MEAN CURVATURE OF RIEMANNIAN HYPERSURFACE UNDER THE EXPONENTIAL MAP

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Abstract. This is an REU paper written for the University of Chicago REU, summer 2020, The purpose of this note is to collect and derive some known but not easily found formulas of mean curvature on Riemannian hypersurfaces.

Contents

1. Introduction

The interest in mean curvature arises as it being identically zero is exactly the condition for the surface to be minimal. And through the first variation formula, people discover that the mean curvature of a Riemannian submanifold encodes the derivative of area as described in [3] and [2]. And the main result of the paper is useful in curvature estimation, the theory of partial differential equations etc.

The paper will remind the reader of concepts and theorems in Riemannian geometry as collected from [4], and apply them in the derivation of the main theorem.

2. Basics on Riemannian Geometry

In this section, we remind the reader of some basic construction on Riemannian manifolds, and derive some important results.

On an abstract manifold, the "second derivative" is not always well-defined. As we need to take difference between vectors with different base points. The connection generalizes the property of "second derivative" on Riemannian manifolds.

Definition 2.1. Let $\pi\colon E \to M$ be a vector bundle over a manifold M. A *connection* ∇ in E is a smooth map $\nabla\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, denoted by $(X, Y) \mapsto \nabla_X Y$, such that

(1) For all $f_1, f_2 \in C^\infty(M)$ and for all $X_1, X_2 \in \Gamma(TM)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y_2$$

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(2) For all $a_1, a_2 \in \mathbb{R}$, and for all $Y_1, Y_2 \in \Gamma(E)$,
$$\nabla_X(a_1Y_1 + a_2Y_2) = a_1\nabla_X Y_1 + a_2\nabla_X Y_2$$
(3) For all $f \in C^\infty(M)$,
$$\nabla_X fY = X(f)Y + f\nabla_X Y$$
.

In this paper, we will be exclusively talking about connections on the tangent bundle. And there are connections that have special properties of interests.

Definition 2.2. Let (M, g) be a Riemannian manifold and ∇ a connection in TM. ∇ is *metric* if for all $X, Y, Z \in \Gamma(TM)$,
$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
∇ is *torsion-free* if for all $X, Y \in \Gamma(TM)$,
$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Theorem 2.3. *Let (M, g) be a Riemannian manifold. Then it admits a unique torsion-free metric connection, called the Levi-Civita connection.*

Example 2.4. Let $M = \mathbb{R}^m$ equipped with the standard metric. Then the Levi-Civita connection on M is just the covariant derivative.

If not stated otherwise, the connection used in this paper is the Levi-Civita connection.

For an oriented Riemannian manifold (M, g), there is a unique n-form dV_g on M, called the *Riemannian volume form*. It can be characterized by the following equivalent conditions:

(1) If $(E_1, \dots, E_m)$ is any oriented local orthonormal frame for TM, then
$$dV_g(E_1, \dots, E_m) = 1$$
(2) If $(x_1, \dots, x_m)$ is any local oriented coordinate, then
$$dV_g = \sqrt{\det g_{ij}}\, dx_1 \wedge \cdots \wedge dx_m$$

Definition 2.5. Let (M, g) be a Riemannian manifold and Σ a submanifold of M. The *normal space of Σ in M* is $N_p\Sigma = \{n \in T_pM \mid g(n, v) = 0\ \forall v \in T_p\Sigma\}$. And the *normal bundle* of Σ is the disjoint union of normal spaces over all points in Σ. Define the *second fundamental form* A of Σ to be the vectored-valued bilinear form on $T\Sigma$. For $X, Y \in T\Sigma$
$$A(X, Y) = (\nabla_X Y)^\perp \tag{2.1}$$
Here the superscript $\perp$ indicates taking the projection of the vector onto the normal space of Σ.

Proposition 2.6. *The second fundamental form is symmetric.*

Proof. For $X, Y \in T\Sigma$, the Lie Bracket of X and Y is tangent to Σ: $[X, Y] \in T\Sigma$. But ∇ is the Levi-Civita connection, hence torsion free, therefore, $\nabla_X Y - \nabla_Y X = [X, Y]$. And so
$$A(X, Y) - A(Y, X) = (\nabla_X Y)^\perp - (\nabla_Y X)^\perp = [X, Y]^\perp = 0 \tag{2.2}$$
∎

Definition 2.7. The *mean curvature* is the trace of the second fundamental form. That is, if x_i form orthonormal coordinates at $x \in \Sigma$, the mean curvature at x is

$$H(x) = \sum_i A\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) \tag{2.3}$$

Let (M, g) be a Riemannian manifold. Then the metric g induces a natural isomorphism between the tangent and the cotangent bundle denoted by $\flat\colon TM \to T^*M$ and $\sharp\colon T^*M \to TM$ defined point-wise: for $p \in M$, $v \in T_pM$ and $\eta \in T_P^*M$, $v^\flat$ is the covector defined by $w \mapsto g(v, w)$; and $\eta^\sharp$ is the unique vector satisfying $g(\eta^\sharp, u) = \eta(u)$ for all $u \in T_pM$.

