ANALYZING THE RUBIK'S CUBE GROUP OF VARIOUS SIZES AND SOLUTION METHODS

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ABSTRACT. The Rubik's cube is a popular puzzle toy; its structure can be understood as a group where the elements are distinct configurations of the cube. We compute the order of the Rubik's cube group as a consequence of the number of legal configurations of the cube, and explain how the most common solution method, CFOP, interacts with the group structure.
In addition to the common 3×3 Rubik's cube, there are also larger and smaller variants. A simpler 2×2 "Pocket Cube" and more complex 4×4 "Rubik's Revenge" can be analyzed in similar ways to the Rubik's Cube. We explain the group structure and solution methods of these cubes by building on the framework established by the 3×3 cube.

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1. INTRODUCTION

The Rubik's cube is one of the most popular puzzles ever invented. Despite its simple appearance, the cube possesses a deep complexity which is not apparent from a cursory inspection, which many people discover when they try to solve one; this complexity is responsible for its ability to capture attention. Mathematically, the cube's structure can be formalized with the Rubik's cube group, which enables usage of the tools of group theory to study that complexity.

There are two natural questions we can ask about the cube. First: *how many different configurations of the cube exist?* Second: *given some configuration, how can we solve the cube?* Both of these questions can be studied by analyzing the structure of the cube group and special elements in the group.

We begin by laying out some important concepts and notation for the cube in section 2. In section 3, we will translate the physical structure of the Rubik's cube into mathematical notation and assemble that structure into a group: the cube group, G_3. The elements of the group are configurations of pieces of the cube, where each configuration has an associated sequence of moves; multiplication is the composition of those associated sequences of moves. The order of the group G_3 is exactly the number of configurations of the cube:

Theorem 1.1. *The order of* G_3 *is* $\frac{1}{2} \times 8! \times 3^7 \times 12! \times 2^{11}$.

In section 4, we will explore the most common cube solving method, called CFOP, and its relationship with the group structure. By reducing the configuration of the cube into smaller and smaller subgroups, it can be solved in simpler stages. This solution method comes from a special filtration of the cube group. We will divide the method into four major steps and explain the details.

In addition to the well-known 3×3 Rubik's cube, there are other cubes with similar mechanisms but different sizes. The two which are nearest to the standard cube are the 2×2 cube ("Pocket Cube") and the 4×4 cube ("Rubik's Revenge"), which differ only in the number of pieces per face. In sections 5 and 6, we will use similar methods to analyze the differently-sized cube groups and derive solution methods.

2. Cube Notation

We first need to define some notation for talking about the basic Rubik's cube as a physical object. The most elementary concepts in the study of the cube are cubies and cubicles:

Definition 2.1. A **cubie** is an individual 3-dimensional piece of the Rubik's cube which is independently manipulated by turns of the cube.

The 3×3 cube has 26 cubies. There are 6 *center cubies* which are the center of each face of the cube, 12 *edge cubies* between the corners, and 8 *corner cubies* located at each corner. (The 27th unit of volume is the center of the cube, which has no visible faces and therefore doesn't need to be counted.) The cubies have one, two, and three faces respectively, which are the 2-dimensional visible sides of the cubies.

Definition 2.2. A **cubicle** is a position on the Rubik's cube occupied by a cubie.

There are 26 cubicles with the same types as the cubies: 6 centers, 8 corners, 12 edges. When we execute a move on the cube, all the cubicles remain in place, while the cubies move to new cubicles. Furthermore, the edge and corner cubies might be flipped or rotated while remaining in the same cubicle. In order to fully describe the positions of cubies, we need the concept of orientation:

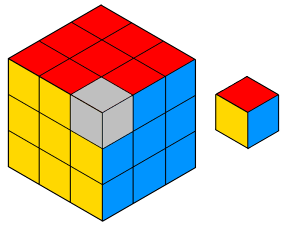

FIGURE 1. A Rubik's cube with one cubie extracted. The grey space left behind is its cubicle.

Definition 2.3. For each corner cubicle, assign the number 0 to any cubicle-face on the up or down face of the cube, then 1 to the cubicle-face clockwise from 0, and 2 to the remaining face. In the solved position of the cube, we mark all corner cubie faces which are on the up or down face. The **orientation of a corner cubie** is then the number of the cubicle-face which contains the marked face of the corner cubie inside of it. We treat this number as an element of $\mathbb{Z}/3\mathbb{Z}$.

