THREE NOTIONS OF DIMENSION AND AN APPLICATION

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ABSTRACT. The Lebesgue outer measure is a notion for the size of a set in $\mathbb{R}^n$. Dimension is a more generalized notion of size which includes non-integer values and helps differentiate between certain sets which outer measure cannot. In this paper, three notions of dimension are introduced and several facts about them are proven. In the last section, we discuss how dimension can be applied to asset prices.

CONTENTS

1. THE LEBESGUE OUTER MEASURE

Definition 1.1. The power set of a set X is the set of all subsets of X, including X and the empty set.

Definition 1.2. Let X be a set and $\mathcal{P}(X)$ be its power set. A subset $\Sigma \subseteq \mathcal{P}(X)$ is called a σ-algebra if it satisfies the following properties:

- **Closure under complements**: If $E \in \Sigma$, then $X \setminus E \in \Sigma$.
- $X \in \Sigma$
- **Closure under countable unions**: If $\{E_k\}_{k=1}^{\infty}$ are in Σ, then

$$\bigcup_{k=1}^{\infty} E_k \in \Sigma.$$

Two consequences of this definition are that sigma algebras contain the empty set and sigma algebras are closed under countable intersections.

Definition 1.3. Let X be a set and Σ be a σ-algebra on X. A function μ from Σ to the real numbers (including positive infinity) is called a measure if it satisfies the following properties:

- **Non-negativity**: For all E in X, $\mu(E) \geq 0$.
- $\mu(\emptyset) = 0$.

Date: Finalized on September 3rd, 2021.

- **Countable additivity**: For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

Remark 1.4. If E_1 and E_2 are measurable sets with $E_1 \subseteq E_2$, then $\mu(E_1) \leq \mu(E_2)$.

Proof. $E_2 = (E_2 \backslash E_1) \cup E_1$. $\mu(E_2) = \mu((E_2 \backslash E_1) \cup E_1) = \mu(E_2 \backslash E_1) + \mu(E_1)$, by the countable additivity property. $E_2 \backslash E_1$ is measurable because it is the complement of $E_2^c \cup E_1$, which means it is in the σ-algebra. By the non-negativity property, $\mu(E_2 \backslash E_1) \geq 0$. Hence, $\mu(E_1) = \mu(E_2) - \mu(E_2 \backslash E_1) \leq \mu(E_2)$. □

Since adding more to a set can only result in its measure increasing (or staying constant), a measure can be thought of as assigning a size to a set. Outer measures do something similar, but unlike measures, they are conveniently defined for all subsets of X.

Definition 1.5. For any open interval $I = (a, b)$, let $l(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the Lebesgue outer measure $\lambda^*(E)$ is defined as

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) : (I_k)_{k \in \mathbb{N}} \text{ is a sequence of open intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The Lebesgue outer measure has countable subadditivity:

$$\lambda^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \lambda^*(E_k).$$

Remark 1.6. $\lambda^*(E)$ remains unchanged if each interval is allowed to be open, half-open, or closed rather than strictly open.

Proof. Suppose α^o is the inf of the total lengths of all the sequences of open intervals that cover E (call this set L^o), and α is the inf of the total lengths of all the sequences of open, half-open, or closed intervals that cover E (call this set L). $\alpha^o \geq \alpha$ because $L^o \subseteq L$. Suppose $\alpha^o > \alpha$. Then there exists some sequence of open, half-open, or closed intervals which covers E but has total length less than α^o. Since there are only a countable number of intervals in total, all the half-open or closed intervals in the sequence can be extended to open intervals, with the total length of the new sequence of strictly open intervals (which are still a cover for E since they cover everything the original sequence covered, plus possibly more) being less than α^o. The existence of a sequence of open intervals that cover E and have a total length less than α^o contradicts the assumption that α^o is the inf of L^o. Thus, α^o is not greater than α. It follows that they are equal. □

For the Lebesgue outer measure of a set $E \subseteq \mathbb{R}^k$, intervals are replaced by products of k intervals, and length is replaced by volume, which is the product of the lengths of the intervals.

For simple sets, the Lebesgue outer measure gives what one would intuit. Intervals in $\mathbb{R}$ have outer measure equal to the length of the interval, shapes (e.g., $[0, 1]$x$[0, 2]$) in $\mathbb{R}^2$ have outer measure equal to the area of the shape ($1 * 2 = 2$), and solids (e.g., $[0, 1]$x$[0, 2]$x$[0, 3]$) in $\mathbb{R}^3$ have outer measure equal to the volume of the solid ($1 * 2 * 3 = 6$).

Proof. Let $A = [0,1]\text{x}[0,2]\text{x}[0,3]$. $\lambda^*(A) \leq 6$ since $[0,1]\text{x}[0,2]\text{x}[0,3]$ is a product of intervals that covers A and has a volume of 6. Now use the definition of Lebesgue outer measure that involves covers by open products of intervals. Given any sequence of open products of intervals, there exists a finite subcover of A since A is compact. The finite subcover must have a total volume of at least 6, since A has a volume of 6. Therefore, the total volume of any sequence of open products of intervals which covers A is greater than or equal to 6. Thus, $\lambda^*(A) \geq 6$. It follows that $\lambda^*(A) = 6$. In general, any product of n intervals will have n-dimensional outer measure equal to the product of the lengths of the intervals, and the proof is similar. □

The Lebesgue outer measure is good at distinguishing the relative sizes of products of k intervals in $\mathbb{R}^k$. However, a line segment and a square both have outer measure 0 in $\mathbb{R}^3$. In this case, there is a need for another notion beyond outer measure which helps indicate the size of a set; this notion is dimension.

2. The self-similarity dimension

The self-similarity dimension is a notion of dimension which is easy to determine for self-similar fractals and other simple sets.

Definition 2.1. Given a bounded, self-similar set $A \subset \mathbb{R}^n$, suppose that scaling A by a factor of s results in a change in the size of A by a factor of c, and let $c = s^d$. Then d, the self-similarity dimension of A, is equal to $\log_s(c)$.