Definition 2.8. For a function f defined on a Riemannian manifold (M, g), the *gradient* of f with respect to the metric g is the vector field $\operatorname{grad}_g f = (df)^\sharp$.

Lemma 2.9. *Let (M, g) be a Riemannian manifold and f be a function defined on M. If x_i are local coordinates, then*

$$\operatorname{grad}_g f = \sum_{i,j=1}^{m} g^{ji} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \tag{2.4}$$

Here $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and g^{ij} is the inverse of the matrix g_{ij}.

Proof. Let $v \in T_pM$ be a tangent vector at p. Then

$$g(\operatorname{grad}_g f, v) = \sum_{i,j=1}^{m} g_{ij} (\operatorname{grad}_g f)_i v_j \tag{2.5}$$

$$(df)_p v = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k} v_k \tag{2.6}$$

Hence, by definition of gradient,

$$\sum_{i,j=1}^{m} g_{ij} (\operatorname{grad}_g f)_i v_j = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k} v_k \tag{2.7}$$

Let $v = \frac{\partial}{\partial x_j}$, for some $j \in \{1, 2, \ldots, m\}$

$$\sum_{i=1}^{m} g_{ij} (\operatorname{grad}_g f)_i = \frac{\partial f}{\partial x_j} \tag{2.8}$$

To uncover the expression of $(\operatorname{grad}_g f)_i$, multiply both sides of the equation by g^{jk} and sum over the index j.

As $\sum_{j=1}^{m} g_{ij} g^{jk} = \delta_{ik}$, by interchanging the index i with k, we obtain the desired expression,

$$(\operatorname{grad}_g f)_i = \sum_{j=1}^{m} g^{ji} \frac{\partial f}{\partial x_j} \tag{2.9}$$

□

Definition 2.10. Let (M, g) be a Riemannian manifold and $X \in \Gamma(TM)$. The *divergence* of X is defined as the function $\operatorname{div} X$ such that $d(\iota_X d\,\mathrm{V}_g) = (\operatorname{div} X) d\,\mathrm{V}_g$.

Here are two important results for later derivation:

Theorem 2.11. *(Divergence Theorem) Let (M, g) be a compact oriented Riemannian manifold. Then for $X \in TM$,*

$$\int_M (\operatorname{div} X) d\,\mathrm{V}_g = \int_{\partial M} g(X, N) d\,\mathrm{V}_{\tilde{g}} \tag{2.10}$$

Here N is unit outward pointing normal and $\tilde{g}$ is the induced metric on the boundary

Proof. By definition, $d(\iota_X d\,\mathrm{V}_g) = (\operatorname{div} X) d\,\mathrm{V}_g$. By applying Stoke's theorem,

$$\int_M (\operatorname{div} X) d\,\mathrm{V}_g = \int_M d\left(\iota_X d\,\mathrm{V}_g\right) = \int_{\partial M} \iota_X d\,\mathrm{V}_g \tag{2.11}$$

If the dimension of M is 1, it's just the fundamental theorem of calculus, if the dimension of M is greater than 1, it suffices to prove $g(X, N) d\,\mathrm{V}_{\tilde{g}} = \iota_X d\,\mathrm{V}_g$ in local coordinates. Without loss of generality, assume that $(N, e_1, \ldots, e_{m-1})$ is an orthonormal basis for $T_x M$ at $x \in \partial M$. Then

$$(\iota_X dV_g)(e_1, \ldots, e_{m-1}) = dV_g(X, e_1, \ldots, e_{m-1}) \tag{2.12}$$

$$= g(X, N)(\iota_N dV_g)(e_1, \ldots, e_{m-1}) \tag{2.13}$$

The second equality follows from the fact that $dV_g(e_i, e_1, \ldots, e_{m-1}) = 0$ for all $1 \leq i \leq m-1$, since $(N, e_1, \ldots, e_{m-1})$ forms a basis and $dV_{\tilde{g}} = \iota_N dV_g$, $g(X, N) d\,\mathrm{V}_{\tilde{g}} = \iota_X d\,\mathrm{V}_g$. ☐

Corollary 2.12. *Let (M, g) be a compact oriented Riemannian manifold. Then for $X \in TM$ and $f \in C^\infty(M)$*

$$\int_M g(\operatorname{grad}_g f, X) d\,\mathrm{V}_g = \int_{\partial M} f g(X, N) d\,\mathrm{V}_{\tilde{g}} - \int_M f(\operatorname{div} X) d\,\mathrm{V}_g \tag{2.14}$$

Proof. We first prove that

$$\operatorname{div}(fX) = f \operatorname{div}(X) + g(\operatorname{grad}_g f, X) \tag{2.15}$$

in local coordinates.

