MIRROR SYMMETRY OF ELLIPTIC CURVES

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Abstract. Following Polishcuk and Zaslow's paper [7] and Kreussler's paper [3], we construct an equivalence of additive categories: $\phi_\tau : D^b(\mathrm{Coh}(E_\tau)) \to \mathcal{FK}^0(E^\tau)$. Here τ in the upper half plane serves as the lattice parameter when defining E_τ and as the complexified Kähler form when defining E^τ. These two manifolds E_τ and E^τ are called mirror manifolds, and the equivalence ϕ_τ turns out to be a preliminary example of so-called mirror symmetry. Examples of mirror symmetry tend to be quite demanding to understand in general, so we hope that this paper could serve as a stepping-stone for those who want to explore the wonders of mirror symmetry.

Contents

1. Introduction

The main goal of this paper is to construct a functor $\phi_\tau : D^b(\mathrm{Coh}(E_\tau)) \to \mathcal{FK}^0(E^\tau)$ that is an equivalence of additive categories and is compatible with the shift functors. First, we introduce elliptic curves in this section. Then, we discuss about $D^b(\mathrm{Coh}(E_\tau))$ in section 2. Theorem 2.1 tells us the structure of a coherent sheaf on an elliptic curve, and Theorem 2.6 gives the structure of $D^b(\mathrm{Coh}(E_\tau))$. Since locally free sheaves come from vector bundles, we discuss about vector bundles and introduce theta functions in the remaining part of section 2. In section 3, we first define a general A_∞-category. Then we show how to get a real category from it. After that, by adding formal finite direct sums, we get the desired abelian category $\mathcal{FK}^0(E^\tau)$ from the **Ab**-category $\mathcal{F}^0(E^\tau)$. Finally, in the last section, we construct the equivalence ϕ_τ by first working on vector bundles of the form $L(\phi) \otimes F(V, \exp(N))$ and then expanding the discussion to arbitrary locally free sheaves.

When we talk about an elliptic curve, we mean a Riemann surface of genus one with a chosen base point. There are also other ways to describe an elliptic curve. For those who prefer an algebraic viewpoint, we can define it by an equation

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$y^2 = x^3 + Ax + B$, where A and B are two complex constants. By embedding $\mathbb{C}^2$ into $\mathbb{CP}^2$, one finds that an elliptic curve is always projective. Meanwhile, for those who are interested in topology, we know that an elliptic curve is always a complex torus $\mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$.

The equivalence between these descriptions of an elliptic curve is well explained in Robert's book [9], which is an excellent reference for those who are interested in elliptic curves.

2. Bounded derived category of Coherent sheaves

The definition of coherent sheaves and thickened skyscraper sheaves and some basic properties can be found in any standard reference of algebraic geometry. In particular, the reader is invited to consult Hartshorne's book [2] when he or she comes across problems while reading this section.

First, we will discuss the decomposition of a coherent sheaf on an elliptic curve.

Theorem 2.1. *Let X be an elliptic curve, and $\mathcal{F}$ be a coherent sheaf on X. Then there exists a decomposition $\mathcal{F} = \mathcal{F}_{tor} \oplus \mathcal{G}$, where the torsion part $\mathcal{F}_{tor}$ is a direct sum of thickened skyscraper sheaves and $\mathcal{G}$ is locally free (a vector bundle).*

Proof. Suppose $U \cong \mathrm{Spec}(A)$ is an open affine subset of X. Since $\mathcal{F}$ is a coherent sheaf, we know that $\mathcal{F}|_U \cong \widetilde{M}$ for some finitely generated A-module M. Since X is an elliptic curve, A is a Dedekind domain. So by the structure theorem of a finitely generated module over a Dedekind domain, M can be decomposed as

$$M \cong A/\mathfrak{p}_1^{a_1} \oplus A/\mathfrak{p}_2^{a_2} \oplus ... \oplus A/\mathfrak{p}_r^{a_r} \oplus P,$$

where $\mathfrak{p}_i$ are prime ideals of A and P is a projective A-module. So the associated sheaf has a similar decomposition

$$\widetilde{M} \cong \widetilde{A/\mathfrak{p}_1^{a_1}} \oplus \widetilde{A/\mathfrak{p}_2^{a_2}} \oplus ... \oplus \widetilde{A/\mathfrak{p}_r^{a_r}} \oplus \widetilde{P}.$$

Let $p,\ q$ be two distinct points in X, and assume that they correspond to prime ideals $\mathfrak{p},\ \mathfrak{q}$ of A respectively. Then for the sheaf $\widetilde{A/\mathfrak{p}^a}$, its stalk at point q is the localization ring $(A/\mathfrak{p}^a)_\mathfrak{q}$. Since A is a Dedekind domain, its Krull dimension is 1, i.e., every non-zero prime ideal of A is maximal. Thus, there exists an element $a \in \mathfrak{p} - \mathfrak{q}$. Then $a \notin \mathfrak{q}$ tells us that a is invertible in $(A/\mathfrak{p}^n)_\mathfrak{q}$. Meanwhile, $a \in \mathfrak{p}$ tells us that a is nilpotent in $(A/\mathfrak{p}^n)_\mathfrak{q}$. Thus, $(A/\mathfrak{p}^n)_\mathfrak{q}$ is equal to 0. Meanwhile, when $q = p$, we have $(A/\mathfrak{p}^n)_\mathfrak{p} \cong (A/\mathfrak{p})^n$, where $A/\mathfrak{p}$ is a field since $\mathfrak{p}$ is a maximal ideal. Therefore, $\widetilde{A/\mathfrak{p}^n}$ is equal to a thickened skyscraper sheaf.

Next, I will prove that $\widetilde{P}$ is a locally free sheaf. In fact, we know that there exists a positive integer r such that $P_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of rank r for every prime ideal $\mathfrak{p}$ of A. Assume that $\mathfrak{p}$ corresponds to the point $p \in U$. Then the stalk of $\widetilde{P}$ at p is isomorphic to r copies of the stalk $\mathcal{O}_{X,p}$. This isomorphism gives us r elements in the stalk of $\widetilde{P}$ at p. By the definition of stalk, these r elements corresponds to r sections defined over an open neighborhood V of p in U. So there is a natural map $\varphi : (\mathcal{O}_X|_V)^r \to \widetilde{P}|_V$, which becomes an isomorphism when restricted to the stalk at p. Now, consider the kernel and cokernel of φ. They are both coherent sheaves with zero stalks at p. However, a coherent sheaf must be supported on a closed subset. Thus, by shrinking V if necessary, we can assume that the kernel and cokernel of φ are zero. Therefore, φ is an isomorphism, and $\widetilde{P}$ is locally free.

What we have proved above is that over every open affine subset $U \cong \operatorname{Spec}(A)$, $\mathcal{F}$ can be decomposed as a direct sum of thickened scyscraper sheaves and a locally free sheaf. Since a thickened scyscraper sheaf is only supported at one point, it can be extended to global skyscraper sheaf defined on X. Then, we define the torsion part $\mathcal{F}_{tor}$ of $\mathcal{F}$ to be the direct sum of these global skyscraper sheaves. Clearly, $\mathcal{F}_{tor}$ is a subsheaf of $\mathcal{F}$, and the quotient sheaf $\mathcal{G} := \mathcal{F}/\mathcal{F}_{tor}$ is locally free. The last part is to see that $\mathcal{F}$ is globally the direct sum of its torsion part $\mathcal{F}_{tor}$ and the locally free sheaf $\mathcal{G}$, i.e., the following short exact sequence splits:

$$0 \longrightarrow \mathcal{F}_{tor} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since an elliptic curve is always quasi-compact, $\mathcal{F}_{tor}|_V$ is supported at only finitely many points in V. Thus $\mathcal{F}_{tor}$ is supported at finitely many points of X, denoted by $\Lambda \subseteq X$. Then, we take any point $p \in X - \Lambda$. Since the stalk of $\mathcal{F}_{tor}$ at p is zero, it is sufficient to prove that the short exact sequence splits over $X - \{p\}$. Now, using the fact that $X - \{p\}$ is open affine and the structure theorem of finitely generated modules over a Dedekind domain, we see that the short exact sequence over $X - \{p\}$ must split. $\square$

Now we will focus on the bounded derived category $D^b(\operatorname{Coh}(X))$ of coherent sheaves on X.

Theorem 2.2 (Global version of Serre theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k admits an n-step resolution ... $\to 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to ... \to \mathcal{F}_1 \to \mathcal{F}_0$ where each $\mathcal{F}_i$ is finitely generated and locally free (thus they come from vector bundles).*

Proof. By Corollary 5.18 of Hartshorne's book [2], we know that for any coherent sheaf $\mathcal{G}$, there exists a locally free sheaf $\mathcal{E}$ and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{G}$. First, we take $\mathcal{G} = \mathcal{F}$ and get a surjection $\mathcal{P}_0 \xrightarrow{d_0} \mathcal{F} \to 0$, where $\mathcal{P}_0$ is locally free. Then we take $\mathcal{G} = \ker(d_0)$ and get a surjection $\mathcal{P}_1 \twoheadrightarrow \ker(d_0)$, where $\mathcal{P}_1$ is also locally free. We compose this map with the embedding $\ker(d_0) \hookrightarrow P_0$ and get an exact sequence $\mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{d_0} \mathcal{F}$. Repeat this procedure, and eventually we will obtain a resolution of $\mathcal{F}$ by locally free sheaves:

$$\cdots \longrightarrow \mathcal{P}_{n+1} \xrightarrow{d_{n+1}} \mathcal{P}_n \xrightarrow{d_n} \mathcal{P}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \longrightarrow \mathcal{F}.$$

We replace $\mathcal{P}_n$ by $\mathcal{P}_n/\operatorname{im}(d_{n+1})$, and get an n-step resolution of $\mathcal{F}$:

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{P}_n/\operatorname{im}(d_{n+1}) \xrightarrow{d_n} \mathcal{P}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \longrightarrow \mathcal{F}.$$

Now, we only need to show that $\mathcal{P}_n/\operatorname{im}(d_{n+1})$ is still a locally free sheaf. Obviously, it is enough to prove this locally at every point. So we can assume that the underlying space is affine. Assume that $X = \operatorname{Spec}(A)$, where A is a regular ring of dimension n. Since there exists a bijection between finitely generated projective modules and locally free coherent sheaves, we can view $\mathcal{P}_i$ as projective A-modules P_i, and we have to prove that $P_n/\operatorname{im} d_{n+1}$ is still projective. We assume that $\mathcal{F}$ corresponds to a A-module M. Since A is a regular ring of dimension n, we know that the projective dimension of M is not greater than n. Therefore $\operatorname{Ext}^{n+k}(M, N) = 0$ for any positive integer k and any A-module N. On the other hand, notice that

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n/\operatorname{im}(d_{n+1}) \longrightarrow 0$$

is a projective resolution of $P_n/\operatorname{im}(d_{n+1})$, one gets

$$\operatorname{Ext}^{n+k}(M,N) = H^{n+k}(\operatorname{Hom}(P_\bullet,N)) = \operatorname{Ext}^k(P_n/\operatorname{im}(d_{n+1}),N).$$

Thus, $\operatorname{Ext}^k(P_n/\operatorname{im}(d_{n+1}),N) = 0$ for any positive integer k and any A-module N, and $P_n/\operatorname{im}(d_{n+1})$ is projective. □

Definition 2.3. An abelian category C is called *hereditary* if $\operatorname{Ext}^2(-,-) = 0$.

