p-ADIC ABSOLUTE VALUES

LOGAN QUICK

ABSTRACT. p-adic absolute values are functions which define magnitudes and distances on the rationals using the multiplicity of primes in the factorization of numbers. This paper will focus on the construction of p-adic absolute values, the new topology formed by these absolute values, and the proof of Ostrowski's theorem, which classifies all absolute values on $\mathbb{Q}$.

CONTENTS

1. INTRODUCTION

Before we construct p-adic absolute values, we must understand what an absolute value is. So, we will begin by looking at the properties all absolute values share.

Definition 1.1. Let $\Bbbk$ be a field. An absolute value is a function $|\cdot| : \Bbbk \to \mathbb{R}_{\geq 0}$. For all absolute values, the following properties hold:

i) $|x| \geq 0$ for all $x \in \Bbbk$.
ii) $|x| = 0$ if and only if $x = 0$.
iii) $|xy| = |x||y|$, for all $x, y \in \Bbbk$.
iv) $|x + y| \leq |x| + |y|$, for all $x, y \in \Bbbk$.

With these general properties, there are two absolute values which arise most easily. The first is the usual absolute value, which maps positive numbers to themselves and negative numbers to their additive inverse:

$$|x|_\infty = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Secondly, there is the trivial absolute value which maps zero to zero and all other numbers to one:

$$|x| = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

While these are the most natural absolute values to construct, far more can be constructed than just these two. As we look at more than just the usual absolute value, there is an idea of non-archimedean absolute values, which satisfy the following property:

v) $|x + y| \leq \max\{|x|, |y|\}$, for all $x, y \in \mathbb{Q}$.

With a basic understanding of these properties in hand, we may begin on the construction of the p-adic absolute value function. In this paper, we will construct the p-adic absolute value and look into some of its interesting properties. We will discuss the product formula and the topology formed on the rationals by the p-adic absolute value. Finally, we will look at Ostrowski's theorem and utilize our constructed absolute value to classify all non-trivial absolute values on $\mathbb{Q}$. To this end, we will mostly follow the exposition of [2].

2. p-adic Absolute Value

Prior to the absolute value itself, we will construct a valuation, which is a function that relates each integer to the multiplicity of a prime in its factorization.

Definition 2.1. A valuation $v : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ fulfills the following properties for all $x, y \in \mathbb{Z}$,

i. $v(xy) = v(x) + v(y)$
ii. $v(x + y) \geq \min\{v(x), v(y)\}$
iii. $v(0) = \infty$

Then, we have that the p-adic valuation function is defined as follows:

Definition 2.2. Fix a prime number $p \in \mathbb{Z}$. For $x \in \mathbb{Z} \setminus \{0\}$, $v_p(x)$ is the unique non-negative integer which satisfies

$$x = p^{v_p(x)}x' \text{ with } x' \in \mathbb{Z} \text{ and } p \nmid x'.$$

This function maps each integer to the multiplicity of the prime in its factorization. As well, $v_p(0) = \infty$ by convention.

To illustrate this function, consider the following examples:

Examples 2.3.
(1) For $p = 3$, $x = 36$, we have that $x = 3^2(4)$, which means $v_3(36) = 2$, as then we have $36 = 3^{v_3(36)}(4)$ where $3 \nmid 4$.
(2) $p = 5$, $x = 17$, then $v_5(17) = 0$, as $17 = 5^0(17)$ where $5 \nmid 17$.
(3) For $60 = 2^2 * 3 * 5$, we have $v_2(60) = 2$, $v_3(60) = 1$, $v_5(60) = 1$, and $v_7(60) = 0$.

This function can be extended to the rational numbers as follows:

$$\text{For } x \in \mathbb{Q} \text{ where } x = a/b,\ v_p(x) = v_p(a) - v_p(b).$$

Examples 2.4.
(1) $v_3(27/5) = v_3(27) - v_3(5) = 3 - 0 = 3$.
(2) $v_5(25/125) = v_5(25) - v_5(125) = 2 - 3 = -1$. Also, $v_5(1/5) = v_5(1) - v_5(5) = 0 - 1 = -1$.
(3) $v_7(49/1) = v_7(49) - v_7(1) = 2 - 0 = 2$.