Let $(x_1, \ldots, x_m)$ be a coordinate chart on M. Then

$$d\iota_X dV_g = d\left(\sum_{i=1}^m (-1)^{i+1} X_i \sqrt{\det g}\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m\right) \tag{2.16}$$

$$= \sum_{i=1}^m \frac{\partial\left(X_i \sqrt{\det g}\right)}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_m \tag{2.17}$$

By comparing the equation with the definition of divergence

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \sum_{i=1}^m \frac{\partial}{\partial x_i}\left(X_i \sqrt{\det g}\right) \tag{2.18}$$

By a similar calculation

$$\operatorname{div} fX = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(f X_i \sqrt{\det g} \right) \tag{2.19}$$

$$= \frac{1}{\sqrt{\det g}} \sum_{i=1}^{m} \left(f \frac{\partial X_i \sqrt{\det g}}{\partial x_i} + \frac{\partial f}{\partial x_i} X_i \sqrt{\det g} \right) \tag{2.20}$$

$$= f \operatorname{div} X + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} X_i \tag{2.21}$$

By Lemma 2.9, since g_{ij} is symmetric

$$g(\operatorname{grad}_g f, X) = \sum_{i,j=1}^{m} g_{ij} g^{ki} \frac{\partial f}{\partial x_k} X_j = \sum_{j=1}^{m} \delta_{kj} \frac{\partial f}{\partial x_k} X_j \tag{2.22}$$

$$= \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} X_j \tag{2.23}$$

Then, by 2.15 and bilinearity of g

$$\int_M \operatorname{div}(fX) d\,\mathrm{V}_g = \int_{\partial M} f g(X, N) d\,\mathrm{V}_{\tilde{g}} \tag{2.24}$$

$$= \int_M f \operatorname{div} X d\,\mathrm{V}_g + \int_M g(\operatorname{grad}_g f, X) d\,\mathrm{V}_g \tag{2.25}$$

□

Definition 2.13. Let (M, g) be a Riemannian manifold, a curve $\gamma\colon [a, b] \to M$ is a *geodesic* if $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$. Here $\dot{\gamma}(t) = \frac{d}{d\tau}\Big|_{\tau=t} \gamma(\tau) = (d\gamma)_t \left(\frac{d}{d\tau} \right)$.

Remark 2.14. In local coordinates, $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$ is a system of linear ODE's. Therefore, for a vector $v \in T_pM$, there exists a geodesic $\gamma_v(t)$ defined about 0 such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.

Definition 2.15. Let (M, g) be a Riemannian manifold, and $v \in T_pM$. The exponential map is defined by $\exp(v) = \gamma_v(1)$ provided that the right hand side of the equation is well-defined.

For an embedded submanifold in a Riemannian manifold, the codimension of the manifold is the same as the rank of the normal bundle, which motivates the notion of a tubular neighborhood.

Lemma 2.16. *Let (M, g) be a Riemannian manifold, and $p \in M$. Then the map $d\exp$ at $0 \in T_pM$ is the identity under the standard isomorphism between $T_0(T_pM)$ and T_pM.*

Proof. Let $v \in T_pM$ be any tangent vector. Define a curve $\alpha\colon (-\epsilon, \epsilon) \to T_pM$ by $\alpha(t) = tv$. So $\alpha(0) = 0$ and $\dot{\alpha}(0) = v$ under the standard isomorphism.

Compute the image of v under the differential of the exponential map,

$$(d\exp)_0 v = \frac{d}{dt}\Big|_{t=0} \exp \circ \alpha(t) = \frac{d}{dt}\Big|_{t=0} \exp(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v \tag{2.26}$$

Here $\gamma_v(t)$ is the geodesic curve for v. □

Definition 2.17. Let Σ be a k-dimensional submanifold of (M, g), and let $V = \{(p, v) \in N\Sigma \mid |v|_g < \delta(p)\}$, where $\delta(p)$ is chosen such that the exponential map is a well-defined diffeomorphism. Then $U = \exp(V)$ is called a *tubular neighborhood* of Σ.

Suppose W_0 is an open set in Σ such that there exists a coordinate chart $\varphi\colon W_0 \to \mathbb{R}^k$, and a local orthonormal frame of the portion of normal bundle NP over W_0: $(E_1, \dots, E_{m-k})$. Let $V_0 = V \cap NP\big|_{W_0}$ and $U_0 = \exp(V_0)$. Define a map $T\colon W \times \mathbb{R}^{m-k} \to NP\big|_{W_0}$ by

$$T(x_1, \dots, x_k, z_1, \dots, z_{m-k}) = \left(\varphi^{-1}(x_1, \dots, x_k), \sum_{i=1}^{m-k} z_i E_i\right) \tag{2.27}$$

It induces a coordinate chart on U_0 defined by $\psi = T^{-1} \circ \exp^{-1}$. Such a chart is called a *Fermi coordinate.*

Let (M, g) be a Riemannian manifold and $p \in M$. The exponential map induces another special coordinate about p.

By Lemma 2.16 and the inverse function theorem, the exponential map is a diffeomorphism on an open, star-shaped neighborhood V of T_pM containing 0. Let (b_i) be an orthonormal basis for T_pM. Then $U = \exp(V)$ is called a *normal neighborhood* of p and the corresponding coordinate chart is defined by $\varphi = T^{-1} \circ \left(\exp\big|_V\right)^{-1}$, where $T\colon V \to \mathbb{R}^m$ defined by $\sum_i x_i b_i \mapsto (x_1, \dots, x_m)$ is an isomprphism onto the image. Such coordinates are called *normal coordinates.*

By a property of the exponential map, if $\varphi\colon V \to \exp(V) = U$ is a normal coordinate chart, then $\varphi(p) = T^{-1} \circ \left(\exp\big|_V\right)^{-1}(0) = T^{-1}(0) = 0$. And if $v \in T_pM$, is a vector with coordinate expression $v = \displaystyle\sum_i v_i \frac{\partial}{\partial x_i}$, the geodesic γ_v starting at p with $\dot{\gamma}_v(0) = v$ has coordinate representation $\gamma_v(t) = (v_1 t, v_2 t, \dots, v_m t)$

3. First Variation Formula

In this section, we derive the first variation formula. The first variation formula gives us the geometric interpretation of the mean curvature as describing the change of volume.