Definition 2.4. For each edge cubicle, assign the number 0 to a cubicle-face on the up/down face, or the front/back face for the four edge cubicles lacking an up or down face, and assign 1 otherwise. In the solved position of the cube, we mark all edge cubie faces which are on the up or down face, or the front or back face for cubies in the middle layer. The **orientation of an edge cubie** is the number of the cubicle-face which contains the marked face of the corner cubie inside of it. We treat this number as an element of $\mathbb{Z}/2\mathbb{Z}$.

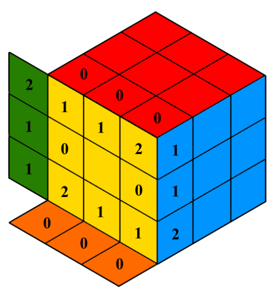

FIGURE 2. The cube with the orientations on one layer indicated. The hidden faces are unfolded to show all assigned values.

We use the following notation to denote the basic moves of the cube. First, we fix the cube in a particular orientation, with one face pointing directly towards the viewing perspective. Then F is the clockwise $90°$ turn of the front face, which is the one pointing at the viewer. Similarly, B is the clockwise turn of the back face, U of the up (top) face, D of the down (bottom) face, and R and L for the right and left faces. A counterclockwise turn is denoted F^{-1} (note that F^{-1} gives the same result as applying F three times). Figure 3 demonstrates the effect of the F move.

Definition 2.5. The **basic moves** on the Rubik's cube are the turns F, B, U, D, R, L and their inverses.

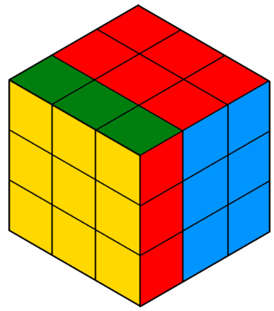

FIGURE 3. The cube after an F turn.

Definition 2.6. A **sequence of moves** is an ordered list of basic moves, applied from left to right.

Example sequences are $RUR^{-1}U^{-1}$, RUL, and $FBFB^{-1}$. A sequence may have the same result on the cube to another sequence: DUD^{-1} has the same result as just U, and applying $RUR^{-1}U^{-1}$ six times is equivalent to the empty sequence.

It is possible to use the permutation and orientation of all cubies to describe the state of the cube. This is what we call a configuration.

Definition 2.7. A **configuration** of the cube is a permutation of the cubies combined with the orientation of each cubie. A **legal configuration** is a configuration for which there exists a sequence of moves transforming the solved cube into the configuration (or equivalently transforming the configuration into the solved cube). In other words, a legal configuration can be actually attained on the Rubik's cube.

The existence of illegal configurations is not immediately clear, but they do exist. One example of an illegal configuration is an otherwise-solved cube with only one flipped edge. We will show in section 3 why this configuration is illegal.

We also need the notion of layers and slices, which will be relevant when solving the cube:

Definition 2.8. A **layer** of the Rubik's cube is a subset of cubicles which can be manipulated by using only a single type of turn. This may be either a basic move or a turn of the middle section of the cube.

Definition 2.9. A **slice** of the Rubik's cube is a layer which does not contain any corner cubicles.

The cube in Figure 3 contains one layer turned from the solved configuration. There are nine layers in the 3×3 Rubik's cube, three along each axis of rotation. Three of these layers are slices.

Remark 2.10. Note that a turn of a central slice of the cube gives the same configuration as the two outer layers in the opposite direction, combined with a rotation of the entire cube. Thus it suffices in the 3×3 case to consider only moves on the outer layers of the cube.

3. The 3×3 Cube Group

The cube group consists of all legal configurations of the Rubik's cube. In order to describe the group structure, we define its elements as equivalence classes of sequences of moves:

Definition 3.1. Let $\mathcal{M}$ be the set of all finite move sequences of basic moves. Define multiplication on $\mathcal{M}$ as concatenation of sequences.

Definition 3.2. Let $\sim$ be an equivalence relation on $\mathcal{M}$; $X \sim Y$ if the result of applying X to a solved cube is identical to the result of applying Y.

Definition 3.3. Define $G_3 = \mathcal{M}/\sim$ as sequences of moves up to equivalence of cube configuration.

Each element of G_3 is an equivalence class of move sequences resulting in the same cube configuration. In a slight abuse of notation, we will use F, B, U, D, L, and R to denote both a move on the cube and the respective move's equivalence class in G_3.

