# THE CONNECTIVE CONSTANT ON HEXGONAL LATTICE

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ABSTRACT. The connective constant is an important value, representing the asymptotic growth rate of the total amount of self-avoiding walks starting from a given origin on a plane. This is an expository paper on reproducing the proofs on the value of the connective constant of the hexagonal lattice.

CONTENTS



## 1. INTRODUCTION

A random walk on a graph is a random path that starts from a vertex and traverses to adjacent vertices. A self-avoiding walk is a special type of random walks. We call a path a self-avoiding walk (SAW) of length $n$ if all of the traversed vertices are distinct. The SAW is a model that has lead to rich mathematical theories that found applications in many other domains of statistical physics.[5]

The main topic of this paper is the number of SAWs with length $n$. We show that it grows exponentially fast as $n$ increases, with a specific rate of growth $\mu$ depending on the lattice. The quantity $\mu$ is called the connective constant of the lattice.

The connective constant exists for all lattices while the value of the connective constant depends on the choice of lattice. That is, its value is not universal. Nonetheless, it is an important quantity in the study of two-dimensional statistical physics models. Studying the connective constant may provide clues to possible approaches for attacking other important open problems in the study of self-avoiding walks and scholastic models in general. For example, the Nobel laureate chemist P. Flory proposed self-avoiding walks on a lattice as a model for polymer chains.[1]

In this paper, we will focus on the value and properties of the connective constant on the hexagonal lattice. The value has been predicted non-rigorously by B. Nienhuis in 1982, using Coulomb gas approach from theoretical physics. It was later rigorously proved by Duminil-Copin and Smirnov. We will reproduce the proof later in this paper. The hexagonal lattice is the first instance that an exact value of the connective constant has been proved for a two-dimensional lattice.

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## 2. Basic Definitions and Theorems

Let $G = (V, E)$ be a finite or countable graph. For $z, w \in V$, let us write $z \sim w$ if $\{z, w\} \in E$. In the context of random walks, $G$ is often chosen to be a lattice. By far the most common choice is the integer lattice $\mathbb{Z}^d$. Another notable example in two dimensions is the hexagonal lattice $\mathbb{H}$, which is drawn in Figure 1.

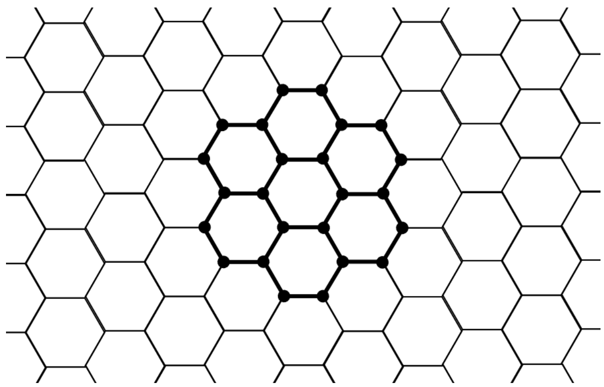

Figure 1. The hexagonal lattice embedded in a plane

**Notation 2.1.** For $A \subset V$, we let $\partial A = \{z \in V \setminus A : z \sim w \text{ for some } w \in A\}$. Also, $\overline{A} = A \cup \partial A$.

**Definition 2.2.** A path $\omega$ of length $n$ in $A$ is a finite sequence of vertices $\omega = [\omega_0, \dots, \omega_n]$ with $\omega_j \in A$ for every $j = 0, \dots, n$.

**Definition 2.3.** A path $\omega = [\omega_0, \dots, \omega_n]$ in $A$ is a self-avoiding walk (SAW) of length $n$ if all of the vertices $\omega_0, \dots, \omega_n$ are distinct, i.e. $\omega_j \neq \omega_k$ for $0 \leq j < k \leq n$.

Specify a vertex $v_r \in V$, which we call the root vertex. If $G = \mathbb{Z}^d$, then we usually take $v_r = 0$.

**Notation 2.4.** Let $\Omega_n$ denote the set of SAWs of length $n$ in $G$ starting at $v_r$. Let $C_n = \#(\Omega_n)$ denote the number of such SAWs.