Proposition 2.4. *The category* $\operatorname{Coh}(X)$ *of coherent sheaves on* X *is hereditary.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves on X. Since X is a smooth projective variety of dimension 1 over the field $\mathbb{C}$, $\mathcal{F}$ admits a 1-step resolution of locally free sheaves $... \to 0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}$ by Theorem 2.2. And the sheaf $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ is the i-th cohomology of the complex

$$0 \to \mathcal{H}om(\mathcal{F}_0,\mathcal{G}) \to \mathcal{H}om(\mathcal{F}_1,\mathcal{G}) \to 0 \to ...$$

Therefore, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$ for $i > 1$.

Now, to show that $\operatorname{Coh}(X)$ is hereditary, we use the local-to-global spectral sequence to compute $\operatorname{Ext}^2(\mathcal{F},\mathcal{G})$. In fact, we have the following result:

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F},\mathcal{G})) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F},\mathcal{G}).$$

By Theorem 2.1, every coherent sheave on X can be decomposed into a direct sum of thickened skyscraper sheaves and a locally free sheaf. Thus, we can assume that $\mathcal{F}$ and $\mathcal{G}$ are thickened skyscraper sheaves or locally free sheaves.

When $\mathcal{F}$ is locally free, $\mathcal{F}$ itself is a 0-step resolution of $\mathcal{F}$. Therefore all sheaves $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ are zero for $i > 0$. Then the spectral sequence is stable at $E_2^{p,q}$, and $\operatorname{Ext}^i(\mathcal{F},\mathcal{G}) = H^i(X, \mathcal{H}om(\mathcal{F},\mathcal{G}))$. Moreover, by Grothendieck's vanishing theorem, $H^i(X, \mathcal{H}om(\mathcal{F},\mathcal{G})) = 0$ for $i > 1$.

When $\mathcal{F}$ is a thickened skyscraper sheaf, $\mathcal{F}$ is supported at only one point $p \in X$. Now, by $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})_x \cong \operatorname{Ext}^i(\mathcal{F}_x,\mathcal{G}_x)$, we know that the sheaves $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})$ are also supported at the point p, i.e., they are again thickened skyscraper sheaves. Notice that thickened skyscraper sheaves are automatically flasque, we know that they have no higher cohomologies. So the spectral sequence is again stable at $E_2^{p,q}$, and $\operatorname{Ext}^i(\mathcal{F},\mathcal{G}) = H^0(X, \mathcal{E}xt^i(\mathcal{F},\mathcal{G})) = \Gamma(X, \mathcal{E}xt^i(\mathcal{F},\mathcal{G}))$. When $i > 1$, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$, and thus $\operatorname{Ext}^i(\mathcal{F},\mathcal{G}) = 0$. □

By Theorem 4.1 of Pakharev's paper [6], we have the following result.

Theorem 2.5. *Suppose that* C *is a hereditary abelian category. Then any object* $L \in D^b(C)$ *is isomorphic to the sum of its cohomologies, i.e.,* $L \cong \oplus_i H^i L[-i]$*. Here* $\mathcal{F}[-i]$ *denotes the complex with the only non-zero term (equal to* $\mathcal{F}$*) in degree* i*.*

Theorem 2.6. *Let* X *be a complex projective curve (certainly an elliptic curve is one such example). Then every object of* $D^b(\operatorname{Coh}(X))$ *is isomorphic to the direct sum of objects of the form* $\mathcal{F}[n]$*, where* $\mathcal{F}$ *is a coherent sheaf on* X *and* $\mathcal{F}[n]$ *denotes the complex with the only non-zero term (equal to* $\mathcal{F}$*) in degree* $-n$*.*

Proof. By Proposition 2.4, $\operatorname{Coh}(X)$ is hereditary. Therefore, for any object $L \in D^b(\operatorname{Coh}(X))$ we can use Theorem 2.5 and get $L \cong \oplus_i H^i L[-i]$, which is the desired expression. □

Remark 2.7. Combining Theorems 2.1 and 2.6, we know that every object of $D^b(\mathrm{Coh}(X))$ is a direct sum of objects of the form $\mathcal{F}[n]$, where $\mathcal{F}$ is a vector bundle or has support at a point (a thickened skyscraper sheaf).

Now, to compute the morphism spaces, we can use a version of Serre Duality that appears as Lemma 2.7 in Kreussler's paper [3]. It says that we have a functional isomorphism

$$\mathrm{Ext}^1(A_1, A_2) \cong \mathrm{Hom}(A_2, A_1)^*.$$

Next, we will discuss vector bundles on an elliptic curve.

We consider an elliptic curve E as a complex torus $E = \mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$. Clearly, Γ has 2 generators which are linearly independent over $\mathbb{R}$. By rescaling, one of the generators can be taken to be $1 \in \mathbb{R}$, while the other is denoted by τ. We denote E_τ to be the quotient space $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. Then, the exponential map $z \mapsto e^{2\pi i z}$ gives an isomorphism between $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ and $\mathbb{C}^*/\sim$, where $u \sim qu$ and $q = e^{2\pi i \tau}$. We use E_q to denote the quotient space $E_q = \mathbb{C}^*/\sim$, and we have $E_q \cong E_\tau$. We will use E_τ and E_q indiscriminately in the following text. We denote $\pi' : \mathbb{C}^* \to E_\tau$ to be the composition of the quotient map $\mathbb{C}^* \to E_q$ and the isomorphism $E_q \cong E_\tau$.

Now, we consider the following short exact sequence of sheaves over $\mathbb{C}^*$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

It induces a long exact sequence:

$$\cdots \to H^1(\mathbb{C}^*, \mathcal{O}) \to H^1(\mathbb{C}^*, \mathcal{O}^*) \to H^2(\mathbb{C}^*, \mathbb{Z}) \to H^2(\mathbb{C}^*, \mathcal{O}) \to \cdots$$

Since $H^1(\mathbb{C}^*, \mathcal{O}) = H^2(\mathbb{C}^*, \mathcal{O}) = 0$, it induces an isomorphism $\mathrm{Pic}(X) \cong H^1(\mathbb{C}^*, \mathcal{O}^*)$ $\cong H^2(\mathbb{C}^*, \mathbb{Z})$, which is exactly the definition of the first Chern class of a complex line bundle. Therefore, a line bundle on $\mathbb{C}^*$ is determined by its first Chern class. But since $\mathbb{C}^*$ is homotopic to S^1 as topological spaces, we have $H^2(\mathbb{C}^*, \mathbb{Z}) \cong H^2(S^1, \mathbb{Z}) = 0$. Thus, all line bundles over $\mathbb{C}^*$ are trivial. In particular, the pull-back of any line bundle L over E_q is trivial over $\mathbb{C}^*$.

To figure out what is happening to general vector bundles, we first prove the following lemma.

Lemma 2.8. *Every vector bundle on an elliptic curve X is obtained as a successive extensions of line bundles.*

Proof. Let $\pi : E \to X$ be a vector bundle on X. Then we consider the associated projective bundle $\pi' : P(E) \to X$. By the definition of vector bundles, there exists an open subset $U \subseteq X$ such that $E|_U$ is trivial. Moreover, since the set of open affine subsets form a topological basis of X, we can take U to be open affine. Since $E|_U$ is trivial, we can take a non-vanishing section $s : U \to E|_U$. This section induces a section of $P(E)$ over U because it is non-vanishing. We denote this induced section by $s' : U \to P(E)$. Since every map from a nonempty open locus in a complete nonsingular curve to a complete variety can be uniquely extended to a regular morphism from the entire curve, we can extend the map s' to a global map $s'' : X \to P(E)$. Now, we consider the composition of s'' and π'. It is a map from E to itself which restricts to the identity over U. Thus, the composition map can be viewed as the extension of the embedding $U \hookrightarrow X$. Using the uniqueness part of the extending theorem used before, we know that the composition map has to be the identity over X. Therefore s'' is a section of $P(E)$ and defines a 1-dimensional subbundle L of E. we quotient E by this subbundle L, and proceed in the same way for E/L. Finally, we get a filtration $0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_r = E$, such that every $L_i = E_i/E_{i-1}$ is a line bundle. This is exactly what the lemma is asking for. $\square$

Recalling that any line bundle on E_q pulls back to the trivial line bundle over $\mathbb{C}^*$, and using Lemma 2.8, we obtain the following proposition.

Proposition 2.9. *The pull-back of every vector bundle on E_q to $\mathbb{C}^*$ is trivial.*

Thus, all vector bundles on E are obtained from gluing the fibers over u and qu in $\mathbb{C}^*$. We denote such a gluing by a holomorphic map $A : \mathbb{C}^* \longrightarrow \mathrm{GL}(V)$, such that the fibers over u and qu are glued by the map $A(u) : V \longrightarrow V$. To be specific, we define the rank r holomorphic vector bundle $F_q(V, A)$ on E by taking the quotient

$$F_q(V, A) = \mathbb{C}^* \times V/(u, v) \sim (qu, A(u) \cdot v).$$

Since we can change the trivialization of every fiber on $\mathbb{C}^*$ by an element in $\mathrm{GL}(V)$, we have $F_q(V, A) \cong F_q(V, \widetilde{A})$ if $\widetilde{A}(u) = B(qu)A(u)B(u)^{-1}$ for some map $B : \mathbb{C}^* \to \mathrm{GL}(V)$.