When a point is scaled by a factor of r, its size changes by a factor of 1 (it's still just a point), so points have self-similarity dimension 0. When a line segment is scaled by a factor of r, its size (length) changes by a factor of r, so line segments have self-similarity dimension 1. When a square is scaled by a factor of r, its size (area) changes by a factor of r^2, so squares have self-similarity dimension 2. Regardless of the space that the line segment and square are embedded in, the self-similarity dimension distinguishes their sizes. Thus, it solves the issue we ran into at the end of the previous section, in which both a square and a line segment had outer measure 0 in $\mathbb{R}^3$, despite obviously not being the same size.

For a more complicated example, take the Vicsek fractal, a subset of $\mathbb{R}^2$ consisting of the intersection of countably many iterates.

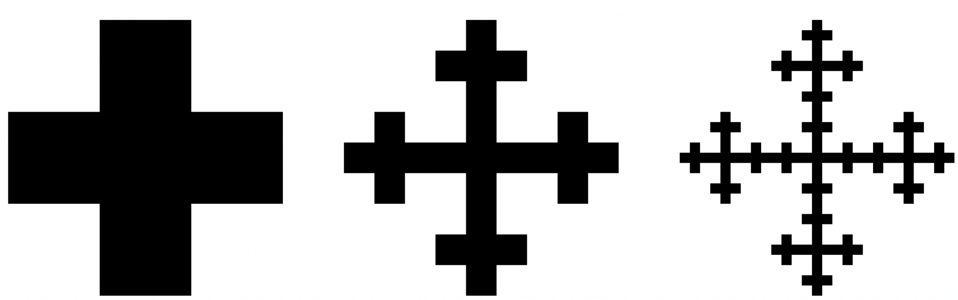

Figure 1. First three iterations of the Vicsek fractal (reference [6])

At each iteration, each cross turns into 5 crosses of $\frac{1}{3}$rd the scale (or $\frac{1}{9}$th the size) of the original. When the original Vicsek fractal is scaled by a factor of 3, the result is a set consisting of the original Vicsek fractal and four translated copies. Thus, the Vicsek fractal has self-similarity dimension $\log_3(5)$, which is greater than 1 but less than 2. This agrees with intuition because the Vicsek fractal has infinite length (the perimeter of each iteration is at least $\frac{11}{9}$ths of the previous iteration), but 0 area (the area of each iteration is $\frac{5}{9}$ths of the previous iteration).

Definition 2.2. The standard Ternary Cantor Set (from now on referred to as the Cantor set) is formed by first removing the middle third of the closed unit interval, leaving two closed intervals of length $\frac{1}{3}$ (the first iterate). Then the middle thirds of these two intervals are removed, leaving four closed intervals of length $\frac{1}{9}$ (the second iterate). The nth iterate contains 2^n closed intervals of length $\frac{1}{3^n}$. The Cantor set is the intersection of all the (countably many) iterates.

When the Cantor set is scaled by a factor of 3, the result is the original Cantor set (consisting of points between 0 and 1) and a copy of the original Cantor set, shifted 2 units right (consisting of points between 2 and 3):

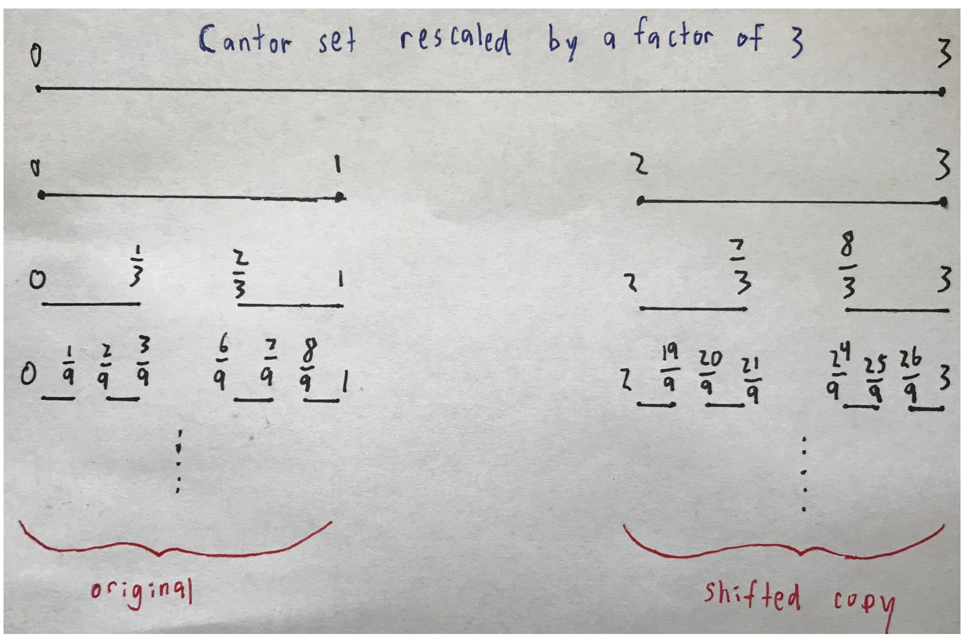

FIGURE 2. Cantor set scaled by a factor of 3

Scaling by a factor of 3 results in a size change by a factor of 2, so the standard Ternary Cantor Set has self-similarity dimension $\log_3(2)$. This agrees with intuition because the Cantor set is somewhere between a point and a line segment (it contains uncountably many points but does not contain a single line segment).

Claim 2.3. For any $\alpha \in (0, 1)$, there exists a Cantor set with self-similarity dimension α.

Proof. Construct such a set as follows: start with the unit interval. At the first iteration, remove $1 - \frac{2}{2^{\frac{1}{\alpha}}}$ from the middle, leaving the first $\frac{1}{2^{\frac{1}{\alpha}}}$ and the last $\frac{1}{2^{\frac{1}{\alpha}}}$. At the second iteration, from each of the two intervals left from the first iteration, remove $\frac{1}{2^{\frac{1}{\alpha}}} - \frac{2}{2^{\frac{2}{\alpha}}}$ from the middle, leaving the first $\frac{1}{2^{\frac{2}{\alpha}}}$ and the last $\frac{1}{2^{\frac{2}{\alpha}}}$. At the kth iteration, from each of the 2^{k-1} intervals left from the $(k-1)$th iteration, remove $\frac{1}{2^{\frac{k-1}{\alpha}}} - \frac{2}{2^{\frac{k}{\alpha}}}$ from the middle, leaving the first $\frac{1}{2^{\frac{k}{\alpha}}}$ and the last $\frac{1}{2^{\frac{k}{\alpha}}}$. Examine the set formed by intersecting all of the (countably many) iterates. When this set is scaled by a factor of $2^{\frac{1}{\alpha}}$, the result is the original set and a copy of the original set, shifted $2^{\frac{1}{\alpha}} - 1$ units to the right. Thus, the set has self-similarity dimension $\log_{2^{\frac{1}{\alpha}}}(2) = \alpha$. □

The self-similarity dimension is an intuitive, useful, and quick way to compare the sizes of self-similar sets. However, most sets are not self-similar. If one wants to be able to assign a dimension to any set, one should turn to the Hausdorff dimension.