Proposition 3.4. *The multiplication on $\mathcal{M}$ makes G_3 a group generated by F, B, U, D, L, and R.*

Proof. There are 54 cubie-faces on the Rubik's cube. Consequently every configuration of the cube may be represented as some element of S_{54}, the symmetric group on 54 elements.

Consider the map $\mathcal{M} \mapsto S_{54}$ which sends each move sequence to the resulting configuration from applying it to the solved cube. The image of this map is an embedding of G_3 inside S_{54}. This map preserves composition of move sequences, and since each basic move has an inverse, it follows that G_3 is closed under inverses. Therefore G_3 is a group generated by the six basic moves. □

Remark 3.5. Since each of the generators of G_3 has an inverse, we can obtain the inverse of any element of G_3 by reversing the order of moves and replacing each move by its inverse. For example, the sequence $FRU^{-1}R^{-1}$ has inverse $RUR^{-1}F^{-1}$. The identity is the solved configuration, associated with the empty sequence.

G_3 is clearly a finite group, since there are only finitely many possible configurations of the cube. It is also non-abelian: compare the move RU to UR. The former sends the bottom-right front cubie to the top-left position, whereas the latter sends it to the top-right position.

Now we will compute the order of G_3, or equivalently the number of legal configurations of the Rubik's cube. We begin by computing the number of all configurations, then reducing that number to only legal configurations.

First, consider the set of all possible configurations, including illegal configurations. This set can be viewed as a subgroup of S_{54} using a similar argument as Proposition 3.4. We can compute its order as follows:

- The centers remain fixed in place, so we don't need to consider permuting them at all.
- The 8 corners are permuted freely, and each has 3 possible orientations, for a total number of $8! \times 3^8$ arrangements.
- The 12 edges are likewise free, and they have $12! \times 2^{12}$ arrangements.

Therefore, if we include illegal configurations, we find that there are

$$8! \times 3^8 \times 12! \times 2^{12} \approx 5.19 \times 10^{20}$$

different configurations of the cube.

We can now view G_3 as a subgroup of this larger group. The index of G_3 inside this group can be found via the following theorem, which describes legal configurations of the Rubik's cube:

Theorem 3.6. [1] *A configuration of the Rubik's cube is legal if and only if:*

(1) the permutation of cubies is even;
(2) the sum of orientations of corners is 0;
(3) the sum of orientations of edges is 0.

Proof. We begin with the *only if* direction: every legal configuration satisfies the three given conditions.

For condition (1), we look at the cycle decomposition of a turn of the Rubik's cube; the decomposition of the F turn is shown in Figure 4. The turn decomposes into two disjoint 4-cycles, which permute the edges and the corners independently. Since a 4-cycle is odd, the total permutation of cubies from an F turn is even; the same is true of all other generators. It follows that all elements of G_3 give an even permutation on the cubies.

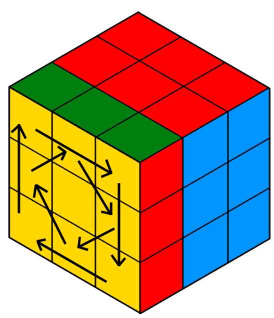

Figure 4. The cycles in an F turn. There are two disjoint 4-cycles: one on the edges, and one on the corners.

For conditions (2) and (3), since G_3 is generated by the six basic moves, it suffices to check that these moves preserve the sum of orientations. Figure 5 shows how the F move affects the orientation of cubies.

After the move F, the orientations of the four corner cubies become 1, 2, 1, 2, and sum to $1 + 2 + 1 + 2 \equiv 0$ in $\mathbb{Z}/3\mathbb{Z}$. The orientations of the four edge cubies become 1, 1, 1, 1 and sum to $1 + 1 + 1 + 1 \equiv 0$ in $\mathbb{Z}/2\mathbb{Z}$. Thus F preserves the sum of corner and edge orientations. Similar arguments can be applied to the other five basic moves.