When $V = \mathbb{C}$ and $A = \varphi$ is a holomorphic function, we denote $L_q(\varphi)$ to be the line bundle constructed in this way. We define $L \equiv L_q(\varphi_0)$ where $\varphi_0(u) = \exp(-\pi i\tau - 2\pi iz) = q^{-\frac{1}{2}}u^{-1}$.

Following chapter I of Robert's book [9], we can define a theta function of type (h, a), where h and a are maps from Γ to $\mathbb{C}$.

Definition 2.10. A *theta function* θ of type (h, a) with respect to Γ is a meromorphic function on the complex line $\mathbb{C}$ satisfying

$$\theta(z + \gamma) \ = \ a(\gamma)e^{\pi h(\gamma)(z+\frac{1}{2}\gamma)}\theta(z), \ \forall\gamma \in \Gamma.$$

In particular, when $a(x + y\tau) = e^{xy\pi i}$ and $h(x + y\tau) = -2iy$, θ is the unique theta function satisfying $\theta(z + 1) = \theta(z)$ and $\theta(z + \tau) = e^{-\pi i(z+2\tau)}\theta(z)$. We call this θ the *classical theta function* or *Jacobi theta function*. The first equation tells us that θ factors through $\exp : \mathbb{C} \to \mathbb{C}^*$, i.e., $\theta(z) = f(e^{2\pi iz})$ for some holomorphic function f on $\mathbb{C}^*$. Now we use u to denote the coordinate on $\mathbb{C}^*$. Then the second equation of θ translates to $f(qu) = q^{-\frac{1}{2}}u^{-1}f(u)$. Therefore f can be viewed as a section of the line bundle $L = L_q(\varphi_0)$ defined above.

Since the complex zeros of the Jacobi theta function θ are given by the orbit of $\frac{1}{2} + \frac{1}{2}\tau$, f has only one zero. Now, we can view f as a section of the line bundle L by the argument above, and we know that the degree of L is 1. This is because the degree of a line bundle S can de defined by assigning 1 or -1 for every vanishing point of S and then summing them up.

The Jacobi theta function θ has an explicit expression:

$$\theta(z) = \sum_{m\in\mathbb{Z}} \exp(\pi im^2\tau + 2\pi imz).$$

One can easily check this fact by showing that the θ defined above by the explecit expression satisfies the two modularity properties of the Jacobi theta function.

The reason why we are interested in the particular line bundle $L = L_q(\varphi_0)$ is because it helps us to classify all holomorphic line bundles on E. To be specific, we have the following proposition:

Proposition 2.11. *Every holomorphic line bundle on E has the form $t_x^* L \otimes L^{n-1}$ for some $n \in \mathbb{Z}$ and $x \in E$, where t_x is the map of translation by x on E (recall that every elliptic curve is isomorphic to a torus, making the curve an abelian group).*

The proof of this proposition relies on the following theorem of the square which gives a description of the group $\mathrm{Pic}^o := \ker(c_1)$, where c_1 is the map of taking the first Chern class. Notice that in the case of a smooth projective curve, the degree map coincides with the first Chern class. Thus Pic^o coincides with the group of degree zero line bundles. Following Beauvilles's paper [1], we have the theorem of the square:

Theorem 2.12 (Theorem of the square)**.** *Let $X \cong \mathbb{C}/\Gamma$ be an elliptic curve, and L be a line bundle on X.*

a) The map

$$\lambda_L : X \to \mathrm{Pic}^o(X), \ \lambda_L(x) = t_x^* L \otimes L^{-1}$$

is a group homomorphism.

b) Let $E \in \mathrm{Alt}^2(\Gamma, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ be the first Chern class of L. If E is non-degenerate, then λ_L is surjective.

Since every line bundle over $X = \mathbb{C}/\Gamma$ pulls back to the trivial line bundle over $\mathbb{C}$, we can recover a line bundle over X from the trivial line bundle by identifying the fibers over the preimages of every point. To be specific, we introduce the notation of *systems of multipliers* following Beauville's paper [1].

Definition 2.13. Let $(e_\gamma)_{\gamma \in \Gamma}$ be a family of holomorphic invertible functions on $\mathbb{C}$. It is called a *system of multipliers* if these functions satisfy

$$e_{\gamma+\delta}(z) = e_\gamma(z+\delta) e_\delta(z), \ \forall \gamma, \delta \in \Gamma \text{ (“cocycle condition”)}.$$

Using $(e_\gamma)_{\gamma \in \Gamma}$, we can define a relation on $\mathbb{C} \times \mathbb{C}$ by

$$(z, t) \sim (z + \gamma, e_\gamma(z) \cdot t), \ \forall \gamma \in \Gamma.$$

Then the cocycle condition guarantees that the relation “ $\sim$ ” is an equivalence relation, and the quotient space $\mathbb{C} \times \mathbb{C}/\sim$ defines a line bundle over $X \cong \mathbb{C}/\Gamma$.

Now, we construct systems of multipliers from Hermitian forms.

We denote by $\mathcal{P}$ the set of pairs (H, α), where H is a Hermitian form on $\mathbb{C}$, and α is a map from Γ to $\mathbb{S}^1 \subset \mathbb{C}$ satisfying following two restrictions:

a) $E(u, v) := \mathrm{Im}(H(u, v)) \in \mathbb{Z}, \ \forall u, v \in \Gamma$
b) $\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)}$

Then the law $(H, \alpha) \cdot (H', \alpha') = (H + H', \alpha\alpha')$ defines a group structure on $\mathcal{P}$.

For $(H, \alpha) \in \mathcal{P}$, we put

$$e_\gamma(z) = \alpha(\gamma) e^{\pi[H(\gamma,z) + \frac{1}{2}H(\gamma,\gamma)]}, \ \gamma \in \Gamma$$

One can easily check that this defines a system of multipliers. And the corresponding line bundle will be denoted by $L(H, \alpha)$.

The benefit of defining a line bundle by a Hermitian form is that we can calculate the first Chern class of the line bundle easily. The following proposition comes from Theorem 2.8 of Beauville's paper [1].

Proposition 2.14. *The first Chern class $c_1(L(H,\alpha))$ is equal to $E \in \mathrm{Alt}^2(\Gamma,\mathbb{Z}) \cong H^2(E,\mathbb{Z})$.*

Corollary 2.15. *Let X be an elliptic curve, and $L = L_q(\varphi_0)$ be the particular line on X defined above. Assume that L' is a holomorphic line bundle on X with zero degree. Then $L' \cong t_x^* L \otimes L^{-1}$ for some $x \in X$.*

Proof. Notice that in the case of a smooth projective curve, the degree map coincides with the first Chern class. Thus, L' has degree 0 tells us that its first Chern class $c_1(L')$ is 0, and $L' \in \mathrm{Pic}^o(X)$. Now it is sufficient to prove that $E = c_1(L)$ is non-degenerate. Then by Theorem 2.12 b), the associated map λ_L is surjective, and L' can be expressed in the desired form. Now, I claim that L comes from the Hermitian form

$$H(u,v) = \frac{2iu\bar{v}}{\tau - \bar{\tau}} = \frac{u\bar{v}}{t}, \text{ where } t = \mathrm{Im}(\tau),$$

up to a normalization. The detailed proof of this claim can be found in chapter 8, section 3 of Lang's book [4]. Since τ lies in the upper half plane, t is not zero, and the equation above makes sense. Now, by Proposition 2.14, the first Chern class E of L is the imaginary part of H. And one can easily check that $E = \mathrm{Im}(H)$ is non-degenerate. □

Now, we get back to the proof of Proposition 2.11.

Proof. Let L' be any holomorphic line bundle on E. Assume that the degree of L' is n. Then we consider the line bundle $L'' = L' \otimes L^{-n}$. The degree of L'' is $\deg(L'') = n + n \times (-1) = 0$. Thus, by Corollary 2.15, $L'' \cong t_x^* L \otimes L^{-1}$ for some $x \in E$. Therefore, $L' \cong L'' \otimes L^n \cong t_x^* L \otimes L^{n-1}$. □

Now, we define theta functions by their explicit expressions.

Definition 2.16. A theta function has three parameters: $\tau \in \mathbb{C}$ for the torus, and $(c', c'') \in \mathbb{R}^2/\mathbb{Z}^2$ for line bundles of the same degree. The theta function is defined by

$$\theta[c', c''](\tau, z) = \sum_{m \in \mathbb{Z}} \exp\{2\pi i[\tau(m + c')^2/2 + (m + c')(z + c'')]\}.$$

If $(c', c'') = (0,0)$, $\theta[0,0](\tau, z)$ becomes the Jacobi theta function, and we will use the notation $\theta(\tau, z)$ for it.

Remark 2.17. The n functions $\widetilde{\theta}_a(z) = \theta[a/n, 0](n\tau, nz), a \in \mathbb{Z}/n\mathbb{Z}$ are the global sections of L^n. The reason is that:

$$\begin{aligned} &\widetilde{\theta}_a(z) = \widetilde{\theta}_a(z+1), \text{and} \\ &\widetilde{\theta}_a(z+\tau) = e^{-n\pi i\tau - 2n\pi i z} \cdot \widetilde{\theta}_a(z) = (q^{-\frac{1}{2}} u^{-1})^n \cdot \widetilde{\theta}_a(z). \end{aligned}$$

Moreover, they form a basis of the space of global sections of L^n.

Now, consider the natural r-fold covering $\pi_r : E_{q^r} \to E_q$ which sends u to u. Then, the preimage of $u \in E_q$ is $\{u, qu, ..., q^{r-1}u\}$. We define the natural functors of pull-back and push-forward associated with π_r.

