Abstract simplicial complex

K $V(K)$ (vertices)

a set of finite subsets of $V(K)$ (simplices)

a subset of a simplex is a simplex

every vertex is a simplex.

Ordered simplicial complex

K $V(K)$ is a poset, which restricts to a total order on each simplex $(v_0 < v_1 < \cdots < v_n)$

A-space $X \mapsto$ ordered simplicial complex

$K = K(X)$ $V(K) = X$ as a ~~set~~ poset

simplices all totally ordered subsets

product $K \times L$? $K(X \times Y)$ is a subdivision of $K(X) \times K(Y)$

product of geometric simplices is not a geometric simplex.

$I \times I =$

ordered simplicial complex $\mapsto$ A-space

Set of simplices ordered by inclusion

$$\Delta_n^{top} = \{(t_0, \cdots, t_n) \mid 0 \le t_i \le 1,\ \textstyle\sum t_i = 1\}$$

etc

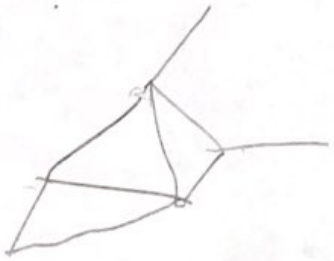
$$\delta_i : \Delta_n^{top} \longrightarrow \Delta_{n+1}^{top}$$

$$\delta_i(t_0, \cdots, t_n) = (t_0, \cdots, \overset{i^{th}}{0}, \cdots t_n)$$

$$\sigma_i : \Delta_{n+1}^{top} \longrightarrow \Delta_n^{top}$$

$$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_i + t_{i+1}, \cdots, t_n)$$

Geometric simplicial complex.
"fill in" abstract simplicial complexes



simplex : simp cx.
$V_0 < V_1 < \cdots < V_n$

associated simplicial set
$V_0 \le V_1 \le \cdots \le V_n$

$$\mathcal{OS}\mathcal{S} \longrightarrow s\mathcal{S}et\mathcal{S}$$

Simplicial set: K_n, $n \geq 0$,

$$d_i : K_n \to K_{n-1}, \quad s_i : K_n \to K_{n+1} \quad 0 \leq i \leq n$$

$$d_i d_j = d_{j-1} d_i \quad i < j$$

$$s_i s_j = s_{j+1} s_j \quad i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ id & i = j, i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

$K : \Delta^{op} \longrightarrow Sets \qquad \Delta_n = (0, 1, \dots, n)$

monotonic functions

$\delta_i : \Delta_{n-1} \to \Delta_n$ skip i in target

$\mu : \Delta_m \to \Delta_n$

$\mu(i) \leq \mu(j)$ if $i < j$

$\sigma_i : \Delta_{n+1} \to \Delta_n$ repeat i in target

$f : K \to L \quad f_n : K_n \to L_n \quad d_i f_n = f_{n-1} d_i \quad s_i f_n = f_{n+1} s_i$

simplicial object in any category

$h : f \simeq g : K \to L \quad h_i : K_n \to L_{n+1}, \quad 0 \leq i \leq n,$

$d_0 h_0 = f \quad d_{n+1} h_n = g$

$$d_i h_j = \begin{cases} h_{j-1} d_i & i < j \\ d_i h_{j-1} & 0 < j \\ h_j d_{i-1} & i \geq j+1 \end{cases} \qquad s_i h_j = \begin{cases} h_{j+1} s_i & i \leq j \\ h_j s_{i-1} & i > j \end{cases}$$

$K \times I \to L$

motivation $I = \Delta_1^{simp}$

$S: \text{Spaces} \longmapsto s\text{Sets}$ $\quad \Delta^{top}_+$

$S_n X = \mathrm{Map}(\Delta^{top}_n, X)$

$\mu: [m] \to [n] \quad \mu^* = \mathrm{Map}(\mu, id)$

$S_n X \to S_m X$

$\Delta^{top}_n = \{(t_0, \cdots, t_n) \mid 0 \le t_i \le 1, \ \sum t_i = 1\}$

$\delta_i(t_0, \cdots, t_n) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_n)$

$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_i + t_{i+1}, \cdots, t_{n+1})$

$T: s\text{Sets} \to \text{Spaces} \qquad TK = K^* \otimes_{\Delta} \Delta^{top}$

$$TK = \coprod_{n \ge 0} K_n \times \Delta^{top}_n / (\mu^* k, t) \sim (k, \mu_* t)$$