**Proposition 2.5** (Sub-mutliplicativity)**.** *For any $n, m \geq 0$, $C_{n+m} \leq C_n C_m$.*

*Proof.* Assume $n, m < \infty$. Separate a path $W = [w_0, \dots, w_{m+n}]$ into two adjoining paths $w = [w_0, \dots, w_n]$ and $w' = [w'_{n+1}, \dots, w'_{n+m}]$ at the $n$th vertex.
Then we can establish an injection from $\{W | W \in \Omega_{n+m}\}$ to $\{(w, w') | w \in \Omega_n, w' \in \Omega_m\}$. Suppose $W_1 \neq W_2$. then clearly $(w_1, w'_1) \neq (w_2, w'_2)$. Therefore, $C_{m+n} < C_n C_m$ □

If $n$ is any integer and $d$ is a positive integer, there exist unique integers $q$ and $r$ such that

$$n = dq + r \quad \text{and} \quad 0 \leq r < d. \tag{2.6}$$

**Theorem 2.7.** *(Sub-additivity Lemma) Let $a_1, a_2, \dots$ be a sequence of non-negative real numbers that are sub-additive, i.e.*

$$a_{i+j} \leq a_i + a_j \quad \textit{for all} \quad i, j \geq 1.$$

*Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\geq 1} \frac{a_n}{n}$.*

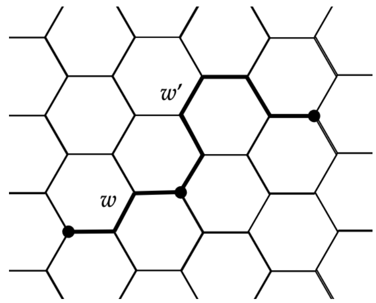


FIGURE 2. An example of separating a path

*Proof.* Since $a_n$ is non-negative, $\frac{a_n}{n} \geq 0$, and the sequence $\frac{a_n}{n}$ is bounded from below by 0. Let $L = \inf_{n\geq 1} \frac{a_n}{n}$. By the definition of infimum, given any $\epsilon > 0$ there exists an integer $n$ such that

$$a_n \leq n(L+\epsilon).$$

Let $b = \max_{1\leq i<n} a_i$. Consider integer $m \geq n$. As in (2.7), we can write $m = qn+r$ with $0 \leq r < q$. By the sub-additivity property,

$$a_{nq+r} = a_{n+n+\cdots+r} \leq a_n + a_n + \cdots + a_n + a_r \leq qa_n + b.$$

Thus

$$\frac{a_m}{m} \leq \frac{qa_n + b}{m} < \frac{qn(L+\epsilon)+b}{m}.$$

Notice that

$$\begin{aligned} \frac{a_m}{m} &\leq \frac{q * a_n + b}{m} \\ &< \frac{qn(L+\epsilon)+b}{m} \\ &= \frac{qn}{m}(L+\epsilon) + \frac{b}{m}. \end{aligned}$$

Since

$$\frac{qn}{m} \to 1 \quad \text{as} \quad m \to \infty,$$

Then

$$\frac{a_m}{m} \to L+\epsilon \quad \text{as} \quad m \to \infty.$$

Since $\epsilon > 0$ was arbitrary,

$$L = \inf_{m\geq 1} \frac{a_m}{m} \leq \liminf_{m\to\infty} \frac{a_m}{m} \leq \limsup_{m\to\infty} \frac{a_m}{m} \leq L.$$

□

From Proposition 2.5, $\log(C_n)$ is a sub-additive sequence. It follows from Theorem 2.7 that

$$\lim_{n\to\infty} \frac{\log(C_n)}{n} = \inf_{n\to\infty} \frac{\log(C_n)}{n} =: \beta$$

**Definition 2.8.** Let $\beta$ be as above. We call $\mu := e^{\beta}$ the connective constant of the graph $G$.

Let us denote this asymptotic relation as $C_n \asymp \exp(n\beta) = \mu^n$.

In general, the value of the connective constant depends on the graph. It is an open problem to find the connective constants of integer lattices. However, for the honeycomb lattice, $\mu$ has been proved to be $\sqrt{2+\sqrt{2}}$ by Duminil-Copin and Smirnov as mentioned in the introduction. In the following section, we will reproduce their proof of this result.