We have only shown so far that the theorem applies in the forwards direction: that every legal configuration of the cube obeys these three conditions. Showing the reverse direction is more difficult; a full treatment can be found in [1]. The basic idea is to show the existence of certain sequences with specific effects on the cube:

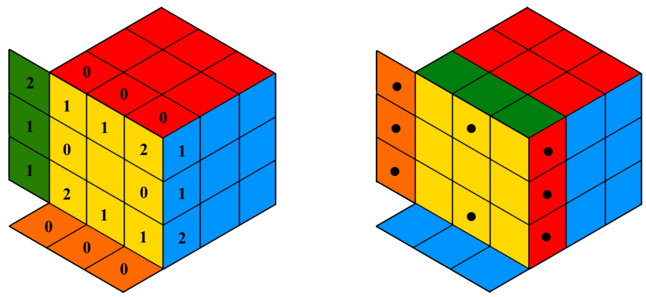

FIGURE 5. The cube with all the assigned orientations shown, and the result of an F turn. Marked faces are indicated with a black dot.

- Permute three edges without modifying orientation
- Permute two edges and two corners without modifying orientation
- Flip two edges in place
- Flip two corners in place, where one gains orientation 1 and the other gains orientation 2

Since these special sequences are in G_3, they generate a subgroup of G_3. Each of these sequences satisfies the three conditions on legality. Furthermore, any configuration satisfying the three conditions can be obtained by some combination of these sequences. It follows that any such configuration is in G_3, which completes the proof of the reverse direction. □

In total, these restrictions reduce the total number of configurations by a factor of 12 from the original number that included illegal positions: a factor of 2 from the sign condition, a factor of 3 from corner orientation, and a factor of 2 from edge orientation. This is exactly the index of the subgroup G_3 in the group of all configurations without regard to legality. We can now obtain the order of G_3 by dividing the number of all configurations by 12:

Theorem 3.7. *The order of G_3 is equal to* $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times 8! \times 3^8 \times 12! \times 2^{12}$.

4. Solving the Rubik's Cube

43 quintillion possible configurations of the cube are obviously too many for any human to remember, so no one is solving the cube just by memorizing all the positions. Typical solution methods focus instead on solving only specific sections of the cube at a time. This allows the usage of moves that don't necessarily fix all other pieces of the cube, which greatly increases the speed of solving. Once a section of the cube is solved, all future moves are chosen to not disrupt the solved section. Mathematically, this corresponds to choosing only moves which are elements of a subgroup of G_3 that fixes all the desired pieces of the cube.

The most common method of solving the Rubik's cube is called CFOP, which is an abbreviation of its four main steps:

- Cross: Recover a correctly permuted cross on one face.
- First Two Layers: Recover the cubies of the layer with the cross and the slice parallel to it.
- Orient Last Layer: Orient the eight cubies of the last layer without disrupting the first two solved layers.
- Permute Last Layer: Permute the eight cubies of the last layer without disrupting the first two layers or the orientation of the last layer.

Furthermore, the latter two steps can be decomposed further, by arranging the last layer's edges and corners in separate steps.

4.1. **Cross.** The first step in CFOP is to assemble a properly-permuted cross on one of the faces of the cube. This is a very simple step, and is typically done purely by observation. The benefits of not restricting ourselves to controlled swaps are most noticeable here, since the cross can be assembled with little effort. At the end of this step, we will have correctly permuted and oriented 4 edge cubies.

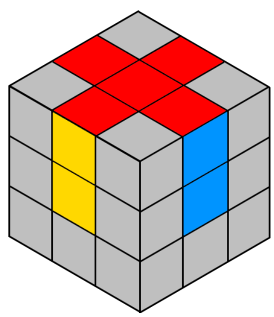

FIGURE 6. The cube after the cross is completed. Grey cubies indicate that the cubicle's contents are not yet fixed and therefore cannot be known.

4.2. **First 2 Layers.** The second step is to recover the two layers which contain the cross from the first step. This consists of four corner cubies and four edge cubies which must be correctly permuted and oriented without disrupting the already-placed edges. There are two useful algorithms for this task:

- RUR^{-1} puts a top-layer corner with orientation 1 into the bottom layer.
- $URU^{-1}R^{-1}U^{-1}F^{-1}UF$ puts an edge from the top layer into the middle layer.

These algorithms can also be mirrored: $L^{-1}U^{-1}L$ manipulates a top-layer corner instead, for instance. The repeated application of these algorithms allows the eight relevant cubies to be placed into their solved positions.