Let Σ be a k-dimensional submanifold of M, and $F\colon \Sigma \times \mathbb{R} \to M$ be a variation of Σ with fixed boundary and compact support, that is $F \equiv id$ outside a compact set and $F\big|_{\partial\Sigma} \equiv id$. The vector field $dF\left(\frac{\partial}{\partial t}\right)$ is called the variation vector field. Denote $\Sigma(t) = F(\Sigma, t)$.

Theorem 3.1. *If Σ is a k-dimensional submanifold of a Riemannian manifold M and $F\colon \Sigma \times \mathbb{R} \to M$ is a variation of Σ as above. Let $\partial_t := dF\left(\dfrac{\partial}{\partial t}\right)$*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\Sigma(t)) = -\int_\Sigma \langle \partial_t, H\rangle \, d\mathrm{V}_\Sigma \tag{3.1}$$

Proof. Let x_i be an oriented local coordinate on Σ. For the purpose of this proof, denote $\partial_i := \dfrac{\partial}{\partial x_i}$. Without loss of generality, we can assume that ∂_i are orthonormal locally with respect to the metric. Denote $g_{ij}(t) = g(dF_t(\partial_i), dF_t(\partial_j))$. Let $g^{ij}(t)$ denote the inverse of the matrix $g_{ij}(t)$

The volume form of Σ is then given by

$$d\,\mathrm{V}_\Sigma = \sqrt{\det(g_{ij}(0))}dx_1 \wedge \cdots \wedge dx_k = dx_1 \wedge \cdots \wedge dx_k \tag{3.2}$$

Then the volume of $\Sigma(t)$ is given by the following formula

$$\mathrm{vol}(\Sigma(t)) = \int_\Sigma \sqrt{\det(g_{ij}(t))}\, d\,\mathrm{V}_\Sigma \tag{3.3}$$

Notice that the partial derivatives ∂_t and ∂_i commute, therefore their Lie bracket is identically 0. By the torsion-free property of the Levi-Civita connection

$$\nabla_{\partial_t}\partial_i - \nabla_{\partial_i}\partial_t = [\partial_i, \partial_t] = 0 \tag{3.4}$$

By the metric property of the Levi-Civita connection

$$\langle \nabla_{\partial_i}\partial_t, \partial_i\rangle - \langle \partial_t, \nabla_{\partial_i}\partial_i\rangle = \partial_i\langle \partial_t, \partial_i\rangle = 0 \tag{3.5}$$

For any function $B\colon (-\epsilon, \epsilon) \to M_{m\times m}(\mathbb{R})$, the Jacobi formula states that

$$\frac{d}{dt}\det(B(t)) = \mathrm{trace}\left(\mathrm{adj}(B(t))\left(\frac{d}{dt}B(t)\right)\right) \tag{3.6}$$

Therefore, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\Sigma(t)) = \int_\Sigma \left.\frac{d}{dt}\right|_{t=0} \sqrt{\det(g_{ij}(t))}\, d\,\mathrm{V}_\Sigma \tag{3.7}$$

$$= \int_\Sigma \frac{\mathrm{trace}\left(\mathrm{adj}(g_{ij}(0))\left(\left.\frac{d}{dt}\right|_{t=0} g_{ij}(t)\right)\right)}{2\sqrt{\det(g_{ij}(0))}}\, d\,\mathrm{V}_\Sigma \tag{3.8}$$

Since we choose the coordinates so that the basis is orthonormal, $g_{ij}(0)$ is the identity matrix, and so is it's adjoint

$$\frac{\mathrm{trace}\left(\mathrm{adj}(g_{ij}(0))\left(\left.\frac{d}{dt}\right|_{t=0} g_{ij}(t)\right)\right)}{2\sqrt{\det(g_{ij}(0))}} = \frac{1}{2}\sum_i \left.\frac{d}{dt}\right|_{t=0} \langle \partial_i, \partial_i\rangle \tag{3.9}$$

$$= \frac{1}{2}\sum_i \nabla_{\partial_t}\langle \partial_i, \partial_i\rangle \tag{3.10}$$

By the Lebnitz property of Levi-Civita connection, we can rewrite the connection

$$\frac{1}{2}\sum_i \nabla_{\partial_t}\langle \partial_i, \partial_i\rangle = \sum_i \langle \nabla_{\partial_t}\partial_i, \partial_i\rangle \tag{3.11}$$