3. The Hausdorff dimension

Definition 3.1. Given $s \geq 0$, $\alpha(s) > 0$, $\delta > 0$, and $A \subseteq \mathbb{R}^n$, let

$$H^s_\delta(A) = \inf\left\{\alpha(s)\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^s : A \subseteq \bigcup_{k=1}^{\infty} E_k \text{ and for every } k,\ \operatorname{diam}(E_k) < \delta\right\},$$

where $\alpha(s)$ is some constant that depends only on s. If the set has no finite values (only contains infinity), then let $H^s_\delta(A) = \infty$.

For any $\delta_1 < \delta_2$, any cover of A with $\operatorname{diam}(E_k) < \delta_1$ for every k is also a cover of A with $\operatorname{diam}(E_k) < \delta_2$ for every k, so $H^s_{\delta_2}(A)$ is the infimum of a larger set than $H^s_{\delta_1}(A)$ is, meaning $H^s_{\delta_2}(A) \leq H^s_{\delta_1}(A)$. Since $H^s_\delta(A)$ is monotone, $\lim_{\delta\to 0} H^s_\delta(A)$ exists, though it may be infinity.

Remark 3.2. If $\lim_{\delta\to 0} H^s_\delta(A) = 0$, then $H^s_c(A) = 0$ for every $c > 0$.

Proof. Suppose for some $c > 0$, $H^s_c(A) > 0$. From above, for all $0 < \delta < c$, $H^s_\delta(A) \geq H^s_c(A) > 0$. This contradicts the assumption that $\lim_{\delta\to 0} H^s_\delta(A) = 0$. Thus, $H^s_c(A) = 0$ for every $c > 0$. □

Define the ***s*-dimensional Hausdorff outer measure of *A***

$$H^s(A) = \lim_{\delta\to 0} H^s_\delta(A).$$

If $\operatorname{diam}(E_k) < 1$ for every k (which occurs eventually since δ is going to 0),

$$\text{then } \sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^s \text{ is monotone decreasing in s.}$$

This suggests that $H^s(A)$ is monotone decreasing in s, which we will soon see.

Claim 3.3. If $H^s(A)$ is finite, $H^{s+c}(A) = 0$ for any $c > 0$.

Proof. Suppose $H^s(A) = l$. In other words,

$$\lim_{\delta\to 0}\inf\left\{\alpha(s)\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^s : A \subseteq \bigcup_{k=1}^{\infty} E_k \text{ and for every } k,\ \operatorname{diam}(E_k) < \delta\right\} = l.$$

Examine $H^{s+c}(A) =$

$$\lim_{\delta\to 0}\inf\left\{\alpha(s+c)\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s+c} : A\subseteq\bigcup_{k=1}^{\infty}E_k \text{ and for every } k,\ \operatorname{diam}(E_k)<\delta\right\}.$$

For every $\delta > 0$,

$$\frac{H_\delta^{s+c}(A)}{\alpha(s+c)} = \inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s+c} : A\subseteq\bigcup_{k=1}^{\infty}E_k \text{ and for every } k,\ \operatorname{diam}(E_k)<\delta\right\}$$

$$\leq \inf\left\{\sum_{k=1}^{\infty}\delta^c(\operatorname{diam}(E_k))^{s} : ...\right\} = \delta^c\cdot\inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s} : ...\right\},$$

since $\operatorname{diam}(E_k) < \delta$ for every k. Also,

$$\lim_{\delta\to 0}\left[\delta^c\cdot\inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s} : ...\right\}\right] =$$

$$\frac{1}{\alpha(s)}\cdot\lim_{\delta\to 0}\left[\delta^c\cdot\inf\left\{\alpha(s)\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s} : ...\right\}\right] =$$

$$\frac{1}{\alpha(s)}\cdot\lim_{\delta\to 0}\delta^c\cdot\lim_{\delta\to 0}H_\delta^s(A) = \frac{1}{\alpha(s)}\cdot 0\cdot l = 0.$$

Since for every $\delta > 0$

$$0 \leq \frac{H_\delta^{s+c}(A)}{\alpha(s+c)} \leq \delta^c\cdot\inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{s} : ...\right\},\quad \lim_{\delta\to 0}\frac{H_\delta^{s+c}(A)}{\alpha(s+c)} = 0.$$

It follows that $\lim_{\delta\to 0} H_\delta^{s+c}(A) = H^{s+c}(A) = 0$. □

Now, given $s_2 > s_1$, if $H^{s_1}(A) = \infty$, then most definitely $H^{s_2}(A) \leq H^{s_1}(A)$. If $H^{s_1}(A)$ is finite, then $H^{s_2}(A) = 0$, and once again $H^{s_2}(A) \leq H^{s_1}(A)$. This shows that $H^s(A)$ is monotone decreasing in s.

Claim 3.4. For any $c > 0$, the $(n+c)$-dimensional Hausdorff outer measure of $[0,1]^n \subset \mathbb{R}^n$ is 0.