4.3. **Orient Last Layer.** The third step is to correct the orientation the eight remaining cubies in the last layer, which is usually called OLL for Orient Last Layer. At the end of this step, the cube will appear as in Figure 8, with the top face entirely one color. Since we know that the orientation of edges and the

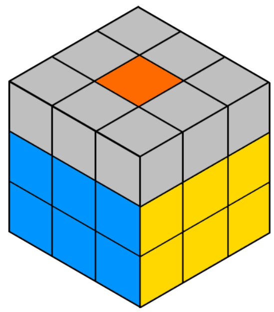

FIGURE 7. The cube after the first two layers are correctly arranged. It has been flipped upside-down from its previous position, to better show the cubies.

orientation of corners are separate, there is actually a simplification of this step called 2-look OLL, which reduces the number of required algorithms.

2-look OLL first orients the edges, and then orients the corners. Since the sum of orientations of edges must be 0, there are only four legal edge orientations at this point:

- No edges correct
- All 4 edges correct
- 2 edges correct in an L shape
- 2 edges correct in a straight line

Since one of these orientations requires no action, there are only 3 algorithms needed to solve this first stage in one step.

We move on to the corners, which have a few more possible orientations. Using the corner orientation sum condition on legal configurations, we can determine all possible scenarios which require algorithms. Let o_1, o_2, o_3, o_4 denote the orientations of the four corners, proceeding clockwise around the upper layer (so that o_1 and o_3 are diagonally opposite orientations). The corner orientation condition states that

$$o_1 + o_2 + o_3 + o_4 \equiv 0 \pmod 3$$

Then we can conclude the following:

- If three of the corners are correctly oriented, $o_1 = o_2 = o_3 = 0$ up to symmetry, which implies that $o_4 = 0$. This means that we cannot flip a single corner by itself.
- If only one corner is correct, then $o_1 = 0 \implies o_2 + o_3 + o_4 \equiv 0$. If none of these are 0, they can only be all 1 or all 2. Hence there are two orientations with one corner correct.
- If two adjacent corners are correct, then $o_3 + o_4 \equiv 0$ up to symmetry. We can have either $o_3 = 1, o_4 = 2$ or $o_3 = 2, o_4 = 1$.
- If two diagonal corners are correct, then $o_2 + o_4 \equiv 0$. Since applying the move sequence UU will swap the diagonal corners, there is only one required algorithm which solves $o_2 = 1, o_4 = 2$.
- Finally, if no corners are correct, then there must be two 1s and two 2s. Equal-orientation cubies will be either diagonal or adjacent; this produces

the two patterns $o_1 = 1, o_2 = 1, o_3 = 2, o_4 = 2$ and $o_1 = 1, o_2 = 2, o_3 = 1, o_4 = 2$ which are all the orientations up to symmetry.

We find that there are 8 possible orientations of the corners, of which one is the solved orientation. Each possible non-solved orientation requires an algorithm, for 7 in total in the second part of OLL. This makes for a total of 10 algorithms required for 2-look OLL; in contrast, 1-look OLL (where the entire layer is oriented in one stage) requires 57 distinct algorithms.

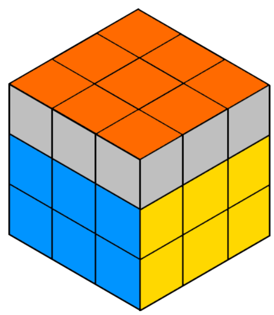

FIGURE 8. The cube after orienting the last layer.

4.4. **Permute Last Layer.** The fourth and last step is to permute the eight cubies in the last layer, which gives this step the acronym PLL. At this point it is feasible to start solving via application of the controlled swaps we used to prove solvability, since most of the cubies need to be fixed in place; however, the typical faster method uses more algorithms to reduce the number of steps needed.

Like with the third step, the PLL step can be solved in either one or two stages. 2-look PLL first solves the corner permutation, followed by the edge permutation. It's possible to solve the corners using only a single algorithm which swaps two corner pieces along the same edge. Note that this algorithm necessarily also swaps two edge pieces, since the signs of the permutations must remain equal, but at this stage we only care about the corner permutation. The following sequence will swap the corners on the right side of the top face:

$$RUR^{-1}U^{-1}R^{-1}FR^2U^{-1}R^{-1}U^{-1}RUR^{-1}F^{-1}$$

After corner permutation, only the last layer edges are not permuted correctly. This stage can be solved with a single algorithm that permutes three edges, since this will reach every possible even permutation of edges on the top layer. The following sequence cycles counterclockwise the three edges not on the back face:

$$RU^{-1}RURURU^{-1}R^{-1}U^{-1}R^2$$

This means that only two algorithms are needed to solve 2-look PLL, although a few more can be employed to speed up the process. Compared to the 21 algorithms of 1-look PLL, the benefits of the subgroup-based approach are clear. Since each stage only requires a few algorithms to solve, the entire cube can be solved without requiring a large amount of memorization.