From these examples, we can see that the extension of the function to all the rationals is well-defined for equivalent fractions, and that representing integers as rational numbers does not change the value of the function.

With a solid understanding of this valuation, we can now finally define the p-adic absolute value:

Definition 2.5. For $x \in \mathbb{Q}$, $|x|_p = p^{-v_p(x)}$ for $x \neq 0$, and $|0|_p = 0$.

Examples 2.6. (1) $|98|_7 = 7^{-v_7(98)} = 7^{-2} = \frac{1}{49}$.
(2) $|47|_5 = 5^{-v_5(47)} = 5^0 = 1$.
(3) $|1/27|_3 = 3^{-v_3(1/27)} = 3^{-(0-3)} = 3^3 = 27$.

Theorem 2.7 ([2, Proposition 2.1.5])**.** *This p-adic absolute value is a non-archimedean absolute value.*

Proof. Fix a prime number $p \in \mathbb{Z}$. Let $x \in \mathbb{Q}$ be an arbitrary non-zero rational number.

i) $|x|_p \geq 0$ for all $x \in \mathbb{Q}$.

ii) $|x|_p = 0$ if and only if $x = 0$.

For all $x \in \mathbb{Q}_{\neq 0}$, $|x|_p = p^{-v_p(x)}$. Since $p > 0$, we have $p^{-v_p(x)} > 0$. As well, $|0|_p = 0$. This shows both i) and ii).

iii) $|xy|_p = |x|_p|y|_p$, for all $x, y \in \mathbb{Q}$.

As before, define $x = p^n x'$ and $y = p^m y'$ for $p \nmid x'y'$. Then, we have

$$\begin{aligned}|xy|_p &= |p^n x' p^m y'|_p = |p^{n+m} x'y'|_p = p^{-(n+m)} = p^{-n}p^{-m} = |p^n x'|_p |p^m y'|_p \\ &= |x|_p|y|_p.\end{aligned}$$

Next, we will prove the condition for a non-archimedean absolute value. Once we have proven this condition, the fourth condition follows naturally, as we will have that $|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$ for all $x, y \in \mathbb{Q}$.

v) $|x+y|_p \leq \max\{|x|_p, |y|_p\}$

Without loss of generality, take $|x|_p \geq |y|_p$. Then, we have that $p^{-v_p(x)} \geq p^{-v_p(y)}$, which means $v_p(x) \leq v_p(y)$. Therefore, $\min\{v_p(x), v_p(y)\} = v_p(x)$. We have already that $v_p(x+y) \geq \min\{v_p(x), v_p(y)\} = v_p(x)$. Therefore, $p^{-v_p(x+y)} \leq p^{-v_p(x)}$, or equivalently, $|x+y|_p \leq |x|_p = \max\{|x|_p, |y|_p\}$. □

Therefore, this p-adic absolute value is a properly constructed, non-archimedean absolute value over the rational numbers.

The following theorem showcases an important property of all non-archimedean absolute values.

Theorem 2.8 ([2, Theorem 2.2.2])**.** *An absolute value on $\mathbb{Q}$ is non-archimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}$.*

Proof. Let $|\cdot|$ be non-archimedean. First note that $|0| = 0 < 1$, for all absolute values. Next, we will prove that $|n| \leq 1$ for all $n \in \mathbb{Z} \setminus \{0\}$ by induction. By property (iii) for all absolute values, we can say that $|1 \times 1| = |1||1|$. Therefore, $|1| = |1|^2$. The only positive real number which satisfies the property that $x = x^2$ is 1, which means we must have $|1| = 1$. We also have $|-1 \times -1| = |-1|^2$, which means that $|-1|^2 = 1$, which implies $|-1| = 1$.