## 3. Self-avoiding walks in the hexagonal lattice

Let us denote the hexagonal lattice as $\mathbb{H}$. Let $H$ be the set of mid-points of the edges of $\mathbb{H}$. Let $\gamma : a \to E$ denote that $\gamma$ is a self-avoiding path that starts at a fixed point $a \in \mathbb{H}$ and ends at some mid-edge in $E \subset H$. If $E = \{b\}$, we shall simply write $\gamma : a \to b$. The length $|\gamma|$ is the number of vertices visited by $\gamma$.

Recall $C_n \asymp \mu^n$. So,

$$\mu = \lim_{n\to\infty} (C_n)^{\frac{1}{n}}.$$

Let

$$Z(x) = \sum_{\gamma : a \to H} x^{|\gamma|} \quad \text{for} \quad x \in (0, \infty). \tag{3.1}$$

By definition, $Z(x)$ includes all finite self-avoiding walks in $H$ with different lengths starting from a fixed point. For SAWs with any fixed length $n$, their aggregated contribution to $Z(x)$ is $C_n x^n$. So we can rewrite $Z(x)$ as

$$Z(x) = \sum_{n=0}^{\infty} C_n x^n.$$

Then, by the root test, the radius of convergence of the above series is

$$\left(\lim_{n\to\infty} (C_n)^{\frac{1}{n}}\right)^{-1} = \mu^{-1}.$$

Thus we only need to show that

$$Z(x_c) = \infty \tag{3.2}$$

and

$$Z(x) < \infty \quad \text{for} \quad 0 < x < x_c \tag{3.3}$$

for

$$x_c := \frac{1}{\sqrt{2+\sqrt{2}}}.$$

To this end, there are some other quantities we need to define.

A SAW $\gamma$ on a hexagonal lattice entering a vertex has possible directions for exit, changing its direction by $\pm\frac{\pi}{3}$ each time. We denote the total turning angle as $\gamma$ traverses from $a \to b$, measured anti-clockwise (in radian), as $T(\gamma)$.

A (hexagonal lattice) domain $M \subset H$ is a union of all mid-edges emanating from a given collection of vertices $S$: a mid-edge $z$ belongs to $M$ if at least one end-point of its associated edge is in $S$, it belongs to $\partial M$ if only one of them is in $S$. In later proofs, we will restrict $M$ to a strip domain $M_{T,L}$ and then let $T, L$ go to infinity.

**Definition 3.4.** The parafermionic observable for $a \in \partial M$ and $z \in M$ is

$$F(z) = \sum_{\gamma\,:\,a\to z} e^{-i\sigma T(\gamma)} x^{|\gamma|}. \tag{3.5}$$

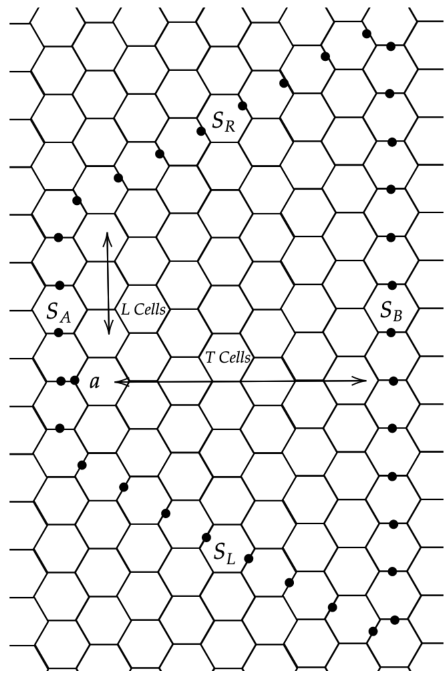


FIGURE 3. The region M

**Lemma 3.6.** *If $x = x_c$ and $\sigma = \frac{5}{8}$, then $F$ satisfies the following relation for every vertex $v \in S$:*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0 \tag{3.7}$$

*where $p, q, r$ are the three mid-points adjacent to $v$.*

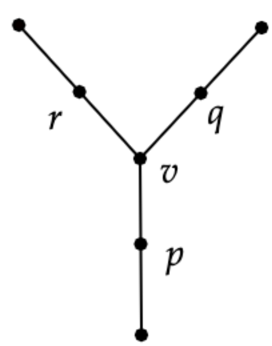


FIGURE 4. The images of $p, q, r, v$.