$\mu: \Delta_m \to \Delta_n$

$k \in K_n, \ t \in \Delta^{top}_m$

$\Delta^{top}_n |k, u| \quad k \in K_n \ u \in \Delta^{top}_n$

$$\coprod_{(m,n)} K_n \times \Delta(m,n) \times \Delta^{top}_m \qquad TK \simeq |K|$$

$\mu^* \times id \downarrow \quad \downarrow id \times \mu_*$

$$\coprod_n K_n \times \Delta^{top}_n$$

$\downarrow$

$TX = |X|$

$\eta: K \to STK$

$\varepsilon: TSX \to X$

$\eta(k)(u) = |k, u|$

$\eta: K \to STK$

$\varepsilon: TSX \to X$

$\varepsilon(|f, u|) = f(u)$

$f: \Delta^{top}_n \to X$

coequalizer

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z \qquad hf = hg$$

$c(f,g)$ — $\exists!$

$\mathrm{Map}(TK, X) \cong \mathrm{map}(K, SX)$

(space) (sSet)

T also works for simplicial spaces

Spaces

$T \uparrow \downarrow S$

simplicial sets

$F \downarrow \uparrow U$

simplicial abelian groups

$\downarrow C_*$

chain complexes

$C_n \xrightarrow{d} C_{n-1}$, $d^2 = 0$

$\downarrow H_*$

→ graded Abelian groups

$$H_n(C_*) = \frac{\operatorname{Ker}(d : C_n \to C_{n-1})}{\operatorname{Im}(d : C_{n+1} \to C_n)}$$

$$H_n(X) = H_n(X; \mathbb{Z})$$

$$H_n(X; A) \equiv H_*(C_*(X) \otimes A)$$

$$\pi_n(X) = [S^n, X]_* \quad n \geq 0$$

based homotopy classes of based maps

X based

$\mathrm{Top}(TK, X) \cong sSet(K, TX)$

$sAb(FK, A) \cong sSet(K, UA)$

$C_*(A)$: Normalized

$C_n(A) = A_n / s A_{n-1}$

$s A_{n-1} = s_0 A_{n-1} + \cdots + s_n A_{n-1}$

$d_i : C_n(A) \to C_{n-1}(A)$

$$d(a) = \sum_{i=0}^{n} (-1)^i d_i(a)$$

$d \cdot d = 0$

A an Ab group

$C_n \otimes A$: chain cx

$H_*(C_*; A) = H_*(C_* \otimes A)$

$\tilde{H}_*(X) = H_*(X)/H_*(pt)$ $\quad * \to X \to *$

$H_q(*) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q\neq 0 \end{cases}$

$\tilde{H}_q(S^n) = \begin{cases} \mathbb{Z} & q=n \\ 0 & q\neq n \end{cases}$ "dimension axiom"

$\Sigma X \cong X \wedge S^1$ $\quad X \wedge Y = X \times Y / X \vee Y$

$\Omega X = \mathrm{map}_*(S^1, X)$ $\quad X \vee Y \cong X \times \{*\} \cup \{*\} \times Y \subset X \times Y$

$\underline{\mathrm{Map}}_*(\Sigma X, Y) \cong \underline{\mathrm{Map}}_*(X, \Omega Y)$ $\quad \Sigma S^n \cong S^{n+1}$

$\pi_{q-1}(\Omega Y) = [S^{q-1}, \Omega Y]_* = [S^q, Y]_* = \pi_q(Y)$

$\tilde{H}_{q+1}(\Sigma X) \cong \tilde{H}_q(X)$ "suspension axiom"

Not so obvious, but True

Hurewicz:

$h : \pi_n(X) \longrightarrow \tilde{H}_n(X)$

$f : S^n \to X \longmapsto H_n(f) : \tilde{H}_n(S^n) \to \tilde{H}_n(X)$

$h_*([f]) := H_n(f)(1)$

Hurewicz theorem: X connected

$h : \pi_1 X \to \tilde{H}_1(X)$ is Abelianization

X $(n-1)$-connected $\Rightarrow h : \pi_n(X) \xrightarrow{\cong} \tilde{H}_n(X)$

$\pi_q(X) = 0,\ q < n,\ n \geq 2$

$$Top(TK, X) \cong sSet(K, SX)$$

$H_*(K) \equiv H(C_*(K))$. $H_*(X) \equiv H_*(SX)$.

How to define $H_*(X)$ without use of (T, S)?