Definition 2.18. The *pull-back* map π_r^* is defined by $\pi_r^* F_q(V,A) = F_q^r(V,A^r)$, and the *push-forward* map π_{r*} is defined by $\pi_{r*}F_{q^r}(V,A) = F_q(V\otimes\mathbb{C}^r, \pi_{r*}A)$, where $\pi_{r*}A(v\otimes e_i) = v\otimes e_{i+1}$ for $i\in\{1,2,...,r-1\}$ and $\pi_{r*}A(v\otimes e_r) = Av\otimes e_1$.

These two maps have the following properties:

Proposition 2.19. *There are natural isomorphisms:*

a) $\pi_{r*}(F_1\otimes\pi_r^*F_2)\cong\pi_{r*}(F_1)\otimes F_2$
b) $(\pi_{r*}(F))^*\cong\pi_{r*}(F^*)$
c) $H^0(E_q,\pi_{r*}(F))\cong H^0(E_{q^r},F)$

Proof. Suppose that $F_1 = F_{q^r}(V,A)$ and $F_2 = F_q(W,B)$. Then $\pi_r^*F_2 = F_{q^r}(W,B^r)$, and $F_1\otimes\pi_r^*F_2 = F_{q^r}(V\otimes W, A\otimes B^r)$.

Thus, we have

$$\pi_{r*}(F_1\otimes\pi_r^*F_2) = F_q(V\otimes W\otimes\mathbb{C}^r, \pi_{r*}(A\otimes B^r))$$

and

$$\pi_{r*}(A\otimes B)(v\otimes w\otimes e_i) = \begin{cases} v\otimes w\otimes e_{i+1} & \text{if } i\in\{1,2,...,r-1\} \\ Av\otimes B^r w\otimes e_1 & \text{if } i=r. \end{cases}$$

On the other hand, $\pi_{r*}(F_1) = F_q(V\otimes\mathbb{C}^r,\pi_{r*}A)$. Thus, we have

$$\pi_{r*}(F_1)\otimes F_2 = F_q(V\otimes\mathbb{C}^r\otimes W, \pi_{r*}A\otimes B)$$

and

$$\pi_{r*}A\otimes B(v\otimes e_i\otimes w) = \begin{cases} v\otimes e_{i+1}\otimes Bw & \text{if } i\in\{1,2,...,r-1\} \\ Av\otimes e_1\otimes Bw & \text{if } i=r. \end{cases}$$

Now, define a map $\sigma : V\otimes W\otimes\mathbb{C}^r\to V\otimes\mathbb{C}^r\otimes W$ by $\sigma(v\otimes w\otimes e_i) = v\otimes e_i\otimes B^i w$. Then one can check that the following diagram commutes:

$$\begin{tikzcd} V\otimes W\otimes\mathbb{C}^r \arrow[r, "\sigma"] \arrow[d, "\pi_{r*}(A\otimes B^r)"'] & V\otimes\mathbb{C}^r\otimes W \arrow[d, "\pi_{r*}A\otimes B"] \\ V\otimes W\otimes\mathbb{C}^r \arrow[r, "\sigma"] & V\otimes\mathbb{C}^r\otimes W. \end{tikzcd}$$

Therefore, we have

$$F_q(V\otimes W\otimes\mathbb{C}^r, \pi_{r*}(A\otimes B^r))\cong F_q(V\otimes\mathbb{C}^r\otimes W, \pi_{r*}A\otimes B)$$

and

$$\pi_{r*}(F_1\otimes\pi_r^*F_2)\cong\pi_{r*}(F_1)\otimes F_2.$$

Now, suppose that $F = F_{q^r}(V,A)$. Then we have

$$\pi_{r*}(F) = F_q(V\otimes\mathbb{C}^r,\pi_{r*}A),$$

and thus

$$(\pi_{r*}(F))^* = F_q(V\otimes\mathbb{C}^r, (\pi_{r*}A)^{-1}),$$

and the associated endomorphism is defined by

$$(\pi_{r*}A)^{-1}(v\otimes e_i) = \begin{cases} v\otimes e_{i-1} & \text{if } i\in\{2,3,...,r\} \\ A^{-1}v\otimes e_r & \text{if } i=1. \end{cases}$$

On the other hand, $F^* = F_{q^r}(V, A^{-1})$, and $\pi_{r*}(F^*) = F_q(V \otimes \mathbb{C}^r, \pi_{r*}(A^{-1}))$. And the associated endomorphism is defined by

$$\pi_{r*}(A^{-1})(v \otimes e_i) = \begin{cases} v \otimes e_{i+1} & \text{if } i \in \{1, 2, ..., r-1\} \\ A^{-1}v \otimes e_1 & \text{if } i = r. \end{cases}$$

Now, we can define a map $\sigma' : V \otimes \mathbb{C}^r \to V \otimes \mathbb{C}^r$ by $\sigma'(v \otimes e_i) = v \otimes e_{r-i+1}$. Then one can check that the following diagram commutes:

$$\begin{tikzcd} V \otimes \mathbb{C}^r \arrow[r, "\sigma'"] \arrow[d, "(\pi_{r*}A)^{-1}"'] & V \otimes \mathbb{C}^r \arrow[d, "\pi_{r*}(A^{-1})"] \\ V \otimes \mathbb{C}^r \arrow[r, "\sigma'"] & V \otimes \mathbb{C}^r \end{tikzcd}$$

Therefore, $F_q(V \otimes \mathbb{C}^r, (\pi_{r*}A)^{-1}) \cong F_q(V \otimes \mathbb{C}^r, \pi_{r*}(A^{-1}))$, and $\pi_{r*}(F) \cong \pi_{r*}(F^*)$.

The last equation can be easily checked since H^0 just means taking global sections. $\square$

Remark 2.20. The pull-back functor commutes with tensor product and duality since one can easily verify that π_r^* is the usually defined pull-back of vector bundles.

Corollary 2.21. *We have the following two isomorphisms, which show the adjointness of π_{r*} and π_r^*:*

$$\operatorname{Hom}(F_1, \pi_{r*}F_2) \cong \operatorname{Hom}(\pi_r^*F_1, F_2),$$
$$\operatorname{Hom}(\pi_{r*}F_1, F_2) \cong \operatorname{Hom}(F_1, \pi_r^*F_2).$$

Proof.

$$\begin{aligned} \operatorname{Hom}(F_1, \pi_{r*}F_2) &\cong H^0(E_q, F_1^* \otimes \pi_{r*}F_2) \\ &\cong H^0(E_q, \pi_{r*}(\pi_r^*F_1^* \otimes F_2)) \\ &\cong H^0(E_{q^r}, \pi_r^*F_1^* \otimes F_2) \\ &\cong H^0(E_{q^r}, (\pi_r^*F_1)^* \otimes F_2) \\ &\cong \operatorname{Hom}(\pi_r^*F_1, F_2) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}(\pi_{r*}F_1, F_2) &\cong H^0(E_q, (\pi_{r*}F_1)^* \otimes F_2) \\ &\cong H^0(E_q, \pi_{r*}(F_1^*) \otimes F_2) \\ &\cong H^0(E_q, \pi_{r*}(F_1^* \otimes \pi_r^*F_2)) \\ &\cong H^0(E_{q^r}, F_1^* \otimes \pi_r^*F_2) \\ &\cong \operatorname{Hom}(F_1, \pi_r^*F_2) \end{aligned}$$

$\square$

The following three useful propositions and their proofs can be found at the end of section 2 of Polishchuk and Zaslow's paper [7], so we will omit the proofs here.

Proposition 2.22. *Every indecomposable bundle on E_q is isomorphic to a bundle of the form $\pi_{r*}(L_{q^r}(\varphi) \otimes F_{q^r}(\mathbb{C}^k, \exp N))$, where N is a constant indecomposable nilpotent matrix, $\varphi = t_X^*\varphi_0 \cdot \varphi_0^{n-1}$ for some $n \in \mathbb{Z}$ and $x \in \mathbb{C}^*$, and t_x represents the translation by x.*

Proposition 2.23. *Let $\varphi = t_x^*\varphi_0 \cdot \varphi_0^{n-1}$, with $n > 0$. Then for any nilpotent endomorphism $N \in \mathrm{End}(V)$, there is a canonical isomorphism*

$$\mathcal{V}_{\varphi,N} : H^0(L(\varphi)) \otimes V \to H^0(L(\varphi) \otimes F(V, \exp N)).$$

And the map $\mathcal{V}_{\varphi,N}$ is defined by

$$\mathcal{V}_{\varphi,N}(f \otimes v) = \exp(DN/n) f \cdot v = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f \cdot N^k v,$$

where $D = -u\frac{d}{du} = -\frac{1}{2\pi i}\frac{d}{dz}$.

Proposition 2.24. *Let $\varphi_1 = t_x^*\varphi_0 \cdot \varphi_0^{n_1-1}$, $\varphi_2 = t_x^*\varphi_0 \cdot \varphi_0^{n_2-1}$, and let $N_i \in \mathrm{End}(V_i), i = 1, 2$, be nilpotent endomorphisms. Then*

$$\begin{aligned}&\mathcal{V}_{\varphi_1,N_1}(f_1 \otimes v_1) \circ \mathcal{V}_{\varphi_2,N_2}(f_2 \otimes v_2)\\ &= \mathcal{V}_{\varphi_1\varphi_2,N_1+N_2}\left[\exp\left(\frac{n_2N_1 - n_1N_2}{n_1 + n_2}\frac{D}{n_1}\right)(f_1)\exp\left(\frac{n_1N_2 - n_2N_1}{n_1 + n_2}\frac{D}{n_2}\right)(f_2)(v_1 \otimes v_2)\right],\end{aligned}$$

where N_1, N_2 denote $N_1 \otimes 1$ and $1 \otimes N_2$ respectively, on the right hand side, and $\circ$ denotes the natural composition of sections

$$\begin{aligned}&H^0(L(\varphi_1) \otimes F(V_1, \exp N_1)) \otimes H^0(L(\varphi_2) \otimes F(V_2, \exp N_2)) \to\\ &\qquad H^0(L(\varphi_1\varphi_2) \otimes F(V_1 \otimes V_2, \exp(N_1 \otimes 1 + 1 \otimes N_2))).\end{aligned}$$