For the inductive step, assume $|n| \leq 1$. Next, note that we already have the base case $n = 1$ shown to be true: $|1| \leq 1$. Then, by the non-archimedean condition we have that $|n+1| \leq \max\{|n|, |1|\}$. So, if we have $|n| \leq 1$, then $\max\{|n|, |1|\} = 1$, which means $|n+1| \leq 1$. Therefore, through induction we have shown this property is true for the natural numbers. Consider for the negatives that $|-a| = |-1||a| = 1 \times |a| = |a|$. Therefore it is also true for negative numbers. We have the forward direction proved.

Assume that for all $n \in \mathbb{Z}$, we have $|n| \leq 1$. This direction requires we show that $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$. For the case where at least one of the terms is 0, this is easy to show: let $y = 0$. Then, $|x+y| = |x| = \max\{|x|, |0|\}$. Therefore, take that both x and y are non-zero. Dividing both sides of $|x+y| \leq \max\{|x|, |y|\}$ by $|y|$ gives an equivalent statement: $|\frac{x}{y} + 1| \leq \max\{|\frac{x}{y}|, 1\}$. Since $\frac{x}{y}$ is a rational number, if we show $|z+1| \leq \max\{|z|, 1\}$ is true for all rational numbers, we will have $|x+y| \leq \max\{|x|, |y|\}$ for all rational numbers.

For an arbitrary positive integer m, we have

$$\begin{aligned} |z+1|^m &= \left| \sum_{0 \leq k \leq m} \binom{m}{k} z^k \right| \\ &\leq \sum_{0 \leq k \leq m} \left| \binom{m}{k} \right| |z^k|. \end{aligned}$$

Since $\binom{m}{k}$ is an integer, we have that $|\binom{m}{k}| \leq 1$:

$$\begin{aligned} &\leq \sum_{0 \leq k \leq m} |z^k| \\ &= \sum_{0 \leq k \leq m} |z|^k. \end{aligned}$$

For $|x| \leq 1$ we have $|x|^k \leq 1$ for $0 \leq k \leq m$, and if $|x| > 1$, then $|x|^k \leq |x|^m$:

$$\leq (m+1) \max\{1, |z|^m\}.$$

Take the m-th root of both sides:

$$|x+1| \leq \sqrt[m]{m+1} \max\{1, |x|\}.$$

Now, this inequality is true for all positive integers m, which means we can consider the limit as m approaches ∞: $\lim_{m \to \infty} \sqrt[m]{m+1} = 1$. Therefore, as $m \to \infty$, we have:

$$|x+1| \leq \max\{1, |x|\}.$$

As we found above, this was a sufficient condition to prove that $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{Q}$ where $x, y \neq 0$. □

3. Product Formula

The following theorem is a fascinating property of the p-adic absolute values that connects all the primes together.

Theorem 3.1 (Product Formula, [2, Proposition 3.1.4]). *For $x \in \mathbb{Q}$ such that $x \neq 0$, we have that*

$$\prod_{p \leq \infty} |x|_p = 1$$

where $p \leq \infty$ means the product over all the primes in the integers and the case where $p = \infty$, which is used to denote the standard absolute value over the rationals.

Proof. First we will consider the case where x is a natural number. In this case, we can consider the prime factorization of x, given by $x = p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}$, where p_j are distinct primes. For each p_j, we have that $v_{p_j}(x) = i_j$, which means $|x|_{p_i} = p_i^{-i_j}$. For the standard absolute value we have $|x|_\infty = p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}$. For any prime $p \notin \{p_j \mid 1 \le j \le n\}$, we have that $v_p(x) = 0$, which means $|x|_p = 1$.

When we multiply all the absolute values together, we get

$$|x|_{p_1} |x|_{p_2} \dots |x|_{p_n} |x|_\infty = p_1^{-i_1} p_2^{-i_2} p_3^{-i_3} \dots p_n^{-i_n} (p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}) = 1$$

Therefore, we have that the product formula holds for the natural numbers. Since $|x| = |-x|$ regardless of the absolute value used, we have that it also holds for negative integers.