CW complex X: $X^0 = [\text{set}]$, $\coprod S^{n-1} \to X^{n-1}$

$X = \bigcup X^n \xrightarrow{f} Y = \bigcup Y^n$ $\quad \coprod D^n \to X^n$

$f(X^n) \subset Y^n$

For any X $\exists$ $\Gamma X \to X$ iso on homotopy groups

"CW approximation" WHE

<u>Theorem</u>: $\varepsilon : TSX \to X$ is a functorial CW approximation.

For CW complexes, have cellular chains

$C_n(X)$ = free Ab. gp on n-cells

$d : C_n(X) \to C_{n-1}(X)$

$$X^n/X^{n-1} \xleftarrow{\cong} X^n \cup CX^{n-1} \to \Sigma X^{n-1} \to \Sigma X^{n-1}/X^{n-2}$$

$\| \wr$ $\qquad\qquad\qquad

Theorem: TSX is a CW-complex such that $C_*(TSX) \cong C_*(SX)$

cellular singular

Symmetry would be clearer using cohomology rather than homology

Just as $H_*(X) = H_*(SX)$, we define $\pi_*(K) = \pi_*(TK)$, where K is based.

Want

Analog of CW spaces with good direct definition of $\pi_* K$, without passage to spaces?

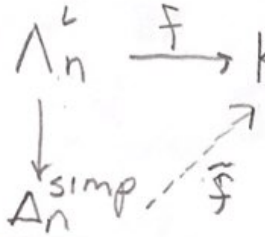
Kan complex $(\Delta_n^{simp})_m = \{\underline{m} \to \underline{n} \text{ in } \Delta\}$

$$sSet(\Delta_n^{simp}, K) \cong K_n$$

$id_n : \underline{n} \to \underline{n}$ is an n-simplex of Δ_n^{simp}

all other simplices are composite,

Λ_n^i = subcomplex of Δ_n^{simp} generated by $d_i id_n$. All but one face (d_i)



given f, $\exists\, \tilde{f}$

"horn filler"

Kan complex : Horn filling property : $\forall f, \exists \tilde{f}$

$$\begin{array}{ccc} \Lambda^i_n & \xrightarrow{f} & K \\ \downarrow u & \nearrow \tilde{f} & \\ \Delta^{simp}_n & & \end{array}$$

Example : $\forall$ spaces X, SX is a Kan complex $(K = SX)$

$$\begin{array}{ccc} (\Lambda^i_n)^{top} = T\Lambda^i_n & \xrightarrow{adf} & X \\ \cap \downarrow \simeq & & \\ \Delta^{top}_n = T\Delta^{simp}_n & \nearrow \tilde{f} & \end{array}$$

$\uparrow$ deform

Δ

Many choices

adjoint

$$\begin{array}{ccc} \Lambda^i_n & \xrightarrow{f} & SX \\ \downarrow & \nearrow \tilde{f} & \\ \Delta^{simp}_n & & \end{array}$$

K a Kan complex $\phi \in K_0$ (and $s_i \phi$)

$$\tilde{K}_n = \{ x \in K_n \mid d_i x = \phi, \ 0 \le i \le n \}$$

$$\tilde{K}_n / (\simeq) = \pi_n K.$$

$x \sim x'$, $x, x' \in K_n$ if $d_i x = d_i x'$ $0 \le i \le n$

and $\exists\ y \in K_{n+1}$ ∋

$$d_n y = x \quad d_{n+1} y = x'$$

$$d_i x = s_{n-1} d_i x = s_{n-1} d_i x' \quad 0 \le i < n$$

Equivalence relation <u>if K is a Kan complex</u>

<u>Prop 3.2 page 5</u> of [5] Gabriel-Zisman Calculus of fractions and homotopy theory

Simplicial objects in algebraic Topology

$\pi_n K$ is a group if $n=1$, Abelian if $n \ge 2$

<u>§ 4 pp. 9-11</u> just combinatorics

<u>Thm 17.1 p. 67.</u> Every simplicial group is a Kan complex

<u>Thm 16.1 p. 62</u> $[T(K), X] \cong [K, SX]$

$$\pi_n(X, *) \cong \pi_n(SX, \underset{*}{s_* *})$$

Prop 16.2, Thm 16.6

$\eta_* : H_*(K) \xrightarrow{\cong} H_*(STK) = H_*(TK)$

any K

$\eta_* : \pi_*(K) \xrightarrow{\cong} \pi_*(STK)$

[Any Kan complex (based)