5. The 2×2 Cube Group

In this section, we will analyze the 2×2 cube group and the strategy for solving configurations. The 2×2 cube is significantly simpler in its physical construction compared to the 3×3 cube: it has only 8 corner cubies, and no face or edge cubies.

Definition 5.1. We define cubies, cubicles, corner orientation, and configurations in the same way as the 3×3 cube.

Remark 5.2. In the 3×3 case, we choose to fix the centers of each face, but there are no centers here; this means we can rotate the entire cube using move sequences. For example, the sequence FB^{-1} on the 2×2 cube rotates the cube clockwise but preserves relative cubie positioning. We will consider these rotations as distinct cube configurations, since doing so will maintain consistency with the 3×3 definition.

Definition 5.3. G_2 is the group of move sequences on the 2×2 cube up to equivalence of cube configuration.

Remark 5.4. We use the same letters F, B, U, D, R, L to denote the six basic moves on the 2×2 cube. These six basic moves generate the group G_2.

Calculating the order of G_2 is straightforward, given that we already know a good deal about legal configurations of the 3×3 cube. Since the 2×2 cube is essentially just the corners of the 3×3 cube, a legal configuration of the 2×2 cube is a legal permutation and orientation of the corners of the 3×3.

Theorem 5.5. *A configuration of the 2×2 cube is legal if and only if the sum of orientations of corners is 0.*

We've already computed the number of legal corner configurations in the 3×3 case, which carries over directly:

Theorem 5.6. $|G_2| = 8! \times 3^7 = 88,179,840$

However, G_2 also includes all rotations of the entire cube, whereas G_3 does not. Since there are 24 different ways to rotate the cube, the index of the subgroup of visually distinct configurations in G_2 is 24. Consequently:

Theorem 5.7. *The number of visually distinct configurations of the 2×2 cube is* $\frac{|G_2|}{24} = 3,674,160$.

Furthermore, G_2 has an additional interesting property in relation to the larger cube group G_3:

Proposition 5.8. *G_2 can be viewed as a quotient group of G_3.*

Proof. Let N be the subgroup of G_3 which fixes all corner cubies. For any moves $n \in N, g \in G_3$, we see that the conjugate gng^{-1} also fixes corner cubies, because n does not modify their position and they are free to return to their original state. Hence N is normal.

Take the quotient group G_3/N. Elements of this quotient group are conjugacy classes of moves which are defined by the configuration of the corners, which is exactly the definition of G_2. Hence $G_2 = G_3/N$. □

5.1. **Solving the 2×2 Cube.** Because G_2 is nicely derived from G_3, we can easily derive a solution method for the 2×2 cube from the method used for the 3×3 cube. If all steps relating to the edge cubies are ignored, the result will be a 2×2 cube solution.

Since there are no edge pieces, the first step (to construct the cross) can be skipped entirely; the first actual moves are to satisfy the First 2 Layers step. In this case, it's actually just one layer of the 2×2 cube, since the "second" layer is the center slice of the 3×3 that does not exist. As such, we need only to correctly permute and orient four of the corners into the bottom layer. This can be done in the same way as the 3×3 cube.

The next step is to orient the top layer. Since there are no edges, we can skip the first part of 2-look OLL and move directly to the corner orientation, which saves needing the 3 algorithms for orienting edges. Finally, we need only permute the top layer, for which only the one algorithm that swaps corners is needed.

In summary, the algorithm for solving the 2×2 cube is as follows:

- Permute and orient the first layer
- Orient the cubies in the second layer
- Permute the cubies in the second layer

This solves the 2×2 cube in far fewer steps than the 3×3 cube.

6. The 4×4 Cube Group

Decreasing the size of the cube clearly results in a much simpler group; it should come as no surprise that increasing the size of the cube also increases the complexity. The 4×4 cube has 8 corners like the other two cubes, but has 24 edge cubies and 24 center cubies as well. These center cubies can be meaningfully permuted by turns of the center slices, unlike the centers of the 3×3 cube.