*Proof.* Let $P_k$ be the set of SAWs, starting at $a$ and ending at $p$, $q$, or $r$ within some bounded region $M \subset H$, whose intersections with the set $\{p, q, r\}$ have $k$ elements (where $k = 1, 2, 3$). We will prove the lemma by showing that the aggregate contribution from $P_1 \cup P_2$ to the left side of (3.7) is zero and similarly for $P_3$.

Consider $\gamma \in P_3$. Let $d_1, d_2, d_3$ be the points $p, q, r$ arranged in the order visited by $\gamma$. By doing so, we can partition the SAW $\gamma$ into three SAWs: $a \to d_1$, $d_1 \to d_2$, and $d_2 \to d_3$. Denote these three paths as $\gamma_1$, $\gamma_2$, and $\gamma_3$, respectively.

Let $\hat{\gamma}_3$ be the path obtained by reversing $\gamma_3$. That is, if $\gamma_3 = [d_2, v_1, v_2, \ldots, v_k, d_3]$ then $\hat{\gamma}_3 = [d_3, v_k, \ldots, v_2, v_1, d_2]$. Now, consider the map $\psi : P_3 \to P_3$ where $\psi(\gamma)$ is obtained by concatenating $\gamma_1$, $[d_1, d_3]$, and $\hat{\gamma}_3$. Note that $\psi$ is bijective and $\psi \circ \psi(\gamma) = \gamma$. Hence, the total contribution of the SAWs in $P_3$ to (3.7) equals $\frac{1}{2}\sum_{\gamma \in P_3} C_\gamma$, where

$$C_\gamma := (d_3(\gamma) - v)e^{-i\sigma T(\gamma)}x^{|\gamma|} + (d_3(\psi(\gamma)) - v)e^{-i\sigma T(\psi(\gamma))}x^{|\psi(\gamma)|}.$$

Let us assume $d_1(\gamma) = p$, $d_2(\gamma) = q$, and $d_3(\gamma) = r$. We then have

$$\begin{aligned} C_\gamma &= (r - v)e^{-i\sigma T(\gamma)}x^{|\gamma|} + (q - v)e^{-i\sigma T(\psi(\gamma))}x^{|\psi(\gamma)|} \\ &= (r - v)e^{-i\sigma (T(\gamma_1) + \frac{4\pi}{3})}x^{|\gamma|} + (q - v)e^{-i\sigma (T(\gamma_1) - \frac{4\pi}{3})}x^{|\gamma|} \\ &= (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma|}\left(\overline{\theta}e^{-i\sigma\frac{4\pi}{3}} + \theta e^{i\sigma\frac{4\pi}{3}}\right) \\ &= c\left(\overline{\theta}e^{-i\sigma\frac{4\pi}{3}} + \theta e^{i\sigma\frac{4\pi}{3}}\right) \end{aligned} \tag{3.8}$$

where

$$c = (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma|}$$

and

$$\theta = \frac{q - v}{p - v} = \exp(2\pi i/3).$$

The contents in the parenthesis of (3.8) is equal to $2\cos(\frac{2}{3}(2\sigma + 1))$, which is 0 when $\sigma = \frac{5}{8}$. Similar calculations give $C_\gamma = 0$ when $\sigma = \frac{5}{8}$ for every $\gamma \in P_3$.

Now consider $P_1 \cup P_2$. For $\gamma \in P_1$, let $d_1(\gamma)$ be the unique point in $\{p, q, r\}$ visited by $\gamma$. (This is the terminal point of $\gamma$.) Let $d_2(\gamma)$ (resp. $d_3(\gamma)$) be the point in $\{p, q, r\}$ that is reached by taking a left (resp. right) turn after heading from $d_1(\gamma)$ to $v$. Let $\eta^2_\gamma \in P_2$ (resp. $\eta^3_\gamma \in P_2$) be the SAW obtained by extending $\gamma$ to $d_2(\gamma)$ (resp. $d_3(\gamma)$). Let us assume $d_1(\gamma) = p$, $d_2(\gamma) = q$, and $d_3(\gamma) = r$. Their aggregate contribution to the left side is