By 3.5 and 3.4

$$\sum_i \langle \nabla_{\partial_t}\partial_i, \partial_i\rangle = \sum_i \langle \nabla_{\partial_i}\partial_t, \partial_i\rangle \tag{3.12}$$

$$= \sum_i \langle \nabla_{\partial_i}(\partial_t)^N, \partial_i\rangle + \sum_i \langle \nabla_{\partial_i}(\partial_t)^T, \partial_i\rangle \tag{3.13}$$

$$= -\sum_i \langle \partial_t^N, \nabla_{\partial_i}\partial_i\rangle + \mathrm{div}_\Sigma\, \partial_t^T \tag{3.14}$$

$$= -\langle \partial_t, H\rangle \; + \mathrm{div}_\Sigma\, \partial_t^T \tag{3.15}$$

Since F is compactly supported, by Stoke's theorem

$$-\int_{\Sigma} \operatorname{div}_{\Sigma} \partial_t^T \, d\,\mathrm{V}_{\Sigma} = 0 \tag{3.16}$$

Therefore

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\Sigma(t)) = -\int_{\Sigma} \langle \partial_t, H\rangle \, d\,\mathrm{V}_{\Sigma} \tag{3.17}$$

□

Example 3.2. Consider the special case that Σ is an oriented hypersurface with unit normal vector field N. Let w and v be real valued functions defined on Σ and define $F\colon \Sigma \times \mathbb{R} \to M$ by $(p, w) \mapsto \exp((w + tv)N) = \gamma_{(w+tv)N}(1) = \gamma_N((w + tv))$. Here γ_N denotes the geodesics curve of N in M.

In the case that $w \equiv 0$, in view of 3.12, since the variation vector field is vN, the first variation formula takes the form

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\Sigma(tv)) = \int_{\Sigma} \sum_i \langle \nabla_{\partial_i}(vN), \partial_i\rangle \, d\,\mathrm{V}_{\Sigma} \tag{3.18}$$

$$= \int_{\Sigma} \sum_i \langle v\nabla_{\partial_i} N + \partial_i(v)N, \partial_i\rangle \, d\,\mathrm{V}_{\Sigma} \tag{3.19}$$

Since N is the unit normal vector on Σ, together with 3.5

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\Sigma(tv)) = -\int_{\Sigma} v \sum_i \langle \nabla_{\partial_i}\partial_i, N\rangle \, d\,\mathrm{V}_{\Sigma} \tag{3.20}$$

$$= -\int_{\Sigma} \langle H, N\rangle v \, d\,\mathrm{V}_{\Sigma} \tag{3.21}$$

Denote the hypersurface $\exp(wN)$ by $\Sigma(w)$.

We can also vary the hypersurface with a vector field X that is transverse to the tangent bundle of Σ. By similar argument as in the proof of the first variation formula, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\exp(tvX)) = \int_{\Sigma} \langle H, X\rangle v \, d\,\mathrm{V}_{\Sigma} \tag{3.22}$$

4. Mean Curvature of $\Sigma(w)$

In this section, we derive the main theorem of the paper, the curvature of the surface $\Sigma(w)$.

Let g_z denote the horizontal part of the induced metric on $\Sigma(z)$. In local Fermi coordinates $(x_1, \ldots, x_{m-1}, z)$ on a tubular neighborhood of Σ,

$$g_z\left(\left.\frac{\partial}{\partial x_i}\right|_{(p,0)}, \left.\frac{\partial}{\partial x_j}\right|_{(p,0)}\right) = g\left(\left.\frac{\partial}{\partial x_i}\right|_{(p,z)}, \left.\frac{\partial}{\partial x_j}\right|_{(p,z)}\right) \tag{4.1}$$

We use $g_{w,ij}$ to indicate the ij entry of the matrix g_w.

Lemma 4.1. *Let $(x_1, \ldots, x_{m-1}, z)$ be the Fermi coordinates in a normal neighborhood about Σ. Let $\tilde{g}$ denote the metric on the hypersurface $\Sigma(w)$. Then the induced metric on $\Sigma(w)$, denoted $\tilde{g}$ decomposes:*

$$\tilde{g} = g_w + dz \otimes dz \tag{4.2}$$

Proof. The Gauss Lemma in Riemannian geometry states that for $v, n \in T_pM$, $g\left((d\exp)_n n, (d\exp)_n v\right) = g(n, v)$. Here we identify T_pM with $T_n(T_pM)$ under the cononical isomorphism. Since $wN = w\dfrac{\partial}{\partial z}$ is normal to the hypersurface Σ, for all points p in Σ, and for all $v \in T_pM$

$$g\left((d\exp)_{wN} wN, (d\exp)_{wN} v\right) = g(wN, v) = 0 \tag{4.3}$$

By a property of Fermi coordinates

$$(d\exp)_{wN} wN = w\frac{\partial}{\partial z}\bigg|_{\exp(wN)} \tag{4.4}$$

Assume that $(x_1, \ldots, x_{m-1})$ is a normal coordinate about p on Σ. Since N is a unit normal on Σ, $(x_1, \ldots, x_{m-1}, z)$ is a normal coordinate about p on M. By property of normal coordinates, if $v = \displaystyle\sum_{i=1}^{m-1} v_i \frac{\partial}{\partial x_i} + v_m \frac{\partial}{\partial z}$, the geodesic can be written in local coordinates as

$$\gamma_v(t) = (v_1 t, v_2 t, \ldots, v_{m-1} t, v_m t) \tag{4.5}$$

Now, let α and β_i for $i = 1, \ldots, m-1$ be curves on T_pM, defined by

$$\alpha(t) = (t+1)\frac{\partial}{\partial z} \tag{4.6}$$

$$\beta_i(t) = t\frac{\partial}{\partial x_i} + \frac{\partial}{\partial z} \tag{4.7}$$