Then, we need to generalize to all rationals. Consider $x = a/b$, where a/b is a completely reduced fraction (the absolute value is independent of choice of a and b). Let $a = (-1)^c \cdot p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}$, and $b = (-1)^d \cdot q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}$, where p_k and q_t are distinct primes. As well, since the fraction is reduced, we have that for all p_k and q_t, $p_k \neq q_t$. For each p_k, we have that $v_{p_k}(a) = i_k$, which means $|a|_{p_k} = p_k^{-i_k}$. Similarly, $v_{q_t}(b) = j_t$ and $|b|_{q_t} = q_t^{-j_t}$. The p-adic absolute value of x for a given prime will fall under one of three categories. If the prime appears in the factorization of a, then it can not also be in the factorization of b:

$$|x|_{p_k} = p_k^{-(v_{p_k}(a) - v_{p_k}(b))} = p_k^{-(i_k - 0)} = p_k^{-i_k}.$$

If the prime appears in the factorization of b, then it is not in the factorization of a:

$$|x|_{q_t} = q_t^{-(v_{q_t}(a) - v_{q_t}(b))} = q_t^{-(0 - j_t)} = q_t^{j_t}.$$

If the prime does not appear in either factorization, then $|x|_p = 1$. So the product of p-adic absolute values where $p < \infty$ gets us:

$$\prod_{p<\infty} |x|_p = (p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n})^{-1} (q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}) = \frac{q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}}{p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}}.$$

For $p = \infty$ we get:

$$|x|_\infty = \frac{|a|_\infty}{|b|_\infty} = \frac{p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}}{q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}}.$$

Therefore, we have that

$$\prod_{p \le \infty} |x|_p = \frac{q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}}{p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}} \cdot \frac{p_1^{i_1} p_2^{i_2} p_3^{i_3} \dots p_n^{i_n}}{q_1^{j_1} q_2^{j_2} q_3^{j_3} \dots q_m^{j_m}} = 1.$$

□

4. Ultrametric Space

With our p-adic absolute value in hand, we can sketch a new idea of closeness in the rational numbers. Like with any absolute value, the distance between two points will be defined by the following function:

$$d(x, y) = |x - y|.$$

This function is called the metric for a specific absolute value and a set with a metric is known as a metric space. Then, there arises the idea of an ultrametric space:

Definition 4.1. An ultrametric space is a metric space $\Bbbk$ with the property that for $x, y, z \in \Bbbk$:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

More properties of non-archimedean absolute values:

Theorem 4.2 ([2, Lemma 2.3.2])**.** *An absolute value is non-archimedean if and only if it induces an ultrametric space.*

Theorem 4.3 ([2, Proposition 2.3.3])**.** *For a non-archimedean absolute value, if $|x| \neq |y|$, then $|x + y| = \max\{|x|, |y|\}$.*

Consult [2] for proofs to 4.2 and 4.3.

Theorem 4.4 ([2, Corollary 2.3.4])**.** *In the ultrametric space, all triangles are isosceles.*

Proof. To begin, consider a triangle in a metric space to be distances between three points which function as vertices. So, for three points x, y, z a triangle is the three distances $d(x, y), d(y, z), d(x, z)$. Assume $d(x, y) \neq d(y, z)$. Then, using Theorem 4.3, we can say then that

$$|x - z| = |(x - y) + (y - z)| = \max\{|x - y|, |y - z|\}.$$

Therefore, $d(x, z) = \max\{d(x, y), d(y, z)\}$, which means that if two sides of a triangle are unequal, the third side must be equal to the larger of the two sides. $\square$

5. Topology

Now that we have described some properties of an ultrametric space, we can create a topology on $\mathbb{Q}$ using a non-archimedean absolute value.