Recalling that $\exp\frac{d}{dz}$ is the generator of translations, we may write formally

$$\exp\left(N \cdot \frac{d}{dz}\right) f(z) = f(z + N).$$

In this notation, the above formula becomes

$$\begin{aligned}\mathcal{V}(f_1 \otimes v_1) \circ \mathcal{V}(f_2 \otimes v_2) &= \mathcal{V}\left(f_1(z + \frac{n_1N_2 - n_2N_1}{2\pi i n_1(n_1 + n_2)})f_2(z + \frac{n_2N_1 - n_1N_2}{2\pi i n_2(n_1 + n_2)})(v_1 \otimes v_2)\right)\\ &= \mathcal{V}\left(f_1(ue^{\frac{n_1N_2 - n_2N_1}{n_1(n_1+n_2)}})f_2(ue^{\frac{n_2N_1 - n_1N_2}{n_2(n_1+n_2)}})(v_1 \otimes v_2)\right).\end{aligned}$$

It is also important to notice that we can write down explicitly the morphism space between two vector bundles. To be specific, we shall need the following lemma:

Lemma 2.25. *If $A \in \mathrm{GL}(V)$, then*

$$H^0(E_\tau, F_\tau(V, A)) = \ker(\boldsymbol{1}_v - A).$$

In particular, we have

$$\mathrm{Hom}(F(V_1, A_1), F(V_2, A_2)) = \{f \in \mathrm{Hom}(V_1, V_2) | f \circ A_1 = A_2 \circ f\}.$$

Proof. The first equation follows easily from the fact that holomorphic sections of a flat vector bundle are covariantly constant. As for the second equation, we have

$$\mathrm{Hom}(F(V_1, A_1), F(V_2, A_2)) = H^0(E_\tau, F(V_1^* \otimes V_2, A)),$$

where $A \in \mathrm{GL}(V_1^* \otimes V_2) = \mathrm{GL}(\mathrm{Hom}(V_1, V_2))$ is defined by $A(f) = A_2 \circ f \circ A_1^{-1}$, for all $f \in \mathrm{Hom}(V_1, V_2)$. Thus, combining it with the first equation, we have:

$$\begin{aligned}\mathrm{Hom}(F(V_1, A_1), F(V_2, A_2)) &= \ker(\mathbf{1}_{V_1^* \otimes V_2} - A)\\ &= \{f \in \mathrm{Hom}(V_1, V_2) | f \circ A_1 = A_2 \circ f\}\end{aligned}$$

$\square$

3. Fukaya Category

Definition 3.1. A complex manifold M of dimension n is a *Calabi-Yau* manifold if M is a compact Kähler manifold with a nowhere-vanishing holomorphic top form Ω, which is called the *Calabi-Yau form* of M.

Now, let $\widetilde{M}$ be a Calabi-Yau manifold . We denote its Kähler form to be k. Then, by Yau's Theorem, $\widetilde{M}$ admits a unique Ricci-flat Kähler metric, which will also be denoted by k. After that, we choose a closed 2-form b and define the *complexified Kähler form* ω to be $\omega = b + ik$. We are interested in the image of ω in the *Kähler moduli space*

$$M_{\text{Kähler}}(\widetilde{M}, J) = (H^2(\widetilde{M}, \mathbb{R}) \oplus i\mathcal{K}(\widetilde{M}, J))/H^2(\widetilde{M}, \mathbb{R}),$$

where J is the complex structure of M and

$$\mathcal{K}(\widetilde{M}, J) = \{[\omega] \in H^2(\widetilde{M}, \mathbb{R}) |\ \omega \text{ is Kähler}\}$$

is called the *Kähler cone* of $\widetilde{M}$.

In the case where $\widetilde{M}$ is a torus, we can compute $H^2(\widetilde{M}, \mathbb{R})$ by Poincare duality: $H^2(\widetilde{M}, \mathbb{R}) \cong H_0(\widetilde{M}, \mathbb{R}) \cong \mathbb{R}$. Thus, we can identify the Kähler form (or the corresponding flat metric) k with a positive real number (it is positive because k induces a metric that is positively definite). Meanwhile, by the same reason, we can also identify b with a real number. Therefore, the complexified Kähler form ω can be identified with an element τ in the upper half-plane. Recall that we have defined the elliptic curve E_τ with the lattice parameter τ. Now we can form E^τ by taking τ to be the complexified Kähler form on the torus. Then E^τ turns out to be the mirror manifold of the elliptic curve E_τ. In this section, we will discuss its Fukaya category $\mathcal{F}^0(E^\tau)$ and enlarge this category to an additive category $\mathcal{FK}^0(E^\tau)$. And in the next section, we will construct an equivalence between $D^b(\text{Coh}(E_\tau))$ and $\mathcal{FK}^0(E^\tau)$ following Polishchuk and Zaslow's paper [7] and Kreussler's paper [3].

First, we will define a general A_∞-category.

Definition 3.2. An A_∞-category $\mathcal{F}$ contains a class of objects $\text{Ob}(\mathcal{F})$. And for any $X, Y \in \mathcal{F}$, their morphism space $\text{Hom}(X, Y)$ is a $\mathbb{Z}$-graded abelian group. Moreover, there are a series of composition maps:

$$m_k : \text{Hom}(X_1, X_2) \otimes \text{Hom}(X_2, X_3) \otimes ... \otimes \text{Hom}(X_k, X_{k+1}) \to \text{Hom}(X_1, X_{k+1}),$$

$k \geq 1$, of degree $2 - k$, satisfying the condition

$$\sum_{r=1}^{n} \sum_{s=1}^{n-r+1} (-1)^\epsilon m_{n-r+1}(a_1 \otimes ... \otimes a_{s-1} \otimes m_r(a_s \otimes ... \otimes a_{s+r-1}) \otimes a_{s+r} \otimes ... \otimes a_n) = 0$$

for all $n \geq 1$, where $\epsilon = (r+1)s + r(n + \sum_{j=1}^{s-1} \deg(a_j))$. All these composition maps and the conditions they satisfy form what is called A_∞-*structure*. In particular, we call the conditions they satisfy the A_∞-relation.

Remark 3.3. We have the following remarks:

a) An A_∞-category with one object is called an A_∞-algebra.
b) When we take $n = 1$ in the A_∞-relation, we get a degree 1 map

$$m_1 : \text{Hom}(X_1, X_2) \to \text{Hom}(X_1, X_2)$$

such that $(m_1)^2 = 0$, making the space $\mathrm{Hom}(X_1, X_2)$ into a chain complex. I will use d to denote m_1 in the following article.

c) When we take $n = 2$ in the A_∞-relation, we find that the degree 0 map
$$m_2 : \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(X_2, X_3) \to \mathrm{Hom}(X_1, X_3)$$
turns out to be a morphism of complexes and induces a product on cohomologies.

d) When we take $n = 3$ in the A_∞-relation, we find that the degree -1 map
$$m_3 : \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(X_2, X_3) \otimes \mathrm{Hom}(X_3, X_4) \to \mathrm{Hom}(X_1, X_4))$$
serves as a homotopy between $m_2(\cdot, m_2(\cdot, \cdot))$ and $m_2(m_2(\cdot, \cdot), \cdot)$. Therefore, the product on cohomologies induced by m_2 is associative.

Now, since the composition map m_2 is not necessarily associative (it is merely associative at the level of cohomologies), the A_∞-category $\mathcal{F}$ is not necessarily a real category. However, we can define a real category $\mathcal{F}^0$ from $\mathcal{F}$ by replacing all morphism spaces by their H^0.

Definition 3.4. Let $\mathcal{F}$ be an A_∞-category. Then we can define a true category $\mathcal{F}^0$. The objects of $\mathcal{F}^0$ is the same as $\mathcal{F}$. The morphism spaces are defined by $\mathrm{Hom}_{\mathcal{F}^0}(X, Y) = H^0(\mathrm{Hom}_{\mathcal{F}}(X, Y))$. Here, we recall that the degree 1 map $d = m_1$ satisfies $d^2 = 0$ and makes the morphism space $\mathrm{Hom}_{\mathcal{F}}(X, Y)$ into a chain complex.

Now, we are able to define the Fukaya category $\mathcal{F}(\widetilde{M})$ for a Calabi-Yau manifold $\widetilde{M}$. To define the objects of this category, we have to introduce the notion of special Lagrangian submanifolds of a Calabi-Yau manifold. The definition of special Lagrangian submanifolds and related properties can be found in Appendix A.

Objects: The objects of $\mathcal{F}(\widetilde{M})$ are special Lagrangian submanifolds of $\widetilde{M}$ endowed with flat bundles with monodromies having eigenvalues of unit modulus. Apart from these, we also have an additional structure that will be discussed later. To summarize, an object $\mathcal{U}$ is a pair $\mathcal{U} = (\mathcal{L}, \alpha, \mathcal{E})$ where $\mathcal{L}$ is a special Lagrangian submanifold, $\mathcal{E}$ is a local system on $\mathcal{L}$ whose monodromy has eigenvalues with unit modulus (we will explain this later), and α is a real number that represents an additional structure (which will also be discussed later). This additional structure will allow us to define a shift functor in the Fukaya category $\mathcal{F}(\widetilde{M})$ and to calculate the Maslov index, which is used to introduce a $\mathbb{Z}$-grading on the spaces of morphisms in this category.

Remark 3.5. According to appendix A, in the case where $\widetilde{M}$ is a torus $\widetilde{M} \cong \mathbb{R}^2/\mathbb{Z}^2$, a special Lagrangian submanifold $\mathcal{L}$ of $\widetilde{M}$ is the image of a line in $\mathbb{R}^2$ with rational slope under the quotient map $\mathbb{R}^2 \to \widetilde{M}$.

Remark 3.6. Here, when we define $\mathcal{E}$ to be a local system, we mean that it is a locally constant sheaf of complex vector spaces, or equivalently, a complex vector bundle equipped with a flat connection.