Proof. Fix any $\delta > 0$, and examine $\frac{H_\delta^{n+c}([0,1]^n)}{\alpha(n+c)} =$

$$\inf\left\{\sum_{k=1}^{\infty}(\operatorname{diam}(E_k))^{n+c} : [0,1]^n\subseteq\bigcup_{k=1}^{\infty}E_k \text{ and for every } k,\ \operatorname{diam}(E_k)<\delta\right\}.$$

Given $\varepsilon > 0$, find $m \in \mathbb{N}$ such that $\frac{\sqrt{n}}{m} < \delta$ and $(\frac{\sqrt{n}}{m})^c < \frac{\varepsilon}{\sqrt{n}^n}$. Cover $[0,1]^n$ with m^n closed cubes each of side length $\frac{1}{m}$. Each of these cubes has diameter $\sqrt{\frac{n}{m^2}} = \frac{\sqrt{n}}{m} < \delta$. Since $(\frac{\sqrt{n}}{m})^c < \frac{\varepsilon}{\sqrt{n}^n}$, the sum of the diameters to the $n+c$ power is equal to

$$m^n\left(\frac{\sqrt{n}}{m}\right)^{n+c} = m^n\left(\frac{\sqrt{n}}{m}\right)^{n}\left(\frac{\sqrt{n}}{m}\right)^{c} = \sqrt{n}^n\left(\frac{\sqrt{n}}{m}\right)^{c} < \sqrt{n}^n\cdot\frac{\varepsilon}{\sqrt{n}^n} = \varepsilon.$$

Thus, there exists a cover of $[0,1]^n$, $\{E_k\}$, such that $\text{diam}(E_k) < \delta$ for every k and $\sum_{k=1}^{\infty}(\text{diam}(E_k))^{n+c} < \varepsilon$. Since ε was arbitrary, $H_\delta^{n+c}([0,1]^n) = 0$. Since this holds for every $\delta > 0$, it follows that $H^{n+c}([0,1]^n) = 0$. □

Claim 3.5. Let $A = \bigcup_{k=1}^{\infty} E_k$. If $H^s(E_k) = 0$ for every k, then $H^s(A) = 0$.

Proof. Fix any $\delta > 0$. By Remark 3.2, $H_\delta^s(A) = 0$. Given $\varepsilon > 0$, for each E_k let $\{E_{ki}\}_{i=1}^{\infty}$ be a countable cover of E_k such that $\text{diam}(E_{ki}) < \delta$ for every i and $\sum_{i=1}^{\infty}(\text{diam}(E_{ki}))^s < \frac{\varepsilon}{2^k}$. Then $\bigcup_{k=1}^{\infty}\{E_{ki}\}$ is a countable collection of sets such that

$$A = \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty} E_{ki}, \text{ and } \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}(\text{diam}(E_{ki}))^s < \sum_{k=1}^{\infty}\frac{\varepsilon}{2^k} = \varepsilon.$$

Since ε was arbitrary, $H_\delta^s(A) = 0$. Since δ was arbitrary, $H_\delta^s(A) = 0$ for every δ, so $H^s(A) = 0$. □

Corollary 3.6. For any $c > 0$, the $(n+c)$-dimensional Hausdorff outer measure of $\mathbb{R}^n \subseteq \mathbb{R}^n$ is 0.

Proof. $\mathbb{R}^n = \bigcup_{k=1}^{\infty} E_k$, where each E_k is some translated copy of $[0,1]^n$. By Claim 3.4 and Claim 3.5, it follows that $H^{n+c}(\mathbb{R}^n) = 0$. □

Remark 3.7. If $A \subseteq B$, then $H^s(A) \leq H^s(B)$.

Proof. Fix any $\delta > 0$. Since $A \subseteq B$, any cover of B is also a cover of A. Thus, $H_\delta^s(A)$ is the infimum of a larger set than $H_\delta^s(B)$ is, meaning $H_\delta^s(A) \leq H_\delta^s(B)$. Since this holds for every $\delta > 0$, it follows that $H^s(A) \leq H^s(B)$. □

Now, given any arbitrary set $A \subseteq \mathbb{R}^n$, examine the set $\{s : H^s(A) = 0\}$. By Corollary 3.6 and Remark 3.7, $H^{n+1}(A) = 0$, so this set is not empty. Moreover, it is bounded from below by 0, so it has an infimum.

Definition 3.8. Given any arbitrary set $A \subseteq \mathbb{R}^n$,
define the **Hausdorff dimension of A**

$$\dim_H(A) = \inf\{s : H^s(A) = 0\}.$$

Now examine the set $\{s : H^s(A) = \infty\}$. By Claim 3.3 and the fact that $H^{n+c}(A) = 0$ for any $c > 0$, it follows that the set is bounded above by n. The set is also not empty since it contains 0. Thus, it has a supremum.

Remark 3.9. $\dim_H(A) = \inf\{s : H^s(A) = 0\} = \sup\{s : H^s(A) = \infty\}$.

Proof. Suppose $\inf\{s : H^s(A) = 0\} < \sup\{s : H^s(A) = \infty\}$. Then there exists $c < d$ such that $H^c(A) = 0$ and $H^d(A) = \infty$. This contradicts Claim 3.3.
On the other hand, suppose $\inf\{s : H^s(A) = 0\} > \sup\{s : H^s(A) = \infty\}$. Let c and d be such that $\sup\{s : H^s(A) = \infty\} < c < d < \inf\{s : H^s(A) = 0\}$. $H^c(A)$ is finite since $c > \sup\{s : H^s(A) = \infty\}$, which means that $H^d(A) = 0$ (by Claim 3.3). This contradicts the assumption that $d < \inf\{s : H^s(A) = 0\}$. □

By Claim 3.3 and Remark 3.9, for any $s > \dim_H(A)$, $H^s(A) = 0$. On the other hand, for any $s < \dim_H(A)$, $H^s(A) = \infty$. Thus, the graph of s and $H^s(A)$ looks like this:

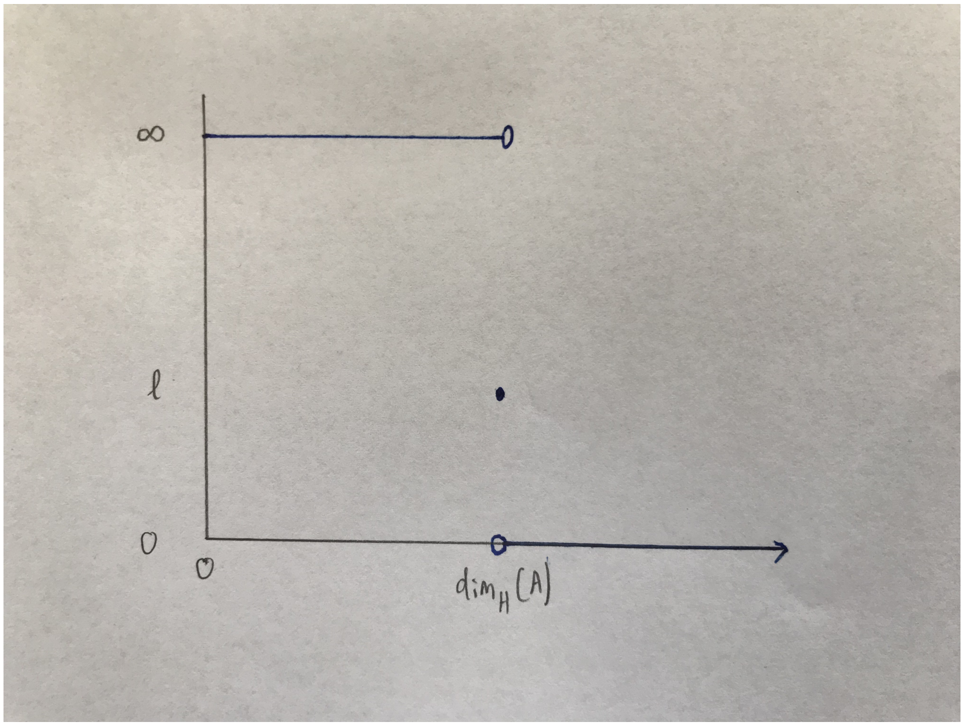

FIGURE 3. Graph of s and $H^s(A)$

This diagram provides the main intuition behind Hausdorff dimension: it is the unique transition value of the Hausdorff outer measure. When $s = \dim_H(A)$, $H^s(A)$ can be infinity, finitely positive (as depicted in the graph), or 0. For any smaller s, the s-dimensional Hausdorff outer measure of A is infinite, and for any larger s, the s-dimensional Hausdorff outer measure of A is zero. Thus, $\dim_H(A)$ is exactly the correct scale at which to measure A.

In addition to existing for every subset of $\mathbb{R}^n$, the Hausdorff dimension has several other desirable properties that make it theoretically useful.

Remark 3.10. If $A \subseteq B$, then $\dim_H(A) \leq \dim_H(B)$.

Proof. Suppose $\dim_H(B) < \dim_H(A)$. Then there exists some s that is greater than $\dim_H(B)$ and less than $\dim_H(A)$, meaning that $H^s(B) = 0$ and $H^s(A) = \infty$. This contradicts Remark 3.7. $\square$

Now we introduce an alternative definition of the Hausdorff dimension which will be used to prove some of its other properties:

Definition 3.11. Let $A \subseteq \mathbb{R}^n$. Define the **Hausdorff dimension of A**

$$\dim_H(A) = \inf\left\{ s : \forall \varepsilon > 0, \exists B_1, B_2, ... \text{ st } A \subseteq \bigcup_{k=1}^{\infty} B_k \text{ and } \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon \right\},$$

where each B_k is a closed ball.

Claim 3.12. Definition 3.11 and Definition 3.8 are equivalent.

Proof. To prove the two definitions are equivalent, we show that

$$\{s : H^s(A) = 0\} = \left\{ s : \forall \varepsilon > 0, \exists \{B_k\} \text{ st } A \subseteq \bigcup_{k=1}^{\infty} B_k \text{ and } \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon \right\},$$

so their infimums are the same. Fix any s.
Suppose for every $\varepsilon > 0$, there exist $B_1, B_2, ...$ such that

$$A \subseteq \bigcup_{k=1}^{\infty} B_k \text{ and } \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon. \text{ Fix any } \delta > 0. \text{ For every } \varepsilon \leq \delta^s,$$

there exist $B_1, B_2, ...$ such that $A \subseteq \bigcup_{k=1}^{\infty} B_k$ and $\sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon$, which means $\operatorname{diam}(B_k) < \delta$ for every k and $\alpha(s) \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \alpha(s)\varepsilon$. Since ε can be chosen arbitrarily small, $H^s_\delta(A) = 0$. This holds for every $\delta > 0$, so $H_s(A) = 0$.
On the other hand, suppose $H^s(A) = 0$. In other words,

$$\lim_{\delta \to 0} \inf \left\{ \alpha(s) \sum_{k=1}^{\infty} (\operatorname{diam}(E_k))^s : A \subseteq \bigcup_{k=1}^{\infty} E_k \text{ and for every } k, \operatorname{diam}(E_k) < \delta \right\} = 0.$$

Given $\varepsilon > 0$, choose $\delta > 0$ such that whenever $0 < x < \delta$,

$$\inf \left\{ \alpha(s) \sum_{k=1}^{\infty} (\operatorname{diam}(E_k))^s : A \subseteq \bigcup_{k=1}^{\infty} E_k \text{ and for every } k, \operatorname{diam}(E_k) < x \right\} < \frac{\alpha(s)}{2^s}\varepsilon.$$

Let $x = \frac{\delta}{2}$. There exist $E_1, E_2, ...$ such that

$$A \subseteq \bigcup_{k=1}^{\infty} E_k \text{ and } \alpha(s) \sum_{k=1}^{\infty} (\operatorname{diam}(E_k))^s < \frac{\alpha(s)}{2^s}\varepsilon.$$

For every k, choose any $p \in E_k$, and let B_k be the closed ball centered at p with radius $\operatorname{diam}(E_k)$. $E_k \subset B_k$ for every k, so $A \subseteq \bigcup_{k=1}^{\infty} B_k$. Moreover, since

$$\alpha(s) \sum_{k=1}^{\infty} (\operatorname{diam}(E_k))^s = \alpha(s) \sum_{k=1}^{\infty} \left(\frac{\operatorname{diam}(B_k)}{2} \right)^s = \frac{\alpha(s)}{2^s} \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \frac{\alpha(s)}{2^s}\varepsilon,$$

$\sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon$. Thus, for any fixed s,

$$H^s(A) = 0 \iff \forall \varepsilon > 0, \exists B_1, B_2, ... \text{ st } A \subseteq \bigcup_{k=1}^{\infty} B_k \text{ and } \sum_{k=1}^{\infty} (\operatorname{diam}(B_k))^s < \varepsilon,$$

so the two definitions are equivalent. □

Remark 3.13. Just as the intervals could be open, half-open, or closed in the definition of the Lebesgue outer measure, these balls can be open or closed.

Recall that the self-similarity dimension is just as able to distinguish between a line segment and a square in $\mathbb{R}^3$ as it is a line segment and a square in $\mathbb{R}^2$. Any good notion of dimension should allow us to compare sets without having to worry about the space each set is embedded in. The Hausdorff dimension meets this requirement.