For a field $\Bbbk$ with the absolute value $|\cdot|$, let $a \in \Bbbk$ and $r \in \mathbb{R}_+$. Then an open ball with center a and radius r is defined as follows:

$$B(x, r) = \{x \in \Bbbk \mid d(x, a) < r\}.$$

A closed ball with the same center and radius is defined as follows:

$$\overline{B}(x, r) = \{x \in \Bbbk \mid d(x, a) \leq r\}.$$

Using these standard definitions of open and closed balls, we can also define open and closed sets in the regular way.

Definition 5.1. An open set is a set $A \subset \Bbbk$, such that for all $a \in A$, there exists $r > 0$ such that $a \in B(a, r) \subset A$. A closed set is the complement of an open set. Under any topology, a closed set is also defined as a set which contains all of its boundary points. A boundary point of a set A is any point such that any open ball U which contains the point also has a non-empty intersection with A. For more on point-set topology, the reader should consult [3, Chapter 2].

With a non-archimedean absolute value, open balls have unique properties:

Theorem 5.2 ([2, Proposition 2.3.6])**.** *The following are true:*

i) All points in a ball are the center of the ball. If $b \in B(a, r)$, then $B(b, r) = B(a, r)$.
ii) All open balls are both open and closed.

Proof. Consider an arbitrary open ball with center a and radius r. Then for $b \in B(a,r)$, we have $|b-a| < r$. Then, for all $x \in B(a,r)$, we also have $|x-a| < r$. Since $x - b = (x-a) + (a-b)$, we can use the non-archimedean property to say $|x-b| = |x-a+a-b| \leq \max\{|x-a|, |a-b|\} < r$, since both of those distances are less than r. Therefore, $x \in B(b,r)$, which proves $B(a,r) \subset B(b,r)$. Showing the subset goes the other way is a similar proof switching b and a. Therefore, $B(a,r) = B(b,r)$.

For an arbitrary open ball $A = B(a,r)$, consider that for $a \in A$, we have $a \in A \subset A$. Therefore, we have that open balls are open. Now, let x be a boundary point of A. Therefore, for $B(x,s)$ where $s \leq r$, we have that $B(x,s) \cap B(a,r) \neq \emptyset$. Take $y \in B(x,s) \cap B(a,r)$. Then we have that $|y-x| < s$ and $|y-a| < r$. Using the non-archimedean property, we can say that

$$|x-a| = |x-y+y-a| \leq \max\{|x-y|, |y-a|\} < \max\{r,s\} = r.$$

Therefore, $x \in B(a,r)$, and A is closed, which means all open balls are closed. □

Returning to our p-adic absolute value rather than an arbitrary non-archimedean absolute value, we can examine the structure of open and closed balls more explicitly.

Theorem 5.3 ([2, Problem 50])**.** *For a p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$, we have that*

$$\overline{B}(0,1) = B(0,1) \cup B(1,1) \cup B(2,1) \cup \cdots \cup B(p-1,1).$$

Proof. Take $x \in \overline{B}(0,1)$. If $|x| < 1$, then $x \in B(0,1)$. If $|x| = 1$, then we have $v_p(x) = 0$. Therefore, for $x = a/b$, we have $v_p(a) - v_p(b) = 0$. This means that we can consider a/b to be the case where neither a nor b is divisible by p. This is because they are both divisible by the same power of p and therefore can be reduced. In this reduced form, we know that b is not divisible by p, which allows us to say that there exists $b^{-1} \in \mathbb{Z}/p\mathbb{Z}$ such that $bb^{-1} \equiv 1 \pmod{p}$. Then we have that $bb^{-1}a \equiv a \pmod{p}$. Let j be the integer where $1 \leq j \leq p-1$ and $b^{-1}a \equiv j \pmod{p}$. Therefore, $jb \equiv a \pmod{p}$. When this is the case we have $p \mid jb - a$, which implies that $|j - a/b|_p = \left|\frac{jb-a}{b}\right|_p < 1$. Therefore, we have that $x \in B(j,1)$. Therefore, $\overline{B}(0,1) \subset B(0,1) \cup B(1,1) \cup B(2,1) \cup \cdots B(p-1,1)$. For more details on modular arithmetic, see [1, Chapter 5].