Under the standard isomorphism between $T_N(T_pM)$ and T_pM, $\alpha(0) = \dfrac{\partial}{\partial z}$, $\dot\alpha(t) = \dfrac{\partial}{\partial z}$, $\beta_i(0) = \dfrac{\partial}{\partial z}$, $\dot\beta_i(t) = \dfrac{\partial}{\partial x_i}$, hence we can compute the image of $\dot\alpha(0)$ and $\dot\beta_i(0)$ under the differential of the exponential map at $N = \frac{\partial}{\partial z}$.

$$(d\exp)_N \frac{\partial}{\partial z} = (d\exp)_N \dot\alpha(0) \tag{4.8}$$

$$= \frac{d}{dt}\bigg|_{t=0} \exp\circ\alpha(t) \tag{4.9}$$

$$= \frac{d}{dt}\bigg|_{t=0} (0, \ldots, 0, 1+t) \tag{4.10}$$

$$= \frac{\partial}{\partial z}\bigg|_{\exp(N)} \tag{4.11}$$

Similarly, we have

$$(d\exp)_N \frac{\partial}{\partial x_i} = \frac{d}{dt}\bigg|_{t=0} (0, \ldots, t, \ldots, 0, 1) \tag{4.12}$$

$$= \frac{\partial}{\partial x_i}\bigg|_{\exp(N)} \tag{4.13}$$

Here the t occurs at the i-th entry of the coordinate. Therefore

$$g\left(\frac{\partial}{\partial x_i}\bigg|_{\exp(N)}, \frac{\partial}{\partial z}\bigg|_{\exp(N)}\right) = g\left(\frac{\partial}{\partial x_i}\bigg|_p, \frac{\partial}{\partial z}\bigg|_p\right) = 0 \tag{4.14}$$

and

$$g\left(\frac{\partial}{\partial z}\Big|_{\exp(N)}, \frac{\partial}{\partial z}\Big|_{\exp(N)}\right) = g\left(\frac{\partial}{\partial z}\Big|_{p}, \frac{\partial}{\partial z}\Big|_{p}\right) = 1 \tag{4.15}$$

By linearity of the differential, the result stands even if the coordinate chart is not a normal coordinate chart. The decomposition follows. □

Lemma 4.2. *The unit normal vector N on the hypersurface $\Sigma(w)$ is given by:*

$$N = \frac{1}{\sqrt{1 + |\operatorname{grad}_{g_w} w|_{g_w}}}\left(\frac{\partial}{\partial z} - \operatorname{grad}_{g_w} w\right) \tag{4.16}$$

Proof. This is a computation using the preceding Lemma.

The tangent space at $p \in \Sigma$ is spanned by $\frac{\partial}{\partial x_i}$. In coordinates, $F(x) = (x, w(x))$. Therefore, the tangent space at $F(p)$ is spanned by $(dF)_p\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} + \frac{\partial w}{\partial x_i}\frac{\partial}{\partial z}$.

By Lemma 2.9, $\operatorname{grad}_{g_w} w = \sum_{i,j} g_w^{ji}\frac{\partial w}{\partial x_j}\frac{\partial}{\partial x_i}$. Therefore, by previous lemma

$$g\left(\frac{\partial}{\partial z} - \operatorname{grad}_{g_w} w, \frac{\partial}{\partial x_k} + \frac{\partial w}{\partial x_k}\frac{\partial}{\partial z}\right) = \frac{\partial w}{\partial x_k} - g\left(\operatorname{grad}_{g_w} w, \frac{\partial}{\partial x_k}\right) \tag{4.17}$$

$$= \frac{\partial w}{\partial x_k} - \sum_{ij} g_{ik} g_w^{ji}\frac{\partial w}{\partial x_j} \tag{4.18}$$

$$= 0 \tag{4.19}$$

This proves that N is in fact a normal vector to $\Sigma(w)$. The lemma then follows from normalizing the vector $\frac{\partial}{\partial z} - \operatorname{grad}_{g_w} w$. □

Theorem 4.3. *The mean curvature of the surface $\Sigma(w)$ has the expression*

$$\begin{aligned} H = {} & \operatorname{div}_{g_w}\left(\frac{\operatorname{grad}_{g_w} w}{\sqrt{1 + |\operatorname{grad}_{g_w} w|^2_{g_w}}}\right) + \frac{\dot g_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} w)}{2\sqrt{1 + |\operatorname{grad}_{g_w} w|^2_{g_2}}} \\ & - \frac{1}{2}\sqrt{1 + |\operatorname{grad}_{g_w} w|^2_{g_2}}\operatorname{trace}\left(g_w^{-1}\dot g_w\right) \end{aligned} \tag{4.20}$$

Here $\dot g_w$ is the tensor $\partial_z g_w$.