Over a contractible space, the flatness of the connection allows us to define a trivialization by parallel transport. Thus, for any flat rank n bundle over X, we have a well-defined map $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}(\mathbb{C}^n)$, which is defined by considering the parallel transport along any loop based at x_0. Moreover, one can prove that the flat bundle and the flat connection are determined by the map φ. In other words, $\mathcal{E}_i$ can be represented as a representation of the fundamental group of the underlying

Lagrangian. In particular, when X is a special Lagrangian manifold of an elliptic curve, it is isomorphic to the circle S^1. And a representation of the fundamental group $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$ is given by a vector space V and an automorphism $M \in \mathrm{GL}(V)$. We use $(\mathcal{L}, \alpha, M)$ to denote the object defined in this way. We can change the orientation of X and replace M by M^{-1}, then we get an isomorphic local system on X. Moreover, the automorphisms that are conjugate to M also define isomorphic local systems. We call M the monodromy of the local system. When we require $\mathcal{E}$'s monodromy to have eigenvalues of unit modulus in the definition of the objects $\mathcal{U} = (\mathcal{L}, \alpha, \mathcal{E})$, it means that we only consider those M whose eigenvalues have modulus one.

Now, let us restrict ourselves to line bundles on X. Then $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}(\mathbb{C})$ is just a map form $\mathbb{Z}$ to $\mathbb{C}$. So we can determine this map by the value of the generator of $\pi_1(X, x_0)$. In other words, we can specify a line bundle $\mathcal{E}$ over the elliptic curve X simply by its monodromy around the circle, which is a complex phase $\exp(2\pi i\beta)$ with $\beta \in \mathbb{R}/\mathbb{Z}$.

The additional structure needed in the definition of the objects is the following. The embedding of a Lagrangian submanifold L into a Calabi-Yau manifold M induces a map from L to V, where V is a fiber bundle over M with fiber at x equal to the space of Lagrangian planes of T_xM (L being Lagrangian implies that T_xL is a Lagrangian plane of T_xM). We call this map from L to V the *Gauss map*. Now, we consider another fiber bundle $\widetilde{V}$ over M. The fiber of $\widetilde{V}$ at $x \in M$ is the universal cover of the fiber of V at x. The additional structure for a special Lagrangian submanifold L is a lift of the Gauss map to $\widetilde{V}$.

In our case where M is an elliptic curve and L is the image of a line, the space of Lagrangian planes of the tangent space is isomorpohic to S^1. So V is a S^1-bundle over M and $\widetilde{V}$ is a $\mathbb{R}$-bundle over M. Now, the Gauss map is a constant map of value equal to the intersection point of the line and the unit circle in $\mathbb{C}$, which can be viewed as a complex phase with rational tangency. We define this phase by $\exp(i\pi\alpha)$. Then, to define a lift of the Gauss map to $\widetilde{V}$, we only have to choose α itself. And we will use $\alpha \in \mathbb{R}$ to represent the additional structure in our case.

Shift functor: We can define a shift functor on the objects of a Fukaya category $\mathcal{F}(\widetilde{M})$ which corresponds to the natural shift functor in the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(M))$. The shift functor on objects is defined by:

$$(\mathcal{L}, \alpha, \mathcal{E})[1] \coloneqq (\mathcal{L}, \alpha + 1, \mathcal{E}).$$

Morphisms: Let $\mathcal{U}_i = (\mathcal{L}_i, \alpha_i, \mathcal{E}_i)$, $i = 1, 2$ be two objects in $F^0(M)$. When $\mathcal{L}_i \neq \mathcal{L}_j$, the morphism space $\mathrm{Hom}(\mathcal{U}_i, \mathcal{U}_j)$ is defined by

$$\mathrm{Hom}(\mathcal{U}_i, \mathcal{U}_j) = \bigoplus_{x \in \mathcal{L}_i \cap \mathcal{L}_j} \mathrm{Hom}(\mathcal{E}_i|_x, \mathcal{E}_j|_x),$$

where the "Hom" means the space of homomorphisms of vector spaces.

We can also define a $\mathbb{Z}$-grading of $\mathrm{Hom}(\mathcal{U}_i, \mathcal{U}_j)$ using Maslov-Viterbo index, which is given by

$$\mu(x) = -[\alpha_j - \alpha_i]$$

in our case. Here, α_i and α_j are the additional structures of $\mathcal{U}_i$ and $\mathcal{U}_j$.

Now, we assume that the lines $\mathcal{L}_i$ and $\mathcal{L}_j$ go through the origin. We also assume that $\tan(\alpha_i) = q/p$ and $\tan(\alpha_j) = s/r$, where (p, q) and (s, r) are both relatively prime pairs. One can easily verify that the intersection points of $\mathcal{L}_i$ and $\mathcal{L}_j$ are those points of the form

$$\left(\frac{pk}{|ps-qr|}, \frac{qk}{|ps-qr|}\right),\ k \in \mathbb{Z}/|ps-qr|\mathbb{Z}.$$

In particular, there are $|ps - qr|$ non-equivalent intersection points.

A_∞**-Structure:** The A_∞-structure on a Fukaya category $\mathcal{F}(\widetilde{M})$ is given by summing over holomorphic maps from the open unit disc D in $\mathbb{C}$ to the elliptic curve. Moreover, these maps should satisfy the boundary condition, and the sum of the maps should be conducted up to projective equivalence (we will define this equivalence after introducing the boundary condition). Now, assume that $\mathcal{L}_i$, $i = 1, 2, ..., k+1$ are different from each other. Then an element in $\mathrm{Hom}(\mathcal{U}_j, \mathcal{U}_{j+1})$ can be represented as a finite sum of elements of the form $u_j = t_j \cdot a_j$, where $a_j \in \mathcal{L}_j \cap \mathcal{L}_{j+1}$ and $t_j \in \mathrm{Hom}(\mathcal{E}_j|_{a_j}, \mathcal{E}_{j+1}|_{a_j})$. And we can define the A_∞-structure as follows:

$$m_k(u_1 \otimes ... \otimes u_k) = \sum_{a_{k+1} \in \mathcal{L}_1 \cap \mathcal{L}_{k+1}} C(u_1, ..., u_k, a_{k+1}) \cdot a_{k+1},$$

where the coefficients C are defined by

$$C(u_1, ..., u_k, a_{k+1}) = \sum_{\phi} \pm e^{2\pi i \int \phi^* \omega} \cdot P e^{\oint \phi^* \beta}.$$

Here, the sum is over holomorphic maps $\phi : D \to M$ that satisfy the following boundary condition: there are $k + 1$ points $p_j = e^{2\pi i \gamma_j} \in \partial D$ such that

$$\phi(p_j) = a_j \quad \text{and} \quad \phi(e^{2\pi i \gamma}) \in \mathcal{L}_j,\ \forall \gamma \in (\gamma_{j-1}, \gamma_j).$$

Here, ω is the complexified Kähler form $\omega = b + ik$, and β is the flat connection of the vector bundles. And the sum is conducted up to projective equivalence. Here, we define two maps ϕ and ϕ' satisfying the boundary condition to be projective equivalence if and only if there exists a hollomorphic automorphism $\rho : D \to D$ such that $\rho(p_j) = p'_j, \forall 1 \le j \le k+1$ and $\phi = \phi' \circ \rho$. One can easily check that this defines an equivalence relation. Next, we will explain what these two integrations $\int \phi^* \omega$ and $\oint \phi^* \beta$ in the coefficient C mean. Obviously, the first integration $\int \phi^* \omega$ is just the symplectic volume of the disc D with respect to the symplectic form $\phi^* \omega$. Now, I will explain what the second integration $\oint \phi^* \beta$ means. In fact, the second integration is defined by composing the integrations on every curve of ∂D divided by the points p_j:

$$P e^{\oint \phi^* \omega} = P e^{\int_{\gamma_k}^{\gamma_{k+1}} \beta_k d\gamma} \cdot t_k \cdot P e^{\int_{\gamma_{k-1}}^{\gamma_k} \beta_{k-1} d\gamma} \cdot t_{k-1} \cdot ... \cdot t_1 \cdot P e^{\int_{\gamma_{k+1}}^{\gamma_1} \beta_1 d\gamma}.$$

Moreover, the integration $\int_{\gamma_i}^{\gamma_{i+1}} \beta_i d\gamma$ can be computed as follows. In a local trivialization, the connection β_i can be represented as $\beta_i = d + N$ where N is a $n_i \times n_i$ matrix of one forms (n_i is the rank of the vector bundle $\mathcal{E}_i$). The intersection of the curve $\phi([\gamma_i, \gamma_{i+1}])$ and the local trivialization gives a local trivialization of the curve. I denote this local trivialization of the curve by $f : [0, 1] \to M$. Now we can integrate the matrix N along f and get a new matrix. We can view this matrix as a linear isomorphism between the fibers at $f(0)$ and $f(1)$. Now we can divide the curve $\phi([\gamma_i, \gamma_{i+1}])$ into finite segments, and we repeat the process of integration on every segment. Then we get finitely many isomorphisms between the fibers at adjacent endpoints. We compose these isomorphisms to get an isomorphism between the fibers at $\phi(\gamma_i)$ and $\phi(\gamma_{i+1})$. Finally, we take the exponential of this isomorphism, resulting in a linear map from $\mathcal{E}_{i+1}|_{a_i}$ to $\mathcal{E}_{i+1}|_{a_{i+1}}$. Since t_i is a linear map from $\mathcal{E}_i|_{a_i}$ to $\mathcal{E}_{i+1}|_{a_i}$, the definition of $Pe^{\oint \phi^*\omega}$ makes sense when the formula

$$Pe^{\oint \phi^*\omega} = Pe^{\int_{\gamma_k}^{\gamma_{k+1}} \beta_k d\gamma} \cdot t_k \cdot Pe^{\int_{\gamma_{k-1}}^{\gamma_k} \beta_{k-1} d\gamma} \cdot t_{k-1} \cdot ... \cdot t_1 \cdot Pe^{\int_{\gamma_{k+1}}^{\gamma_1} \beta_1 d\gamma}.$$

is read from right to left. And $Pe^{\oint \phi^*\omega}$ is an element of $\mathrm{Hom}(\mathcal{E}_1|_{a_{k+1}}, \mathcal{E}_{k+1}|_{a_{k+1}})$.