Next, pick some $x \in B(0,1) \cup B(1,1) \cup B(2,1) \cup \cdots B(p-1,1)$. First, if $x \in B(0,1)$, then $x \in \overline{B}(0,1)$, since $B(0,1) \subset \overline{B}(0,1)$. If $x \in B(i,1)$ for $1 \leq i \leq p-1$, then we have that $|i - a/b|_p = |\frac{ib-a}{b}| < 1$. This means we have that $v_p(ib-a) > v_p(b)$.

Since $ib - a$ is divisible by a higher power of p than b, we can use cancellation to find an equivalent fraction where the numerator is divisible by p but the denominator is not. For convenience and because they are equivalent fractions, we will treat $\frac{ib-a}{b}$ as though it is the reduced form. In this case, we would see that this reduced fraction would have $p \nmid a$, since if $p \mid a$, then we would have that $p \mid ib - a$ which implies that $p \mid ib$, which is false since neither i nor b is divisible by p. Therefore, for the fraction a/b, we have that neither a nor b is divisible by p, which implies that $|a/b| = 1$. Therefore $x = a/b \in \overline{B}(0,1)$. □

In the bizarre topology created by the p-adic absolute value, every closed ball is the union of a finite set of open balls. Then, we have that all open and closed balls are clopen.

6. Equivalent Absolute Values

While absolute values can be defined in different ways, there is an idea that separate absolute values can be "equivalent."

Definition 6.1 ([2, Definition 3.1.3])**.** Two absolute values are called equivalent when they define the same topology on $\mathbb{Q}$, which means that all sets which are open with respect to one absolute value are open with respect to the other.

To aid in determining whether two absolute values are equivalent, we have a couple of criteria. Consult [2] for a proof:

Theorem 6.2 ([2, Lemma 3.1.2])**.** *The following are equivalent:*

i) $|\cdot|_1$ *and* $|\cdot|_2$ *are equivalent absolute values.*
ii) For any x *in a field* $\Bbbk$*,* $|x|_1 < 1$ *if and only if* $|x|_2 < 1$*.*
iii) There exists a positive real number α *such that for all* $x \in \Bbbk$*,* $|x|_1 = |x|_2^{\alpha}$*.*

Theorem 6.3 ([2, Problems 69 & 70])**.** *The following are true:*

(1) For p, q *distinct primes,* $|\cdot|_p$ *is not equivalent to* $|\cdot|_q$
(2) For p *prime,* $|\cdot|_p$ *is not equivalent to* $|\cdot|_\infty$

Proof. 1: Let p, q be distinct primes. Then, we have that since $q \nmid p$, $|p|_q = 1$. However, $|p|_p = \frac{1}{p} < 1$. Therefore, by Theorem 6.2 we have that these are not equivalent absolute values.

2: Let p be prime. $|p|_p = \frac{1}{p} < 1$ and $|p|_\infty = p > 1$. Therefore, we also have that $|\cdot|_p$ is not equivalent to $|\cdot|_\infty$. □

7. Ostrowski's Theorem

Now, for the final theorem of the paper, a theorem that classifies all non-equivalent absolute values on $\mathbb{Q}$.