Proof. We prove the theorem by considering the variation of volume of the hypersurface. By Lemma 4.1, the volume form on $\Sigma(w)$ can be expressed as

$$d\,\mathrm{V}_{\tilde g} = \sqrt{1 + |\operatorname{grad}_{g_w} w|^2_{g_w}}\, d\,\mathrm{V}_{g_w} \tag{4.21}$$

Let h_t denote the tensor g_{w+tv} to simplify notation. Let v be a compactly supported function

$$\frac{d}{dt}\Big|_{t=0}\operatorname{vol}(\Sigma(w+tv)) = \int_\Sigma \frac{d}{dt}\Big|_{t=0}\sqrt{1 + |\operatorname{grad}_{h_t} w + tv|^2_{h_t}}\, d\,\mathrm{V}_{h_t} \tag{4.22}$$

That is, we are varying the surface $\Sigma(w)$ in the direction of $\frac{\partial}{\partial z}$. We can express $|\operatorname{grad}_{h_t} w + tv|^2_{h_t}$ in local coordinates x_i as

$$|\operatorname{grad}_{h_t} w + tv|^2_{h_t} = \sum_{k,l} h_{t,kl}(\operatorname{grad}_{h_t} w + tv)_k(\operatorname{grad}_{h_t} w + tv)_l \tag{4.23}$$

$$= \sum_{i,j,k,l} h_{t,kl}\left(h_t^{ik}\frac{\partial(w+tv)}{\partial x_i}\right)\left(h_t^{jl}\frac{\partial(w+tv)}{\partial x_j}\right) \tag{4.24}$$

$$= \sum_{i,j} h_t^{ji}\left(\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + 2t\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + t^2\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right) \tag{4.25}$$

And the volume form has expression

$$d\,\mathrm{V}_{h_t} = \sqrt{\det h_t}\, dx_1 \wedge \cdots \wedge dx_{m-1} \tag{4.26}$$

Notice that $h_0 = g_w$ and so we can compute the integrand in 4.22 in coordinates

$$\left.\frac{d}{dt}\right|_{t=0}\sqrt{1 + |\operatorname{grad}_{h_t} w + tv|^2_{h_t}}\, d\,\mathrm{V}_{h_t} =$$

$$\left.\frac{d}{dt}\right|_{t=0}\sum_{i,j} h_t^{ji}\left(\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + 2t\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + t^2\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right)\frac{d\,\mathrm{V}_{g_w}}{2\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}} \tag{4.27}$$

$$+\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}\left(\left.\frac{d}{dt}\right|_{t=0}\sqrt{\det(h_t)}\right) dx_1 \wedge \cdots \wedge dx_{m-1}$$

Computing each of the derivatives, we get

$$\left.\frac{d}{dt}\right|_{t=0}\sqrt{\det(h_t)} = \frac{1}{2\sqrt{\det g_w}}\operatorname{trace}(\operatorname{adj}(g_w)\dot{g}_w) \tag{4.28}$$

$$= \frac{\sqrt{\det g_w}}{2}\operatorname{trace}(g_w^{-1}\dot{g}_w) \tag{4.29}$$

The second equality follows from the fact $\operatorname{adj}(g_w) = (\det g_w)g_w^{-1}$.

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\sum_{i,j} h_t^{ji}\left(\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + 2t\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + t^2\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right) &= \sum_{i,j} 2g_w^{ij}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} \\ &+ \sum_{i,j} v\dot{g}_w^{ji}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\end{aligned} \tag{4.30}$$

Notice that the first term on the right hand side of 4.30 is the metric g_w applied to vectors $\operatorname{grad}_{g_w} w$ and $\operatorname{grad}_{g_w} v$

$$\sum_{i,j} 2g_w^{ij}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} = \sum_{i,j,k,l} 2g_{w,kl}g_w^{ik}\frac{\partial w}{\partial x_i}g_w^{jl}\frac{\partial v}{\partial x_j} \tag{4.31}$$

$$= 2g_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} v) \tag{4.32}$$

Now we proceed to compute the second term on the right hand side of 4.30. To do so, we must first understand the expression $\dot{g}_w^{ji}$.

As $h_{t,ij}$ and h_t^{ij} are inverses of each other. We have

$$0 = \sum_k \left.\frac{d}{dt}\right|_{t=0} h_{t,ik}h_t^{kj} \tag{4.33}$$

By the product rule

$$0 = \sum_k v\dot{g}_{w,ik} g_w^{kj} + v g_{w,ik}\dot{g}_w^{kj} \tag{4.34}$$

Multiply both sides by g_w^{li} and sum over i

$$0 = \sum_{k,i} v g_w^{li}\dot{g}_{w,ik} g_w^{kj} + v g_w^{li} g_{w,ik}\dot{g}_w^{kj} \tag{4.35}$$

$$= \sum_{k,i} v g_w^{li}\dot{g}_{w,ik} g_w^{kj} + v\dot{g}_w^{kj}\delta_{lk} \tag{4.36}$$