There is also an alternative interpretation of the integration of the connection from the geometric perspective. Since the connection β_i is flat, the parallel transport between any two points of the curve $\phi([\gamma_i, \gamma_{i+1}])$ is well defined, i.e., it is independent of the choice of the path connecting these two points. In particular, the parallel transport from the point $\phi(\gamma_i)$ to $\phi(\gamma_{i+1})$ gives us an element in $\mathrm{Hom}(\mathcal{E}_{i+1}|_{a_i}, \mathcal{E}_{i+1}|_{a_{i+1}})$. This element should equal to the element $\int_{\gamma_i}^{\gamma_{i+1}} \beta_{i+1} d\gamma$ defined above using integration of the matrix. In fact, this equivalence reveals the idea that "connection is the derivative of parallel transport."

Fact 3.7. *The compositions defined above satisfy the A_∞-relation, making $\mathcal{F}(M)$ into an A_∞-category. $\mathcal{F}(M)$ is called the* Fukaya category *of M.*

However, as I have mentioned in the beginning of this section, the Fukaya category $\mathcal{F}(M)$ is not a real category because the composition map m_2 is not associative. Instead, we can define a real category $\mathcal{F}^0(M)$ by taking the 0-cohomology of $\mathcal{F}(M)$. In our case, M is an elliptic curve, and one can check that $m_1 = d = 0$. So the 0-cohomology is just the zero-degree part of the morphism groups. Recalling that m_2 is associative at the level of cohomologies, we know that m_2 is truly associative at the level of the original groups as well. Moreover, the higher m's are also zero in $\mathcal{F}(M)$. And the equivalence that I am going to prove is between $D^b(\mathrm{Coh}(E_\tau))$ and $\mathcal{FK}^0(E^\tau)$, where E^τ is the mirror manifold of E_τ, and $\mathcal{FK}^0(E^\tau)$ is constructed from the Fukaya category $\mathcal{F}^0(E^\tau)$ by adding formal finite direct sums. We will give the explicit definition of $\mathcal{FK}^0(E^\tau)$ later.

Since the morphism space in $\mathcal{F}^0(M)$ is just the zero-graded part of the morphism space in $\mathcal{F}(M)$ and the grading is given by $-[\alpha_j - \alpha_i]$, we can write down explicitly the morphism spaces in $\mathcal{F}^0(M)$: when $\mathcal{L}_i \neq \mathcal{L}_j$, the morphism space $\mathrm{Hom}_{\mathcal{F}^0(M)}(\mathcal{U}_i, \mathcal{U}_j)$ is defined by

$$\mathrm{Hom}_{\mathcal{F}^0(M)}(\mathcal{U}_i, \mathcal{U}_j) = \begin{cases} 0 & \text{if } \alpha_j - \alpha_i \notin [0,1); \\ \bigoplus\limits_{x \in \mathcal{L}_i \cap \mathcal{L}_j} \mathrm{Hom}(\mathcal{E}_i|_x, \mathcal{E}_j|_x) & \text{if } \alpha_j - \alpha_i \in [0,1) \end{cases}$$

and when $\mathcal{L}_i = \mathcal{L}_j$, we know that $\alpha_j - \alpha_i \in \mathbb{Z}$ and we define

$$\mathrm{Hom}_{\mathcal{F}^0(M)}(\mathcal{U}_i, \mathcal{U}_j) = \begin{cases} 0 & \text{if } \alpha_j - \alpha_i \notin \{0,1\}; \\ H^0(\mathcal{L}_i, \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)) & \text{if } \alpha_j = \alpha_i; \\ H^1(\mathcal{L}_i, \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)) & \text{if } \alpha_j = \alpha_i + 1. \end{cases}$$

Here, the "Hom" in the former case is the space of homomorphisms of vector spaces, and the "$\mathcal{H}om$" in the latter case is the sheaf of homomorphisms of local systems (which are regarded as locally constant sheaves of complex vector spaces).

Let me explain the latter $\mathcal{H}$om more explicitly. We can represent $\mathcal{E}_i$ and $\mathcal{E}_j$ by two automorphisms $M_i \in \mathrm{GL}(V_i)$ and $M_j \in \mathrm{GL}(V_j)$. Then one can easily check that the resulting local system $\mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)$ corresponds to the automorphism M in $V = \mathrm{Hom}(V_i, V_j)$, where M is defined by $M(f) = M_j \circ f \circ M_i^{-1}$ for $f \in V$.

Moreover, since $\mathcal{L}_i \cong S^1$, we can compute the sheaf cohomology above. In fact, one can compute that

$$H^0(\mathcal{L}_i, M) \cong \ker(M - \mathbf{1}_V) \cong \{f \in \mathrm{Hom}(V_i, V_j) | M_j \circ f = f \circ M_i\}$$

and

$$H^1(\mathcal{L}_i, M) \cong \mathrm{coker}(M - \mathbf{1}_V) \cong \mathrm{Hom}(V_i, V_j)/M_j \circ \mathrm{Hom}(V_i, V_j) \circ M_i^{-1}.$$

Now, since $\ker(M - \mathbf{1}_V)^* \cong \mathrm{coker}({}^tM - \mathbf{1}_V) = \mathrm{coker}({}^tM^{-1} - \mathbf{1}_V)$, we get a canonical isomorphism $H^0(\mathcal{L}_i, M)^* \cong H^1(\mathcal{L}_i, M^\vee)$. Here, $M^\vee$ is the dual local system, which is given by the automorphism ${}^tM^{-1}$. Combining this with the definition of morphism spaces, we get the following "Symplectic Serre Duality:"

Lemma 3.8 (Compare to Serre Duality)**.** *Let $(\mathcal{L}_i, \alpha_i, \mathcal{E}_i)$ and $(\mathcal{L}_j, \alpha_j, \mathcal{E}_j)$ be objects in $\mathcal{F}^0(M)$. Then there exists a canonical isomorphism*

$$\mathrm{Hom}((\mathcal{L}_i, \alpha_i, \mathcal{E}_i), (\mathcal{L}_j, \alpha_j, \mathcal{E}_j)[1]) \cong \mathrm{Hom}((\mathcal{L}_j, \alpha_j, \mathcal{E}_j), (\mathcal{L}_i, \alpha_i, \mathcal{E}_i))^*.$$

Now we define the composition in $\mathcal{F}^0(M)$.

Let $(\mathcal{L}_i, \alpha_i, \mathcal{E}_i)$, $i = 1, 2, 3$ be three objects in $\mathcal{F}^0(M)$. We will use Λ_i to denote $(\mathcal{L}_i, \alpha_i, \mathcal{E}_i)$. Let $u \in \mathrm{Hom}(\Lambda_1, \Lambda_2)$, $v \in \mathrm{Hom}(\Lambda_2, \Lambda_3)$ be two non-zero morphisms in $\mathcal{F}^0(M)$. Then we have, by definition, $\alpha_1 \le \alpha_2 \le \alpha_3$, $\alpha_2 \le \alpha_1 + 1$ and $\alpha_3 \le \alpha_2 + 1$. To define $v \circ u$, we have to consider the following different cases.

Case 1. $\alpha_3 > \alpha_1 + 1$.

Then $\mathrm{Hom}(\Lambda_1, \Lambda_3) = 0$. So we define $v \circ u = 0$.

Case 2. $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_1 + 1$.

In this case, we know that the $\mathcal{L}_i$ are different from each other. And the definition is exactly what we have discussed before when we define the A_∞-structure of $\mathcal{F}(M)$.

If we are not in *Case 1* or *Case 2*, then we have $\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_1 + 1$ and at least one of these inequalities is actually equal. The case where $\alpha_3 = \alpha_1 + 1$ will be discussed in *Case 3.* And the remaining cases will be discussed in *Case 4* and *Case 5.*

Case 3. $\alpha_1 \le \alpha_2 \le \alpha_3 = \alpha_1 + 1$.

If $\alpha_1 < \alpha_2 < \alpha_3 = \alpha_1 + 1$, then we know that the composition map

$$\mathrm{Hom}(\Lambda_1, \Lambda_2) \otimes \mathrm{Hom}(\Lambda_2, \Lambda_3) \to \mathrm{Hom}(\Lambda_1, \Lambda_3)$$

is equivalent to

$$\mathrm{Hom}(\Lambda_1, \Lambda_2) \otimes \mathrm{Hom}(\Lambda_3[-1], \Lambda_2)^* \to \mathrm{Hom}(\Lambda_3[-1], \Lambda_1)^*,$$

and is equivalent to

$$\mathrm{Hom}(\Lambda_3[-1], \Lambda_1) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2) \to \mathrm{Hom}(\Lambda_3[-1], \Lambda_2).$$

Here, the first equivalence comes from symplectic Serre duality: $\mathrm{Hom}(A, B[1]) \cong \mathrm{Hom}(B, A)^*$, and the second equivalence comes from the canonical isomorphism $\mathrm{Hom}(V \otimes W^*, S^*) \cong \mathrm{Hom}(S \otimes V, W)$. Now we have $\alpha_3 - 1 = \alpha_1 < \alpha_2 < (\alpha_3 - 1) + 1$, and this case can be reduced to *Case 4.*

If $\alpha_1 < \alpha_2 = \alpha_3 = \alpha_1 + 1$ or $\alpha_1 = \alpha_2 < \alpha_3 = \alpha_1 + 1$, then we have $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3$. Both cases can be reduced to *Case 5* by symplectic Serre duality and the canonical isomorphism $\mathrm{Hom}(V \otimes W^*, S^*) \cong \mathrm{Hom}(S \otimes V, W)$.

Case 4. Precisely two of the α_k coincides and $\alpha_1 + 1 > \alpha_3$.