Theorem 7.1 (Ostrowski's Theorem, [2, Theorem 3.1.3])**.** *Every non-trivial absolute value on* $\mathbb{Q}$ *is equivalent to either* $|\cdot|_p$ *for some prime or* $|\cdot|_\infty$*. More specifically, all archimedean absolute values are equivalent to* $|\cdot|_\infty$*, while non-archimedean absolute values are equivalent to some* $|\cdot|_p$*.*

Proof. First, we want to show that for an arbitrary archimedean absolute value $|\cdot|$, that $|\cdot|$ is equivalent to $|\cdot|_\infty$. Let n_0 be the least positive integer such that $|n_0| > 1$. We know this exists because of the contrapositive of Theorem 2.8. Since $|\cdot|$ is archimedean, there must be some $z \in \mathbb{Z}$ such that $|z| > 1$. We know that this is non-zero because $|0| = 0 < 1$. There exists positive numbers that satisfy this as well, because if $z < 0$, then $-z > 0$ and $|-z| = |z| > 1$. Therefore, there must be some least positive integer n_0 that satisfies $|n_0| > 1$. Let $\alpha = \frac{log(|n_0|)}{n_0}$ so that $|n_0| = n_0^{\alpha} = |n_0|_\infty^{\alpha}$.

First, assume that we know $|x| = |x|_\infty^{\alpha}$ for all $x \in \mathbb{N}$. Then it will be true for all integers, as for a negative integer y, $|y| = |-y| = |-y|_\infty^{\alpha} = |y|_\infty^{\alpha}$. For rational numbers in general:

$$|\frac{a}{b}| = \frac{|a|}{|b|} = \frac{|a|_\infty^{\alpha}}{|b|_\infty^{\alpha}} = |\frac{a}{b}|_\infty^{\alpha}.$$

Therefore, once we have shown this property is true for all integers, we have shown it in general.

Take an arbitrary natural number n. Then, we will write n as the sum of powers of n_0, essentially writing it in "base n_0":

$$n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_k n_0^k,$$

with $0 \leq a_i \leq n_0 - 1$ and $a_i \in \mathbb{N}$ for all i. Since k is the largest natural number with $a_k > 0$, we find that k is the unique natural number satisfying $n_0^k \leq n < n_0^{k+1}$. By the triangle inequality:

$$\begin{aligned} |n| &= |a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_k n_0^k| \\ &\leq |a_0| + |a_1 n_0| + |a_2 n_0^2| + \cdots + |a_k n_0^k| \\ &= |a_0| + |a_1| n_0^\alpha + |a_2| n_0^{2\alpha} + \cdots + |a_k| n_0^{k\alpha}. \end{aligned}$$

Since n_0 is the smallest positive integer such that $|n_0| > 1$ and since $a_i < n_0$ for all i, we must have that $|a_i| \leq 1$. Therefore:

$$|n| \leq 1 + n_0^\alpha + n_0^{2\alpha} + \cdots + n_0^{k\alpha} = n_0^{k\alpha}(1 + n_0^{-\alpha} + n_0^{-2\alpha} + \cdots + n_0^{-k\alpha}) = n_0^{k\alpha} \sum_{i=0}^{k} n_0^{-i\alpha}.$$

Since $n_0 > 1$:

$$|n| \leq n_0^{k\alpha} \sum_{i=0}^{\infty} n_0^{-i\alpha} = n_0^{k\alpha} \frac{n_0^\alpha}{n_0^\alpha - 1}.$$

Let $C = \frac{n_0^\alpha}{n_0^\alpha - 1}$. We know that $n_0 > 1$, since n_0 is a positive integer and $|n_0| > 1$ whereas $|1| = 1$. Consequently, we have $n_0^\alpha > 1$ because $\alpha > 0$, which means $n_0^\alpha - 1 > 0$ and $C > 0$. Remember that $n \leq n_0^k$, so $|n| \leq C n_0^{k\alpha} \leq C n^\alpha$. The inequality $|n| \leq C n^\alpha$ holds for all $n \in \mathbb{N}$, since n is arbitrary. Therefore, it also holds for n^N where N is a positive integer: $|n^N| \leq C n^{N\alpha}$. Taking the N-th roots of both sides: $|n| \leq \sqrt[N]{C} n^\alpha$. This too is true for all N, which means we can consider it as $N \to \infty$ and see that

$$|n| \leq n^\alpha, \tag{1}$$

since C is positive and $\lim_{N \to \infty} \sqrt[N]{C} = 1$.