This holds for all functions v and all indices l. Interchanging the index i and l, we obtain the desired expression

$$\dot{g}_w^{ij} = -\sum_{k,i} g_w^{il}\dot{g}_{w,lk} g_w^{kj} \tag{4.37}$$

Therefore, we can rewrite the second term on the right hand side of 4.30 as

$$\sum_{i,j} v\dot{g}_w^{ji}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} = -\sum_{i,j,k,l} v\dot{g}_{w,lk} g_w^{il}\frac{\partial w}{\partial x_i} g_w^{kj}\frac{\partial w}{\partial x_j} \tag{4.38}$$

$$= v\dot{g}_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} w) \tag{4.39}$$

We recover the variation of volume of $\Sigma(w + tv)$ as the sum of three integrals

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \operatorname{vol}(\Sigma(w+tv)) = &\int_\Sigma \frac{g_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} v)}{\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}}\, d\,\mathrm{V}_{g_w} \\ &- \int_\Sigma v\frac{\dot{g}_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} w)}{2\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}}\, d\,\mathrm{V}_{g_w} \\ &+ \int_\Sigma v\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}\frac{\operatorname{trace}(g_w^{-1}\dot{g}_w)}{2}\, d\,\mathrm{V}_{g_w} \end{aligned} \tag{4.40}$$

By Corollary 2.12, as v is compactly supported, the first term on the right hand side can be re-written as

$$\int_\Sigma \frac{g_w(\operatorname{grad}_{g_w} w, \operatorname{grad}_{g_w} v)}{\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}}\, d\,\mathrm{V}_{g_w} = -\int_\Sigma v \operatorname{div}_{g_w} \frac{\operatorname{grad}_{g_w} w}{\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}}\, d\,\mathrm{V}_{g_w} \tag{4.41}$$

By Lemma 4.2

$$g\left(N, \frac{\partial}{\partial z}\right) = \frac{1}{\sqrt{1+|\operatorname{grad}_{g_w} w|^2_{g_w}}} \tag{4.42}$$

Using the expression in 4.21 and 4.41, we can then rewrite 4.40,

(4.43)

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(\Sigma(w+tv)) = & -\int_\Sigma v \,\mathrm{div}_{g_w}\left(\frac{\mathrm{grad}_{g_w} w}{\sqrt{1+|\,\mathrm{grad}_{g_w} w|^2_{g_w}}}\right) g\left(N, \frac{\partial}{\partial z}\right)\, d\,\mathrm{V}_{\tilde{g}} \\ & -\int_\Sigma v \frac{\dot{g}_w(\mathrm{grad}_{g_w} w, \mathrm{grad}_{g_w} w)}{2\sqrt{1+|\,\mathrm{grad}_{g_w} w|^2_{g_w}}} g\left(N, \frac{\partial}{\partial z}\right)\, d\,\mathrm{V}_{\tilde{g}} \\ & +\int_\Sigma v \sqrt{1+|\,\mathrm{grad}_{g_w} w|^2_{g_w}}\, \frac{\mathrm{trace}(g_w^{-1}\dot{g}_w)}{2} g\left(N, \frac{\partial}{\partial z}\right)\, d\,\mathrm{V}_{\tilde{g}}\end{aligned}$$

Comparing 4.43 with the first variation formula as stated in 3.22, the theorem follows.

□

Remark 4.4. Take $M = \mathbb{R}^m$ endowed with the standard metric and $\Sigma = U \times \{0\}$ for some $U \subseteq R^{m-1}$. Then for a function w defined on Σ, choosing the upward normal, $\Sigma(w)$ is just the graph of w: $\Sigma(w) = \{(x_1, \ldots, x_{m-1}, w) \mid (x_1, \ldots, x_{m-1}) \in U\}$. Meanwhile, $\dot{g}_w$ vanishes identically. Therefore, the mean curvature of such a graph is given by:

$$H = \mathrm{div}\left(\frac{\mathrm{grad}\, w}{\sqrt{1+|\,\mathrm{grad}\, w|^2}}\right) \tag{4.44}$$

The subscripts are omitted as the metric is understood.

Definition 4.5. An immersed Riemannian submanifold $\Sigma \subseteq M$ is said to be *minimal* if the mean curvature of Σ vanishes identically.

Remark 4.6. Using calibration form, it can be determined that an immersed Riemannian submanifold $\Sigma \subseteq M$ with identically vanishing mean curvature is area minimizing, among all compactly supported variations in the same relative homology class.

Example 4.7. Consider the parabola, defined by $\{(x, y, x^2 + y^2) \mid (x, y) \in \mathbb{R}^2\}$. It is a surface. It is the graph of the function $w : (x, y) \mapsto x^2 + y^2$. We can apply 4.44 to compute the mean curvature of the parabola.

$$\mathrm{grad}\, w = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \tag{4.45}$$

$$\sqrt{1+|\,\mathrm{grad}\, w|^2} = \sqrt{1+4x^2+4y^2} \tag{4.46}$$

And so, the mean curvature is given by,

$$H = \mathrm{div}\left(\frac{2x}{\sqrt{1+4x^2+4y^2}}\frac{\partial}{\partial x} + \frac{2y}{\sqrt{1+4x^2+4y^2}}\frac{\partial}{\partial y}\right) \tag{4.47}$$

$$= \frac{4+8x^2+8y^2}{(1+4x^2+4y^2)^{\frac{3}{2}}} \tag{4.48}$$

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