Remark 3.14. Let $A \subset \mathbb{R}^n$ and $A^0 \subset \mathbb{R}^{n+1}$, where A^0 consists of all the points in A with 0 as the $(n+1)$st coordinate. Then $\dim_H(A) = \dim_H(A^0)$.

Extending a ball from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ by adding 0 as the $(n+1)$st coordinate does not change the ball's diameter. Similarly, reducing a ball in $\mathbb{R}^{n+1}$ to a ball in $\mathbb{R}^n$ by removing the $(n+1)$st coordinate does not change the ball's diameter.

The Hausdorff dimension also has the useful property of being countably stable (Claim 3.15).

Claim 3.15. Let $A = \bigcup_{k=1}^{\infty} E_k$. Then $\dim_H(A) = \sup\{\dim_H(E_k) : k \in \mathbb{N}\}$.

Proof. Suppose $\dim_H(A) > \sup\{\dim_H(E_k) : k \in \mathbb{N}\}$. Choose $\alpha \in \mathbb{R}$, $\sup\{\dim_H(E_k) : k \in \mathbb{N}\} < \alpha < \dim_H(A)$. Since $\alpha > \dim_H(E_k)$ for every k, $H^\alpha(E_k) = 0$ for every k. By Claim 3.5, $H^\alpha(A) = 0$. This is a contradiction because $\alpha < \dim_H(A)$.
On the other hand, suppose $\dim_H(A) < \sup\{\dim_H(E_k) : k \in \mathbb{N}\}$. Then there exists $\alpha \in \mathbb{R}$ such that $\dim_H(A) < \alpha < \sup\{\dim_H(E_k) : k \in \mathbb{N}\}$, so for some $m \in \mathbb{N}$, $\alpha < \dim_H(E_m)$ and thus $H^\alpha(E_m) = \infty$. Since $E_m \subseteq A$, $H^\alpha(A) = \infty$ by Remark 3.10. This is a contradiction because $\alpha > \dim_H(A)$. □

Claim 3.16. The Hausdorff dimension of any countable set is 0.

Proof. First, we show that the Hausdorff dimension of a point p is 0. Fix any $s > 0$. Given any $\varepsilon > 0$, take the ball with diameter $(\frac{\varepsilon}{2})^{\frac{1}{s}}$, centered at p. This ball covers p, and its diameter to the power of s is $\frac{\varepsilon}{2}$, which is less than ε. Thus, for every $\varepsilon > 0$, there exists a countable collection of balls which covers p and for which the sum of the diameters to the s power is less than ε. Since this holds for every $s > 0$, $\dim_H(p) = 0$. Applying Claim 3.15, we see that the Hausdorff dimension of any countable set is 0. □

The Hausdorff dimension has many nice properties. However, the box-counting dimension is preferable for some applications because it is easier to calculate.

4. The Minkowski-Bouligand (box-counting) dimension

Definition 4.1. Given a bounded set $A \subset \mathbb{R}^n$ and $\varepsilon > 0$, let $N(\varepsilon)$ be the minimum number of closed cubes of side length ε needed to cover A.

Define the **Minkowski-Bouligand (box-counting) dimension of *A***

$$\dim_{box}(A) = \lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N(\varepsilon)),$$

if the limit exists (see example 4.11 for a set where it does not).

Claim 4.2. If it exists, the box-counting dimension is the same regardless of whether closed or open cubes are used to cover the set.

Proof. Given $A \subset \mathbb{R}^n$, let $N(\varepsilon)$ be the minimum number of closed cubes of side length ε needed to cover A and let $N^o(\varepsilon)$ be the minimum number of open cubes of side length ε needed to cover A. For every $\varepsilon > 0$, $\frac{N^o(\varepsilon)}{2^n} \leq N(\varepsilon) \leq N^o(\varepsilon) \leq 2^n \cdot N(\varepsilon)$. Suppose $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} N(\varepsilon) = a$. Then for every $c > 0$, $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(c \cdot N(\varepsilon)) = a$. Since $N(\varepsilon) \leq N^o(\varepsilon) \leq 2^n \cdot N(\varepsilon)$, $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} N^o(\varepsilon) = a$. A similar argument can be used for the other direction (going from open cubes to closed cubes). □

Remark 4.3. If $A \subseteq B$, then $N_A(\varepsilon) \leq N_B(\varepsilon)$ for every ε, so $\dim_{box}(A) \leq \dim_{box}(B)$ if they both exist.

Remark 4.4. Remark 3.14 also applies to box-counting dimension.

Remark 4.5. If $\varepsilon_1 < \varepsilon_2$, then $N_A(\varepsilon_1) \geq N_A(\varepsilon_2)$.

Intuitively, the larger or more dense a set, the more $N(\varepsilon)$ increases in response to a decrease in ε, and the greater the box-counting dimension. For example, a single point, which is the smallest nonempty set, has box-counting dimension 0 since $N(\varepsilon) = 1$ for every $\varepsilon > 0$.

Claim 4.6. Given any $n \in \mathbb{N}$, $\dim_{box}([0, \frac{1}{n}]) = 1$.

Proof. $N(\frac{1}{kn}) = k$ for every $k \in \mathbb{N}$. Given any $0 < \varepsilon < \frac{1}{n}$, there exists $k \in \mathbb{N}$ such that $\frac{1}{(k+1)n} \leq \varepsilon \leq \frac{1}{kn}$. Thus, $k \leq N(\varepsilon) \leq (k+1)$ and $kn \leq \frac{1}{\varepsilon} \leq (k+1)n$, so $\log_{(k+1)n}(k) \leq \log_{\frac{1}{\varepsilon}}(N(\varepsilon)) \leq \log_{kn}(k+1)$. Also,

$$\lim_{k \to \infty} \log_{(k+1)n}(k) = \lim_{k \to \infty} \log_{kn}(k+1) = 1.$$

Given $\sigma > 0$, find $m \in \mathbb{N}$ such that for all $k \geq m$, $|\log_{(k+1)n}(k) - 1| < \sigma$ and $|\log_{kn}(k+1) - 1| < \sigma$. Then for all ε such that $0 < \varepsilon < \frac{1}{m}$, $|\log_{\frac{1}{\varepsilon}}(N(\varepsilon)) - 1| < \sigma$. Since σ was arbitrary, $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N(\varepsilon)) = 1$. □

Claim 4.7. Let $A = \bigcup_{k=1}^{n} E_k$. If $\dim_{box}(E_1), ..., \dim_{box}(E_n)$ exist, then $\dim_{box}(A) = \max\{\dim_{box}(E_1), ..., \dim_{box}(E_n)\}$.