Now we want to show the opposite inequality. Consider the following, utilizing the triangle inequality:

$$n_0^{(k+1)\alpha} = |n_0^{k+1}| = |n + n_0^{k+1} - n| \leq |n| + |n_0^{k+1} - n|.$$

Therefore,

$$|n| \geq n_0^{(k+1)\alpha} - |n_0^{k+1} - n|.$$

Then, we can use (1) to say that

$$|n| \geq n_0^{(k+1)\alpha} - (n_0^{k+1} - n)^\alpha.$$

Since $n_0^k \leq n$, we can say that $-(n_0^{k+1} - n)^\alpha \leq -(n_0^{k+1} - n_0^k)^\alpha$, and so:

$$|n| \geq n_0^{(k+1)\alpha} - (n_0^{k+1} - n_0^k)^\alpha = n_0^{(k+1)\alpha}(1 - (1 - \frac{1}{n_0})^\alpha).$$

Let $C' = (1 - (1 - \frac{1}{n_0})^\alpha)$ which is positive:

$$|n| \geq C' n_0^{(k+1)\alpha}.$$

Then, let d be a positive number such that $C' < d \leq \frac{C' n_0^{(k+1)\alpha}}{n^\alpha}$. Therefore, we have

$$dn^\alpha \leq \frac{C' n_0^{(k+1)\alpha}}{n^\alpha} \cdot n^\alpha = C' n_0^{(k+1)\alpha} \leq |n|.$$

As before, since n is an arbitrary positive integer, we know it is true for n^N for some positive integer N. Therefore, $|n| \geq \sqrt[N]{d} n^\alpha$, for all $n \in \mathbb{N}$, and $\lim\limits_{n \to \infty} \sqrt[N]{d} = 1$; this shows that $|n| \geq n^\alpha$. Therefore, $|n| = n^\alpha = |n|_\infty^\alpha$, as both inequalities have been proven.

Now, let $|\cdot|$ be a non-archimedean absolute value. Since it is not the trivial absolute value, we know that there exists a least positive integer such that $|n| < 1$, since if all positive integers had $|n| = 1$, then it would follow that all rational numbers besides zero satisfy $|x| = 1$. Let p be this smallest positive integer. If p were composite, we would have $|a||b| = |p|$ for some $a, b \in \mathbb{N}$. However, in this case we would have $a, b < p$, which would mean $|a| = |b| = 1$, and so $|p| = 1$. Therefore, p is prime. Let $\alpha = \frac{\log(|p|)}{\log(|p|_p)}$ so that $|p| = |p|_p^\alpha$.

Let $n \in \mathbb{Z}$ be not divisible by p. We can use division with remainder to say that $n = rp + s$ with $r \in \mathbb{Z}$ and $1 \leq s \leq p - 1$. Equivalently, $rp = n - s$. Since $s < p$, we have that $|s| = 1$. As well, we have that $|rp| = |r||p| \leq 1 \cdot |p| < 1$. Therefore we have $|rp| = |n - s| \neq |s|$. For a triangle with vertices of $0, s, n$, we can apply Theorem 4.4 to say that $|n| = \max\{|s|, |n - s|\} = |s| = 1$. Therefore, for $p \nmid n$, we have that $|n| = 1$.

Now, let $n \in \mathbb{Z}$ be arbitrary. Let v be the positive number such that $n = p^v n'$ with $n' \in \mathbb{Z}$ and $p \nmid n'$. Therefore,

$$|n| = |p^v n'| = |p^v||n'| = |p|^v = |p|_p^{v\alpha} = p^{-v\alpha} = |n|_p^\alpha,$$

by the definition of the p-adic absolute value. Therefore, each non-archimedean absolute value is equivalent to some p-adic absolute value. $\square$

Now, we can see that all non-trivial absolute values of $\mathbb{Q}$ are actually equivalent to one of the p-adic absolute values we have constructed, or the standard absolute value.

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