$\eta : K \xrightarrow{\simeq} STK$

$\varepsilon_* : H_*(TSX) \xrightarrow{\cong} H_*(X)$

any X

$\varepsilon_* : \pi_*(TSX) \xrightarrow{\cong} \pi_*(X)$

[$\varepsilon : TSX \xrightarrow{\simeq} X$ if X is CW

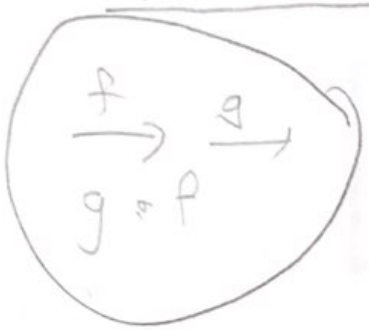
Nerve of a category $N\mathcal{C}$

S = source
T = Target
I = Identity
C = composition

$SI = Id = TI$

$N_0\mathcal{C} = Ob\mathcal{C}$

$N_1\mathcal{C} = Mor\mathcal{C}$



$X_0 \to X_1$

$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_n$

$X_0 \to X_1 \to X_{i-2} \to X_{i-1} \to X_{i+1} \to X_{i+2} \to X_n$, with X_i mapping to X_{i-1} and X_{i+1}

All inner horns of K have unique fillers if and only if

$K \cong N\mathcal{C}$ where $\mathcal{C}$ is given by

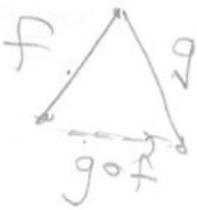
$K_0 = \text{objects } \mathcal{C}$

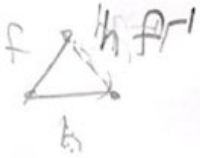
$K_1 = \text{morphisms } \mathcal{C}$

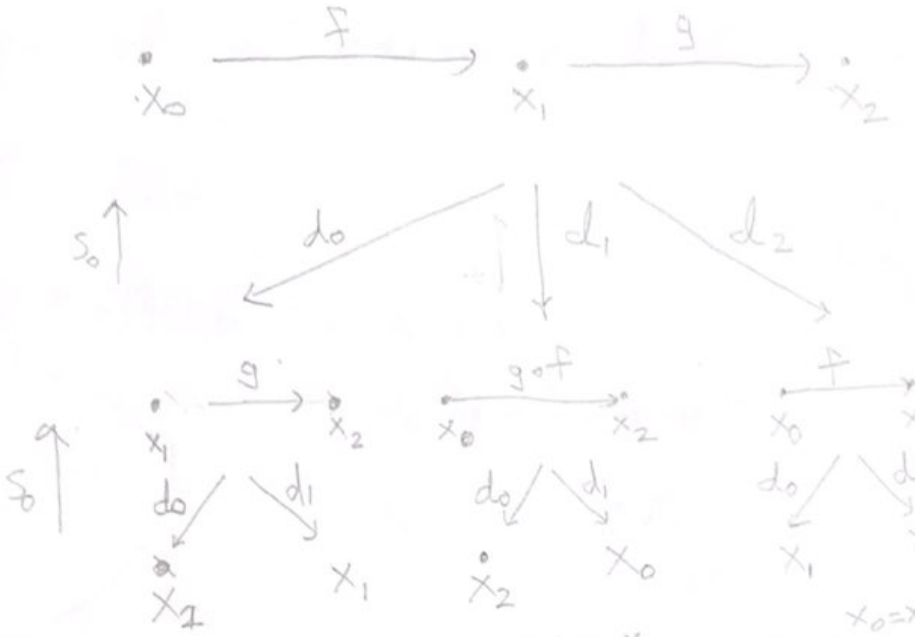
$d_1 : K_2 \to K_1$ define $\mathcal{C}$

$s_0 : K_0 \to K_1$ gives I

$d_0, d_1 : K_1 \to K_0$ gives T, S







$x_2 = x_2$ $\quad x_1 = x_1$ $\quad x_0 = x_0$

$d_0 d_1 = d_0 d_0$ $\quad d_0 d_2 = d_1 d_0$ $\quad d_1 d_2 = d_1 d_1$

$TC = TT$ $\quad TS = ST$ $\quad SS = SC$

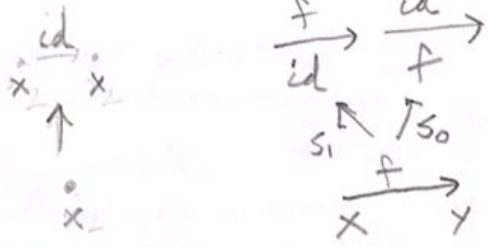
$d_0 s_0 = id = d_1 s_0$ $\quad d_2 s_0 = s_0 d_1$ $\quad d_1 s_1 = id = d_2 s_1$