Proof. It suffices to show this for $n = 2$. Note that $N_A(\varepsilon) \leq N_{E_1}(\varepsilon) + N_{E_2}(\varepsilon)$ for all $\varepsilon > 0$, and $N_{E_1}(\varepsilon), N_{E_2}(\varepsilon) \leq N_A(\varepsilon)$ by Remark 4.3. Thus, for all $\varepsilon > 0$,

either $N_{E_1}(\varepsilon) \leq N_A(\varepsilon) \leq 2 \cdot N_{E_1}(\varepsilon)$ [1] or $N_{E_2}(\varepsilon) \leq N_A(\varepsilon) \leq 2 \cdot N_{E_2}(\varepsilon)$ [2].

Also,

$$\dim_{box}(E_1) = \lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N_{E_1}(\varepsilon)) = \lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(2 \cdot N_{E_1}(\varepsilon)) \text{ [*]},$$

and the same is true for E_2.

First, take the case in which $\dim_{box}(E_1) \neq \dim_{box}(E_2)$. WLOG, assume $\dim_{box}(E_1) > \dim_{box}(E_2)$. Find $x > 0$ such that whenever $0 < \varepsilon < x$, $|\log_{\frac{1}{\varepsilon}}(N_{E_1}(\varepsilon)) - \dim_{box}(E_1)|$ and $|\log_{\frac{1}{\varepsilon}}(N_{E_2}(\varepsilon)) - \dim_{box}(E_2)|$ are both less than

$$\frac{\dim_{box}(E_1) - \dim_{box}(E_2)}{4}.$$ Then for all $0 < \varepsilon < x$, $N_{E_1}(\varepsilon) > N_{E_2}(\varepsilon)$.

This means that $N_{E_1}(\varepsilon) \leq N_A(\varepsilon) \leq 2 \cdot N_{E_1}(\varepsilon)$ for all $0 < \varepsilon < x$. By [*], it follows that $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N_A(\varepsilon)) = \dim_{box}(E_1)$.

For the other case, assume $\dim_{box}(E_1) = \dim_{box}(E_2)$, and call this value a. Given $\sigma > 0$, choose $x > 0$ such that whenever $0 < \varepsilon < x$, $|\log_{\frac{1}{\varepsilon}}(N_{E_1}(\varepsilon)) - a|$, $|\log_{\frac{1}{\varepsilon}}(2 \cdot N_{E_1}(\varepsilon)) - a|$, $|\log_{\frac{1}{\varepsilon}}(N_{E_2}(\varepsilon)) - a|$, and $|\log_{\frac{1}{\varepsilon}}(2 \cdot N_{E_2}(\varepsilon)) - a|$ are all less than σ. For every ε between 0 and x, either [1] or [2] must be true. In both cases, $|\log_{\frac{1}{\varepsilon}}(N_A(\varepsilon)) - a| < \sigma$. Thus, whenever $0 < \varepsilon < x$, $|\log_{\frac{1}{\varepsilon}}(N_A(\varepsilon)) - a| < \sigma$. Since σ was arbitrary, $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N_A(\varepsilon)) = a$. □

Remark 4.8. From Claim 4.6, Claim 4.7, and Remark 4.3, it follows that the box-counting dimension of any interval $[a, b]$ is 1. By a similar argument, the box-counting dimension of any product of k intervals is k.

Although box-counting dimension is finitely stable (Claim 4.7), it is not countably stable, and that is one of its limitations compared to Hausdorff dimension. For example, the rationals in $[0, 1]$ have box-counting dimension 1, even though each individual rational has dimension 0.

Claim 4.9. Another example is the set $\{\frac{1}{n} : n \in \mathbb{N}\}$, which contains no limit points but has box-counting dimension $\frac{1}{2}$.

Proof. For every $n \in \mathbb{N}$ such that $\frac{1}{n^2} \leq \frac{1}{4}$, at least n boxes of side length $\frac{1}{n^2}$ are required to cover the points $1, ..., \frac{1}{n}$, and at least 1 more box is required to cover the rest. Thus, $N(\frac{1}{n^2}) \geq n + 1$.

$$\text{For every } \varepsilon < \frac{1}{4},\ N(\varepsilon) \geq \frac{1}{\sqrt{\varepsilon}} \text{ (1).}$$

To show this, note that for every such ε, there exists $k \in \mathbb{N}$ such that $\frac{1}{(k+1)^2} \leq \varepsilon \leq \frac{1}{k^2}$, so $k \leq \frac{1}{\sqrt{\varepsilon}} \leq k + 1$. Since $\varepsilon \leq \frac{1}{k^2}$, $N(\varepsilon) \geq N(\frac{1}{k^2}) \geq k + 1 \geq \frac{1}{\sqrt{\varepsilon}}$.

$$\text{Also, for every } \varepsilon < \frac{1}{4},\ N(\varepsilon) \leq \frac{3}{\sqrt{\varepsilon}} \text{ (2).}$$

To show this, examine $N(\frac{1}{n^2})$ for some $n \in \mathbb{N}$. To cover the points $1, ..., \frac{1}{n-1}$, $n - 1$ boxes is sufficient. To cover all points less than $\frac{1}{n^2}$, 1 box is sufficient. To cover all points in $[\frac{1}{n^2}, \frac{1}{n}]$, $n - 1$ boxes of side length $\frac{1}{n^2}$ is sufficient since $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$. Thus, $N(\frac{1}{n^2}) \leq 2n - 1$. Given $\varepsilon < \frac{1}{4}$, find $k \in \mathbb{N}$ such that $\frac{1}{(k+1)^2} \leq \varepsilon \leq \frac{1}{k^2}$. $N(\varepsilon) \leq N(\frac{1}{(k+1)^2}) \leq 2k + 1 \leq 3k \leq \frac{3}{\sqrt{\varepsilon}}$. By (1), (2), and the fact that $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(\frac{1}{\sqrt{\varepsilon}}) = \lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(\frac{3}{\sqrt{\varepsilon}}) = \frac{1}{2}$, $\lim_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}}(N(\varepsilon)) = \frac{1}{2}$. □

For fractals, the box-counting dimension is often the same as the self-similarity dimension.