$$\begin{aligned} C_\gamma &= (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma_1|} + (q - v)e^{-i\sigma T(\eta^2_\gamma)}x^{|\eta^2_\gamma|} + (r - v)e^{-i\sigma T(\eta^3_\gamma)}x^{|\eta^3_\gamma|} \\ &= (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma_1|} + (q - v)e^{-i\sigma T(\gamma_1 + \pi/3)}x^{|\gamma_1|+1} + (r - v)e^{-i\sigma T(\gamma_1 - \pi/3)}x^{|\gamma_1|+1} \\ &= (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma_1|} + (p - v)\theta e^{-i\sigma T(\gamma_1 + \pi/3)}x^{|\gamma_1|+1} + (p - v)\overline{\theta}e^{-i\sigma T(\gamma_1 - \pi/3)}x^{|\gamma_1|+1} \\ &= (p - v)e^{-i\sigma T(\gamma_1)}x^{|\gamma_1|}(1 + x\theta e^{i\sigma\frac{\pi}{3}}) \\ &= c(1 + x\theta e^{i\sigma\frac{\pi}{3}} + x\overline{\theta}e^{-i\sigma\frac{\pi}{3}}) \end{aligned} \tag{3.9}$$

where

$$c = (p - v)e^{-i\sigma t(\gamma_1)x^{l(\gamma_1)}}$$

and

$$\theta = \frac{q - v}{p - v} = \exp(2\pi i/3).$$

Each of these is zero when $\sigma = \frac{5}{8}$, whence the entire LHS of (3.7) is zero for this value of $\sigma$. So we can find $x_c = \frac{1}{2\cos(\pi/8)}$. □

We will now use Lemma 3.6 to show (3.2) and (3.3).

Consider $M = M_{T,L}$, which is the hexagonal lattice cut at heights $\pm L$ at angles $\frac{\pi}{3}$ as in Figure 3:

Let $S_A, S_B, S_L, S_R$ be the set of mid-points on the boundaries and $a \in S_A$ be some point on $S_A$. Let $S_A$ be the be the upper boundary, $S_B$ be the lower one, $S_L$ be the left one, $S_R$ be the right one. Fix $a$, and define the following functions as the sum of all SAWs from $a$ to any point in $S_A, S_B, S_L, S_R$ except $a$ itself.

$$A_{T,L} := \sum_{\gamma\,:\,a\to S_A\setminus\{a\}} x^{|\gamma|} \quad \text{where} \quad T(\gamma) = \pm\pi$$

$$B_{T,L} = \sum_{\gamma\,:\,a\to S_B} x^{|\gamma|} \quad \text{where} \quad T(\gamma) = 0$$

$$C_{T,L} = \sum_{\gamma\,:\,a\to S_L\cup S_R} x^{|\gamma|} \quad \text{where} \quad T(\gamma) = \mp\frac{2\pi}{3}$$

$$D_{T,L} = \sum_{\gamma\,:\,a\to S_L} x^{|\gamma|} \quad \text{where} \quad T(\gamma) = -\frac{2\pi}{3}$$

$$E_{T,L} = \sum_{\gamma\,:\,a\to S_R} x^{|\gamma|} \quad \text{where} \quad T(\gamma) = +\frac{2\pi}{3}$$

For an internal vertex $v$ in $M$, let $f(v)$ be the left hand side of (3.7) with $p, q, r$ as the vertices incident to $v$ and $x = x_c$, $\sigma = \frac{5}{8}$. On one hand, by Lemma 3.6, the sum of $f(v)$ over all vertices $v \in M$ equals 0. On the other hand, the terms coming from the internal vertices cancel each other, so that the sum is over SAWs terminating on the boundary. That is,

$$\begin{aligned} 0 &= \sum_{v\in V(M)} f(v) \\ &= -\sum_{w\in S_A} F(w) + \sum_{w\in S_B} F(w) + \theta \sum_{w\in S_L} F(w) + \overline{\theta} \sum_{w\in S_R} F(w) \qquad (3.10)\\ &= -iF(\alpha) - i\,\mathrm{Re}(e^{i\sigma\pi})A_{T,L} + iB_{T,L} + i\theta e^{-i\sigma\frac{2\pi}{3}} D_{T,L} + i\overline{\theta} e^{i\sigma\frac{2\pi}{3}} E_{T,L} \end{aligned}$$

where

$$\theta = \exp(2\pi i/3).$$

By the left-right symmetry of the domain, $F(\overline{w}) = F(w)$, where $\overline{w}$ is the mirror image of $w$ flipped with respect to the middle vertical line.

Observe that the winding equals $\pi$ if it ends on the top boundary on the right side of $a$, $-\pi$ if on the left side. The winding of any path ending in the bottom boundary is 0. From $S_A$ to $S_R$, the winding is $4\pi/3$. $S_A$ to $S_L$: winding is $-4\pi/3$.