$s_0 s_0 = s_1 s_0$ $\quad s_0 s_1 = s_2 s_0$

Functors(C, D)

$\downarrow$ N

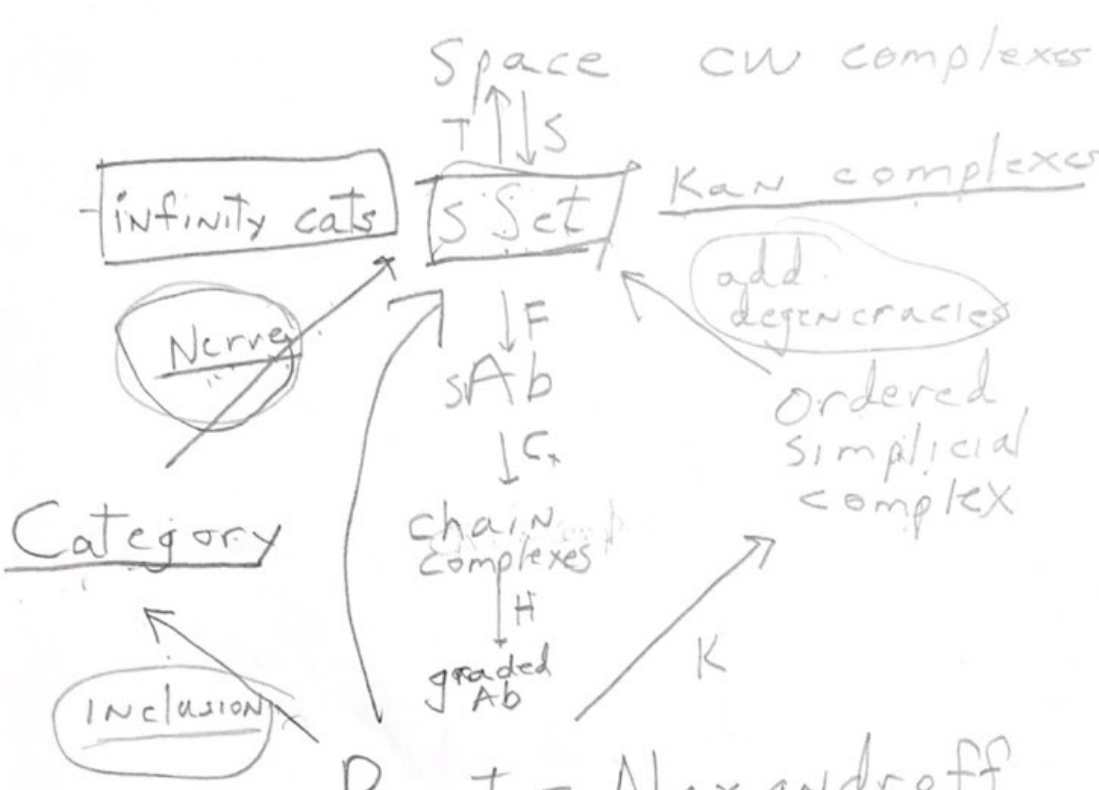
sSets(NC, ND)



$\Delta_n^{cat} = 0 \to 1 \to 2 \to \cdots \to n$

add identities to get category

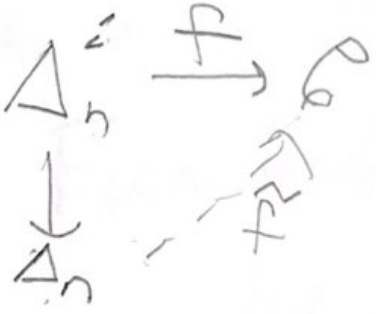
$N_n \mathcal{C} = \mathrm{Funct}(\Delta_n, \mathcal{C})$



Poset = Alexandroff
To space

$X \quad x_1 \to x_2$

$x_1 \leq x_2$



$x_0 \to x_1 \to x_2 \to \cdots \to x_{n-1} < x_n$

$0 \leq 1 \leq 2 \cdots \leq n$

Δ^{cat} $\quad \Delta^{op} \to Sets$

$\Delta \to Cat$

$[n] \mapsto \Delta_n^{cat}$

$N_n \mathcal{C} = Functors(\Delta_n^{cat}, \mathcal{C})$

$N_* \mathcal{C}$