Definition 6.1. We denote by cursive $\mathcal{F}$ the clockwise turn of the center slice directly behind the F turn (likewise for the other basic turns). This gives 12 basic moves for the 4×4 cube: the six moves F, B, U, D, R, L that it shares with the smaller cubes, and the six new moves: $\mathcal{F}, \mathcal{B}, \mathcal{U}, \mathcal{D}, \mathcal{R}$, and $\mathcal{L}$.

We also need to define edge orientation in a slightly different way:

Definition 6.2. Each face of an edge cubicle is adjacent to exactly one other edge cubicle on the same face. Referring to Figure 9, assign 0 to the faces of edges marked with a black dot, and assign 1 otherwise. Mark the cubie-faces with a 0 in the solved position; the **orientation of an edge cubie** is the element of $\mathbb{Z}/2\mathbb{Z}$ corresponding to the marked face.

Like the 2×2 cube, the 4×4 also has no fixed cubies. The centers may all be individually permuted, which means that we must similarly consider full rotations as distinct configurations. This will allow us to easily embed the non-rotating 3×3 cube group as a subgroup, which will be important later on.

Definition 6.3. G_4 is the group of moves on the 4×4 cube up to equivalence of cube configuration.

Remark 6.4. G_4 is generated by the 12 basic turns of the 4×4 cube.

To obtain the number of legal configurations and thereby compute the order of G_4, we will separate the cube into its three types of cubies: corners, centers, and edges.

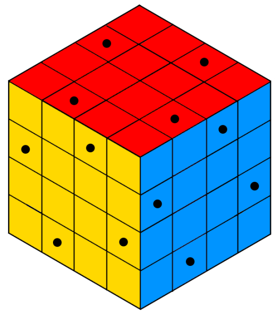

FIGURE 9. Edge faces assigned 0 are marked with a black dot. Any turn of the cube will carry every black dot onto another black dot, so no cubie's orientation can be modified.

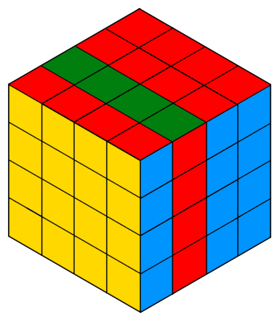

FIGURE 10. The $\mathcal{F}$ turn of the 4×4 cube. Note that it does not affect any corners.

Theorem 6.5. *A configuration of the* 4×4 *cube is legal if and only if:*

- *the sum of corner orientations is* $0 \pmod 3$*;*
- *every edge orientation is exactly 0;*
- *the sign of the permutation of corner and center cubies is even.*

Proof. We will only show the *only if* direction here. The full treatment of the other direction may be found in [2].

The corner cubies are the easiest to analyze, as we've already done all the work. They behave identically to smaller cubes: they can be permuted freely (with restrictions based on the sign of the overall permutation), but only 1/3 of all orientations are legal. This immediately gives us the first condition on corner orientation.

For the second condition, note that the orientation of any edge cubie is not changed by any of the basic turns, as shown in Figure 9. As a result, the orientation of a cubie is determined completely by its position.

The center cubies and edge cubies can be permuted freely. The basic idea is that a 3-cycle of some centers (likewise edges) can be constructed; then by taking all conjugates of the 3-cycle, all centers (and edges) can be permuted. However, there is a restriction on the sign of the permutation of centers when combined with the permutation of corners. The F turn induces a 4-cycle on corners, a 4-cycle on centers, and two 4-cycles on edges, as shown in Figure 11. Since 4-cycles are odd permutations, F changes the sign of corner and center permutations, and preserve the sign of edge permutations. The inner slice turn $\mathcal{F}$ instead induces a 4-cycle on edges and two 4-cycles on centers, which means that $\mathcal{F}$ changes the sign of edge permutations and preserves the sign of corner and center permutations. The same holds for all other generators by symmetry. This implies that the sign of the corner and center permutations must always be the same, while the sign of the edge permutation may change freely. This gives the final condition on the sign of the permutation.

□

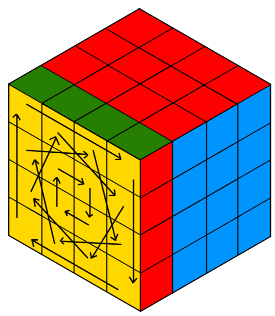

FIGURE 11. The cycles in an F turn. There are 4 disjoint 4-cycles; one on corners, one on centers, and two on edges.