Therefore

$$\begin{aligned}
\sum_{w\in S_A} F(w) &= \sum_{w\in S_A\setminus a} F(w) + F(a) \\
&= F(a) + \frac{1}{2}\sum_{x\in S_A\setminus\{a\}} (F(w)+F(\overline{w})) \\
&= 1 + \frac{e^{-i\sigma\pi}+e^{i\sigma\pi}}{2} A_{T,L} \\
&= 1 + (\cos\sigma)\, A_{T,L} \\
&= 1 - \alpha A_{T,L} \quad \text{where} \quad \alpha = \cos(3\pi/8).
\end{aligned} \tag{3.11}$$

Notice that the only SAW from $a$ to $a$ is a path of length 0, whence $F(a) = 1$. Therefore,

$$\sum_{w\in U} F(w) = 1 - \left(\cos\frac{3\pi}{8}\right) A_{T,L},$$

$$\sum_{w\in L} F(w) = B_{T,L},$$

$$\theta \sum_{w\in S_L} F(w) + \overline{\theta}\sum_{w\in S_R} F(w) = \left(\cos\frac{\pi}{4}\right) C_{T,L}.$$

Let $\beta = \cos\frac{\pi}{4}$. We then have

$$1 = \alpha A_{T,L} + B_{T,L} + \beta C_{T,L}. \tag{3.12}$$

Notice that $A_{T,L}$ and $B_{T,L}$ increase as $L$ increases. Furthermore, since $A_{T,L}$, $B_{T,L}$, and $C_{T,L}$ are all positive when $x = x_c$, from (3.12) we see that

$$A_T := \lim_{L\to\infty} A_{T,L} \quad \text{and} \quad B_T := \lim_{L\to\infty} B_{T,L}$$

exist. Since $A_{T,L}$ and $B_{T,L}$ increase and $C_{T,L}$ decreases monotonically as $L$ increases,

$$C_T := \lim_{L\to\infty} C_{T,L}$$

exists. Clearly,

$$\alpha A_T + B_T + \beta C_T = 1. \tag{3.13}$$

With (3.13), we can prove (3.2) and (3.3). We start by proving the former, that $Z(x_c) = \infty$. Suppose first that for some $T$, $C_T > 0$. We have shown that $C_{T,L}$ decreases monotonically in $L$, so $C_{T,L} \geq C_T$ for all $L > 0$. Then

$$Z(x_c) \geq \sum_{L>0}^{\infty} C_{T,L} \geq \sum_{L>0}^{\infty} C_T = \infty.$$

Now suppose that $C_T = 0$ for all $T$. Then, by (3.13), we have

$$\alpha A_T + B_T = 1$$

for all $T$.

Observe that a walk $\gamma \in A_{T+1}$, where $\gamma \notin A_T$, has to visit some vertex adjacent to the right edge of $D_{T+1}$. So the difference between $A_{T+1}$ and $A_T$ is the sum of all such $\gamma$ lie on $M_{T+1}$. Because one must add at least one edge between the two portions pulled up from $B_{T+1}$,

$$A_{T+1} - A_T \leq \frac{1}{x_c}(B_{T+1})^2.$$

Then, by combining the above equations, we have

$$\begin{aligned} 0 &= (\alpha A_{T+1} + B_{T+1}) - (\alpha A_T + B_T) \\ &= \alpha(A_{T+1} - A_T) + (B_{T+1} - B_T) \\ &\leq \alpha x_c (B_{T+1})^2 + B_{T+1} - B_T. \end{aligned} \tag{3.14}$$

It follows that

$$\frac{x}{x_c}(B_{T+1})^2 + B_{T+1} \geq B_T.$$

By induction on $T$,

$$B_T \geq \frac{\min(B_1, x_c/\alpha)}{T}.$$

Therefore,

$$Z(x) \geq \sum_{T>0} B_T = \infty.$$

By the above we have proven $\mu \geq \sqrt{2+\sqrt{2}}$ and hence proved (3.2).

Let us now show (3.3). We will use the following concept to decompose self-avoiding walks into two parts. A bridge of width $T$ is a self-avoiding walk in $M_T$ between midpoints of edges.