Claim 4.10. The Cantor set has box-counting dimension $\log_3(2)$.

Proof. For every $n \in \mathbb{N}$, $N(\frac{1}{3^n}) = 2^n$. Given any $0 < \varepsilon < \frac{1}{3}$, there exists $k \in \mathbb{N}$ such that $\frac{1}{3^{k+1}} \leq \varepsilon \leq \frac{1}{3^k}$. Thus, $2^k \leq N(\varepsilon) \leq 2^{k+1}$ and $3^k \leq \frac{1}{\varepsilon} \leq 3^{k+1}$, so

$$\log_{3^{k+1}}(2^k) = \frac{k}{k+1} \log_3(2) \leq \log_{\frac{1}{\varepsilon}}(N(\varepsilon)) \leq \frac{k+1}{k} \log_3(2) = \log_{3^k}(2^{k+1}).\ \text{Also,}$$

$$\lim_{k \to \infty} \frac{k}{k+1} \log_3(2) = \lim_{k \to \infty} \frac{k+1}{k} \log_3(2) = \log_3(2).$$

Given $\sigma > 0$, find m such that for all $k \geq m$, $|\frac{k}{k+1}\log_3(2)) - \log_3(2)| < \sigma$ and $|\frac{k+1}{k}\log_3(2) - \log_3(2)| < \sigma$. Then for all ε such that $0 < \varepsilon < \frac{1}{3^m}$, $|\log_{\frac{1}{\varepsilon}}(N(\varepsilon)) - \log_3(2)| < \sigma$. Since σ was arbitrary, $\lim_{\varepsilon\to 0}\log_{\frac{1}{\varepsilon}}(N(\varepsilon)) = \log_3(2)$. □

As mentioned in Definition 4.1, the box-counting definition is not defined for every bounded subset of $\mathbb{R}^n$. For example, we can construct an irregular (not self-similar) Cantor set for which $\lim_{\varepsilon\to 0}\log_{\frac{1}{\varepsilon}}(N(\varepsilon))$ does not exist by taking out different amounts from the middles of the intervals at each iteration.

Example 4.11. For the first iteration, remove the middle third. This leaves 2 intervals each of length $\frac{1}{3}$, so $N(\frac{1}{3}) = 2$. For the second iteration, remove $\frac{5}{24}$ from the middle of each interval left over from the first iteration. This leaves 4 intervals each of length $\frac{1}{16}$, so $N(\frac{1}{16}) = 4$. Now, it is not possible to have 8 intervals each of length $\frac{1}{27}$ after the third iteration, because $\frac{2}{27} > \frac{1}{16}$. Instead, find the smallest $n \in \mathbb{N}$ such that $\frac{2^{n-2}}{3^n} < \frac{1}{16}$, which is $n = 4$. For the third iteration, remove a negligible amount from each of the 4 intervals left over from the second iteration. For the fourth iteration, remove just enough from each of the 8 intervals left over from the third iteration to leave 16 intervals each of length $\frac{1}{81}$, so $N(\frac{1}{81}) = 16$. For the fifth iteration, remove the necessary amount to leave 2^5 intervals each of length $\frac{1}{4^5}$, so $N(\frac{1}{4^5}) = 2^5$. Now, after the sixth iteration it is not possible to have 2^6 intervals each of length $\frac{1}{3^6}$, because $\frac{2}{3^6} > \frac{1}{4^5}$. Instead, find the smallest $n \in \mathbb{N}$ such that $\frac{2^{n-5}}{3^n} < \frac{1}{4^5}$, which is $n = 9$. For the sixth, seventh, and eight iterations, remove negligible amounts from each of the intervals left over from the previous iteration. For the ninth iteration, remove just enough from each of the 2^8 intervals left over from the 8th iteration to leave 2^9 intervals each of length $\frac{1}{3^9}$, so $N(\frac{1}{3^9}) = 2^9$. This process can be repeated to infinity because for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\frac{2^{n-k}}{3^n} < \frac{1}{4^k}$. The resulting set is one such that for every $x > 0$, there exist $0 < a < x$ and $0 < b < x$ such that $\log_{\frac{1}{a}} N(a) = \log_3(2)$ and $\log_{\frac{1}{b}} N(b) = \log_4(2)$. It follows that $\lim_{\varepsilon\to 0}\log_{\frac{1}{\varepsilon}}(N(\varepsilon))$ does not exist.

5. Applying dimension to asset prices

The Black-Scholes model assumes that stock prices follow Brownian motion. Graphs of Brownian motion have Hausdorff dimension $\frac{3}{2}$ almost surely (reference [5], Theorem 4.29). Thus, calculating the dimension of the graph of a stock's price versus time can tell us how close that asset is to exhibiting Brownian motion. This is where the box-counting dimension comes in. For most of these graphs, it should be pretty similar to the Hausdorff dimension and easier to calculate.

To calculate the box-counting dimension of a stock, we need data on prices, preferably by a small-enough time scale to provide enough detail (milliseconds, for example). Though the prices will always be discrete, we can pretend as if they are connected by line segments to approximate continuous prices. Then, we select several values of ε which go to 0 and calculate $N(\varepsilon)$ for each ε. When selecting the values, we must keep in mind how fine our data is, so we select scales that are not too big or too small (if too big, then ε won't be getting close enough to 0 to give an accurate calculation, and if too small, the dimension will be too close to 1 since we only have finite resolution). After plotting $\log(\frac{1}{\varepsilon})$ and $\log(N(\varepsilon))$, we find a line of best fit through those points. The slope of this line should approximate the dimension of the graph, if it were continuous.

In general, the graphs of assets with greater price fluctuations (volatility) exhibit greater dimension since more boxes are required to cover big jumps in prices. Thus, dimension may be a useful measure of an asset's volatility compared to other assets.

Acknowledgments

I thank Jake Fiedler for teaching me about dimension, giving me intellectually stimulating exercises, and being a cool person to talk to and learn from. I also thank Peter May for his endless energy in organizing the program.

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