We can now compute the order of G_4. We multiply the number of corner, center, and edge permutations together, multiply by corner orientation, then divide by 2 due to the sign restriction on the corner and center permutations:

Theorem 6.6. $|G_4| = 8! \times 24! \times 24! \times 3^7 \times \frac{1}{2} \approx 1.70 \times 10^{55}$

However, not all of these configurations are visually distinct. The order of G_4 includes an additional factor of 24 from rotations of the cube. In addition, since many cubies in the 4×4 cube have identical appearance, we also need to divide by the number of moves that meaningfully permute cubies, but keep the appearance of the cube identical; these are "invisible" moves because they cannot be visually distinguished from the identity element, even though they do actually swap cubies.

As explained previously, edge cubies do not have meaningful orientation in the 4×4 cube, so swapping them will not lead to any "invisible" moves. However, the centers of identical color can be permuted among themselves. There are six colors, each with 4 centers; hence there are 24 permutations for each color and 24^6 permutations of all centers among those of the same color. Since the sign of this permutation must be even, only half of these can be counted.

Theorem 6.7. *The number of visually distinct configurations of the* 4×4 *cube is*

$$\frac{|G_4|}{24 \times 24^6 \times \frac{1}{2}} \approx 7.40 \times 10^{45}$$

Remark 6.8. The G_2 group is a quotient group of G_4, in the same way that G_2 is a quotient group of G_3.

Remark 6.9. G_3 is contained in G_4 as a subgroup. The group of moves generated by $F, B, U, D, L, \text{and} R$ is clearly a copy of G_3 embedded in G_4. This fact will be important for solving the 4×4 cube.

6.1. **Solving the 4×4 Cube.** To solve the 4×4 cube, we will attempt to reduce the problem to the known case of the 3×3 cube. The main steps are as follows:

(1) Recover the centers of the 4×4 cube in a legal permutation for the 3×3 cube.
(2) Pair the edge cubies together to form 12 edge blocks.
(3) Solve the cube as if it were a 3×3 cube, treating the center slices as joined together.
(4) While orienting and permuting edges, repair the edge configuration to a legal configuration in G_3, if necessary.

6.2. **Permute Centers.** The first step in reducing the 4×4 to the 3×3 is to arrange the center cubies. Since there is significant freedom in the early stages to manipulate the cube, this can be done without much trouble, similar to the cross. However, it is important that the resulting arrangement of centers is a legal permutation for the 3×3 cube. Recall that the centers of the 3×3 cube cannot be permuted relative to each other; since the 4×4 cube's centers can be permuted freely, it's possible to produce an illegal center configuration. For example, on a standard Rubik's cube, the red and orange faces are opposite each other. It's possible to arrange the red and orange centers to be on adjacent faces on a 4×4 cube, but since there do not exist any edges colored red and orange, that configuration of centers would be impossible to solve. This can be avoided easily by choosing the correct colors for each face when solving the centers.

6.3. **Pair Edges.** The next step in the reduction is to pair identically-colored edges together. After this step, both the centers and edges will be linked, which makes the cube functionally a 3×3 cube and solvable using the CFOP method. The result of this step, however, is not necessarily an element of the 3×3 subgroup described in Remark 8.4.

6.4. **Repair Edge Orientation and Permutation.** Because edges on the 4×4 cube can be permuted freely, it's possible to assemble an "illegal" 3×3 configuration in two ways:

- Two adjacent edges are transposed, causing the combined edge in the 3×3 embedding to be flipped
- Two pairs of edges are transposed, causing the edge permutation to have odd sign rather than even sign

Both of these scenarios result in a cube configuration that is not in the copy of G_3 inside G_4, but instead inside a subgroup with different edge orientation or permutation conditions. The problem would become noticeable while solving the cube as a 3×3 and discovering an impossible scenario during either the OLL or PLL step, and is fixable using an algorithm using the inner slices to move the configuration into the G_3 subgroup.

This reductive method of solving the cube is a generic method for solving all larger-sized cubes. By first arranging the centers and edges, even very large cubes can be transformed into near-3×3 cubes. The issues with edges can even be avoided by computing the sign of the permutation and orientation during the edge-arrangement step, which prevents needing to develop algorithms to fix larger edge sizes.

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References

[1] Janet Chen. Group Theory and the Rubik's Cube. http://people.math.harvard.edu/ jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf
[2] Mogens Esrom Larsen. Rubik's Revenge: The Group Theoretical Solution. https://www.maa.org/sites/default/files/pdf/pubs/Rubiks8.pdf