A SAW between two midpoints $c, d$ is a bridge if

(1) $c, d$ lies on horizontal edges (as shown in the graph below),
(2) $c$ is the lowest midpoint on $\gamma$ and $d$ is the highest.

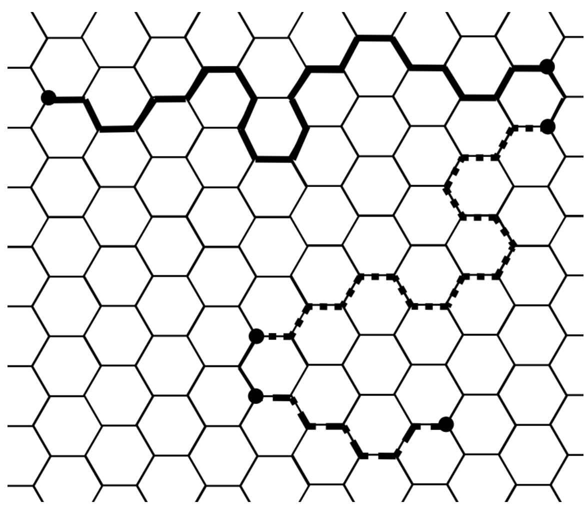

FIGURE 5. An example of decomposing SAWs

Noting that a bridge of width $T$ has length at least $T$ , we obtain for $x < x_c$,

$$B_T^x \leq \left(\frac{x}{x_c}\right)^T B_T^{x_c} \leq \left(\frac{x}{x_c}\right)^T.$$

Note that for $y > 0$, $\log(1+y) < y$. Hence,

$$\log Z(x) \le \log 2 + 2\sum_{T=1}^{\infty} \log(1+B_T)$$
$$\le \log 2 + 2\sum_{T=1}^{\infty} B_T \le \log 2 + 2\sum_{T=1}^{\infty}\left(\frac{x}{x_c}\right)^T < \infty \quad \text{for} \quad 0 < x < x_c.$$

Therefore, $Z(x) < \infty$ whenever $0 < x < x_c$. This implies that the connective constant $\mu$ is smaller than $\frac{1}{x_c} = \sqrt{2+\sqrt{2}}$.

In the above proof, we have assumed the fact that any SAW can be uniquely decomposed into a sequence of bridges of widths $T_{-i} < \cdots < T_{-1}$ and $T_0 > \cdots > T_j$ if one fixes a starting mid-edge and the first vertex. So now, it only remains to prove this.

First assume that SAW $\gamma$ is in a half plane $M_{\infty,\infty}$, that the vertices of $\gamma$ are strictly on one side of the starting point $a$. Without loss of generality, assume that all the vertices are on the right of $a$. Assume $\gamma$ itself is not a bridge. Denote its highest left-most vertex as $v$, and the mid-edge left to $v$ as $c$. Let $\gamma_1$ be the bridge from the starting point $a$ to $c$. Denote the earliest horizontal mid-edge after $v$ as $c'$. And let $\gamma_2$ be the SAW from $c'$ to the end point $b$.

Define the width of a SAW $\gamma$ as the number of complete horizontal edges $\gamma$ traversed plus or minus one depending on its direction.

Notice that by our definition, the width of $\gamma_1$ is strictly larger than that of $\gamma_2$. Repeat the process on sub-SAW $\gamma_2$, until we reach the end point $b$, or its earliest horizontal mid-edge. We have a ordered set of bridges with width $T_0 > \cdots > T_j$.

We have shown that a half-plane walk can be decomposed into a ordered set of bridges. So the rest to prove is every SAW can be decomposed into a combination of half-plane walks. Now assume that SAW $\gamma$ is not a half-plane walk. Denote its earliest left-most vertex as $v$, and the mid-edge left to $v$ as $c$. Let $\gamma_1$ be the bridge from the starting point $a$ to $c$. Denote the earliest horizontal mid-edge after $v$ as $c'$. Let $\gamma_2$ be the SAW from $c'$ to the end point $b$. By doing so, we obtain half-plane walks $\gamma_1$ and $\gamma_2$. By decomposing $\gamma_1$ and $\gamma_2$, we decompose $\gamma$ into a sequence of bridges of widths $T_{-i} < \cdots < T_{-1}$ and $T_0 > \cdots > Tj$. So we have completed the proof.

## 4. Acknowledgements

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