank m as presented by Definition 5.1,

$$M(m, n) = N_V(m, n), \tag{5.34}$$

which is what we wanted to show. □

This theorem gives that for all $n > 1$, $M(m, n)$ shares all of its values with $N_V(m, n)$. For these n, then, $M(m, n)$ inherits the patterns and congruences of $N_V(m, n)$, including those described by the Vector Crank Theorem. Remembering that that theorem, Theorem 4.17, gave that the crank on vector partitions fulfilled their analogue of Dyson's Crank Conjecture, we have that our proof of Theorem 5.14 has shown that the integer crank of Definition 5.1 fulfills Dyson's Crank Conjecture itself. Thus we have shown the long-sought combinatoric explanation of Ramanujan's Congruences.

6. Further Reading

The study of partitions reaches far and wide, so this section will deal primarily with subtopics of direct relevance to what was discussed in this paper. However, for a thorough and rigorous introduction to the study of partitions up to 1998, see George E. Andrews' textbook "The Theory of Partitions" [4]. In addition to being the first author of the paper this paper was centered upon, Andrews has been among the foremost researchers into partitions for the past half century and has been the doctoral advisor of many of the others, including Garvan.

The discovery, by Andrews, and analysis, in this case by Manjil P. Saika, of Ramanujan's lost notebook revealed that its author had been investigating the very combinatorial interpretations of his congruences with which this paper is concerned [8]. Ramanujan, however, had approached the problem with dissections, a method of analyzing generating functions which decomposes their polynomial forms into the sum of multiple, more manageable generating functions. Many of his results can be viewed as the first steps of an unfinished, independent discovery of the crank.

The definition of vector partitions given in this paper is not the only definition analyzed by mathematicians today. In fact, this particular definition of vector partitions is not even the predominant definition. The more common definition is more in line with the intuitive interpretation of the name: an unordered collection of nonzero k-vectors with no negative components such that their sum is equal to a given k-vector. For an introduction to this definition of vector partitions, see Jennifer French's "Vector Partitions" [6]. This is also an excellent source for examples of simple generating function-based proofs about both integer partitions and these alternate vector partitions.

Similarly, the given definition of the rank of a partition is not the only one. In this case, however, it is the alternative definition which is less commonly referred to by this name. This alternative definition is that the rank of a partition is equal to the side-length of a partition's Durfee square. Informally, this is the largest n such that an n by n square fits within the Ferrers or Young diagrams of the partition. Formally, it is the greatest natural k such that at least k parts of the partition are of value at least k.

The results discussed in the body of this paper are over thirty years old. Predictably, a number of limited extensions have since been proven. For example, for any natural $n \geq 5$ such that $\gcd(n, 6) = 1$, there exist naturals A, B such that for all natural k, $p(Ak + B) \equiv 0 \pmod{n}$ [10]. Additionally, for any prime q there exist infinitely many non-nested sequences $\{Ak + B\}_{k=0}^{\infty}$ such that we have $N_V(m, q, Ak + B) \equiv 0 \pmod{q}$ for any m from zero to $(q - 1)$ [7]. In other words, for any prime q there exist infinitely many Ramanujan-like congruences combinatorially explained by the crank. Remember, however, that Ramanujan's congruences are known to be the "simplest" such congruences, the only ones where $A > B$. An informal and miscellaneous catalogue of such results, including those just described, can be found in Daniel Glasscock's lecture notes, "What is the crank of a partition?" [1].

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References

[1] Daniel Glasscock. What is the crank of a partition? https://math.osu.edu/sites/math.osu.edu/files/whatisthecrank.pdf

[2] F. J. Dyson. Some Guesses in the Theory of Partitions. https://www.archim.org.uk/eureka/archive/Eureka-8.pdf

[3] Francis Gerard Garvan. Generalization's of Dyson's Rank. http://qseries.org/fgarvan/phd-thesis/index.html

[4] George E. Andrews. The Theory of Partitions. Cambridge University Press. 1984.

[5] George E. Andrews and F. G. Garvan. Dyson's Crank of a Partition. https://www.ams.org/journals/bull/1988-18-02/S0273-0979-1988-15637-6/S0273-0979-1988-15637-6.pdf

[6] Jennifer French. Vector Partitions. Electronic Theses and Dissertations. 2018. htts://dc.etsu.edu/etd/3392

[7] Karl Mahlburg. Congruences for the Coefficients of Modular Forms and Applications to Number Theory. https://www.math.lsu.edu/ mahlburg/preprints/Mahlburg-Thesis05.pdf

[8] Manjil P. Saikia. Cranks of Ramanujan's Lost Notebook. Journal of the Assam Academy of Mathematics. 2013. https://arxiv.org/abs/1402.6644

[9] S. Ramanujan. Congruence Properties of Partitions. http://doi.org/10.1007/bf01378341

[10] Scott Ahlgren and Ken Ono. Congruence Properties for the Partition Function. https://www.pnas.org/content/98/23/12882