If $\alpha_1 = \alpha_2 < \alpha_3$, then we have $\mathcal{L}_1 = \mathcal{L}_2 \neq \mathcal{L}_3$. And we have

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = H^0(\mathcal{L}_1, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$$

and

$$\mathrm{Hom}((\mathcal{L}_2, \alpha_2, \mathcal{E}_2), (\mathcal{L}_3, \alpha_3, \mathcal{E}_3)) = \bigoplus_{x \in \mathcal{L}_2 \cap \mathcal{L}_3} \mathrm{Hom}(\mathcal{E}_2|_x, \mathcal{E}_3|_x)$$

and

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_3, \alpha_3, \mathcal{E}_3)) = \bigoplus_{x \in \mathcal{L}_1 \cap \mathcal{L}_3} \mathrm{Hom}(\mathcal{E}_1|_x, \mathcal{E}_3|_x).$$

Any $\varphi \in \mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = H^0(\mathcal{L}_1, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$ induces maps on stalks $\varphi_x : \mathcal{E}_1|_x \to \mathcal{E}_2|_x$. Let $(f_x)_x \in \mathrm{Hom}((\mathcal{L}_2, \alpha_2, \mathcal{E}_2), (\mathcal{L}_3, \alpha_3, \mathcal{E}_3))$, where $x \in \mathcal{L}_2 \cap \mathcal{L}_3$ and $f_x \in \mathrm{Hom}(\mathcal{E}_2|_x, \mathcal{E}_3|_x)$. Then we define the composition by

$$\varphi \otimes (f_x)_x \mapsto (f_x \circ \varphi_x)_x, \ x \in \mathcal{L}_2 \cap \mathcal{L}_3 = \mathcal{L}_1 \cap \mathcal{L}_3.$$

Here, $f_x \circ \varphi_x$ is a map in $\mathrm{Hom}(\mathcal{E}_1|_x, \mathcal{E}_2|_x)$ while x runs through $\mathcal{L}_2 \cap \mathcal{L}_3 = \mathcal{L}_1 \cap \mathcal{L}_3$. Therefore, $(f_x \circ \varphi_x)_x$ is indeed an element in $\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_3, \alpha_3, \mathcal{E}_3))$. Similarly, one can define the composition in the case where $\alpha_1 < \alpha_2 = \alpha_3$.

Case 5. $\alpha_1 = \alpha_2 = \alpha_3 < \alpha_1 + 1$.

In this case, we have $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3$, and the composition in $\mathcal{F}^0(M)$ is just the composition of homomorphisms between local systems.

Now we have finished the definition of the category $\mathcal{F}^0(M)$. However, it is impossible to define an equivalence between $D^b(\mathrm{Coh}(E_\tau))$ and $\mathcal{F}^0(E^\tau)$, where E_τ and E^τ are mirror elliptic curves. Since a derived category is always additive, it contains, in particular, finite direct sums and a zero object. However, the Fukaya category $\mathcal{F}^0(M)$ is merely an **Ab**-category (or a preadditive category), which means that it does not necessarily contain all finite direct sums. In fact, we can only define the direct sum for a pair of objects with the same underlying Lagrangians and the same α by

$$(\mathcal{L}, \alpha, M_1) \oplus (\mathcal{L}, \alpha, M_2) \coloneqq (\mathcal{L}, \alpha, M_1 \oplus M_2).$$

Therefore, to construct an equivalence between $D^b(\mathrm{Coh}(E_\tau))$, which is additive, and $\mathcal{F}^0(E^\tau)$, which is merely preadditive, we have to allow the formal direct sums in $\mathcal{F}^0(E^\tau)$. By adding formal finite direct sums in $\mathcal{F}^0(E^\tau)$, we get the desired abelian category $\mathcal{FK}^0(E^\tau)$, which is called the *Fukaya-Kontsevich* category. In fact, there is a general construction to enlarge an **Ab**-category to an additive category. This construction can be found in Kreussler's paper [3].

Similar to the isogeny π_r and the associated push-forward π_{r*} and pull-back π_r^* functors, we introduce a map p_r and define the associated functors in the Fukaya side. The map p_r from the tours $E^{r\tau}$ to E^τ is defined by $p_r(x, y) = (rx, y)$. The push-forward and pull-back functors associated to p_r are defined as follows.

Push-forward p_{r*}

Let $(\mathcal{L}, \alpha, \mathcal{E})$ be an object in $\mathcal{FK}^0(E^\tau)$. We define

$$p_{r*}((\mathcal{L}, \alpha, \mathcal{E})) := (p_r(\mathcal{L}), \alpha', p_{r*}\mathcal{E}),$$

where α' is the unique possible value (it is determined by the slope of $p_r(\mathcal{L})$ up to an integer) such that it lies in the same interval $(k - \frac{1}{2}, k + \frac{1}{2}]$ with $k \in \mathbb{Z}$ as α lies, and $p_{r*}\mathcal{E}$ is the direct image of the local system $\mathcal{E}$. If we represent $\mathcal{E}$ by a matrix $M \in \mathrm{GL}(V)$, then $p_{r*}\mathcal{E}$ is represented by $p_{r*}M \in \mathrm{GL}(V^{\oplus d})$, where d is the degree of the map p_r. (Notice that p_r is a map from S^1 to S^1, so we can define its degree by the induced map on $\pi_1(S^1) \cong \mathbb{Z}$), and $p_{r*}M$ is defined by $p_{r*}M(v_1, v_2, ..., v_d) = (v_2, v_3, ...v_d, Mv_1)$. (One can compare this to the definition of $\pi_{r*}A$ in the definition of π_{r*}.)

Next, we will define the functor for morphisms. Let $(\mathcal{L}_1, \alpha_1, \mathcal{E}_1)$ and $(\mathcal{L}_2, \alpha_2, \mathcal{E}_2)$ be two objects in $\mathcal{FK}^0(E^\tau)$.

Case 1. $p_r(\mathcal{L}_1) \neq p_r(\mathcal{L}_2)$: In this case, we have

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = \bigoplus_{x \in \mathcal{L}_1 \cap \mathcal{L}_2} \mathrm{Hom}(\mathcal{E}_1|_x, \mathcal{E}_2|_x),$$

and

$$\mathrm{Hom}(p_{r*}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1)), p_{r*}((\mathcal{L}_2, \alpha_2, \mathcal{E}_2))) = \bigoplus_{y \in p_r(\mathcal{L}_1) \cap p_r(\mathcal{L}_2)} \mathrm{Hom}(p_{r*}(\mathcal{E}_1)|_y, p_{r*}(\mathcal{E}_2)|_y).$$

Notice that $p_{r*}(\mathcal{E})|_y = \bigoplus_{x \in \mathcal{L}, p_r(x) = y} \mathcal{E}|_x$, thus we can define the functor p_{r*} in an obvious way.

Case 2. $\mathcal{L}_1 = \mathcal{L}_2$: In this case, we have

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = H^v(\mathcal{L}_1, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$$

for some $v \in \{0, 1\}$, and

$$\mathrm{Hom}(p_{r*}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1)), p_{r*}((\mathcal{L}_2, \alpha_2, \mathcal{E}_2))) = H^v(p_r(\mathcal{L}_1), \mathcal{H}om(p_{r*}(\mathcal{E}_1), p_{r*}(\mathcal{E}_2))).$$

Then we can use the canonical homomorphism of sheaves

$$p_{r*}\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \to \mathcal{H}om(p_{r*}(\mathcal{E}_1), p_{r*}(\mathcal{E}_2))$$

and the fact that p_r is a local homomorphism to obtain the required map.

Case 3. $\mathcal{L}_1 \neq \mathcal{L}_2$ *but* $p_r(\mathcal{L}_1) = p_r(\mathcal{L}_2)$: In this case, $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$. And the map p_{r*} is just zero.

Now, we have to verify that p_{r*} defined above is indeed a functor, i.e., we have to verify the compatibility of p_{r*} with compositions. Let $(\mathcal{L}_k, \alpha_k, \mathcal{E}_k)$, $k \in \{1, 2, 3\}$ be three objects in $\mathcal{FK}^0(E^{r\tau})$. If at least two of these three objects have the same underlying Lagrangian submanifold, then the compatibility can be easily verified from the definition. If $\mathcal{L}_k$, $k \in 1, 2, 3$ are differ from each other, then we have to compare two sums. Recall that the composition in E^τ sums over $\phi^\tau : D \to E^\tau$, and the composition in $E^{r\tau}$ sums over $\phi^{r\tau} : D \to E^{r\tau}$. If we lift both maps to $\mathbb{R}^2$, then their images are triangles with Euclidian areas A_{ϕ^τ} and $A_{\phi^{r\tau}}$. Notice that the map p_r defines a bijection between these triangles and that $A_{\phi^\tau} = rA_{\phi^{r\tau}}$, so we can verify the compatibility easily. Next, we will define the pull-back functor p_r^*.

Pull-back p_r^*

Let $(\mathcal{L}, \alpha, \mathcal{E})$ be an object in $\mathcal{FK}^0(E^\tau)$. Assume that the preimage $p_r^{-1}(\mathcal{L})$ consists of n connected components $\mathcal{L}^{(1)}, \mathcal{L}^{(2)}, ..., \mathcal{L}^{(n)}$, i.e., $p_r^{-1}(\mathcal{L}) = \coprod_{k=1}^{n} \mathcal{L}^{(k)}$. Then the restrictions $p_r^{(k)} : \mathcal{L}^{(k)} \to \mathcal{L}$ are of degree $d \coloneqq r/n$. And we define

$$p_r^*(\mathcal{L}, \alpha, \mathcal{E}) \coloneqq \bigoplus_{k=1}^{n} (\mathcal{L}^{(k)}, \alpha', (p_r^{(k)})^* \mathcal{E}),$$

where α' is the unique possible value such that it lies in the same interval $(k - \frac{1}{2}, k + \frac{1}{2}]$ with $k \in \mathbb{Z}$ as α is, and $(p_r^{(k)})^* \mathcal{E}$ is the pull-back of a local system. If the local system $\mathcal{E}$ is represented by a matrix $M \in \mathrm{GL}(V)$, then $(p_r^{(k)})^* \mathcal{E}$ is represented by $M^d \in \mathrm{GL}(V)$. The definition of p_r^* also tells us why we have to consider the additive category $\mathcal{FK}^0(E)$ instead of $F^0(E)$. One observes that the preimage of a line in the torus may consist of several disconnected lines. Since these lines can be transformed to each other by translations, there is no line that is more important than the others. Thus, we have to contain all of these lines in the pull-back object. In other words, we have to allow finite direct sums in our category, which leads to the definition of the category $\mathcal{FK}^0(E)$.

Next, we will define p_r^* on morphisms.

Case 1. $\mathcal{L}_1 = \mathcal{L}_2$: In this case, we have

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = H^v(\mathcal{L}_1, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)),$$

for some $v \in \{0, 1\}$, and

$$\mathrm{Hom}(p_r^*((\mathcal{L}_1, \alpha_1, \mathcal{E}_1)), p_r^*((\mathcal{L}_2, \alpha_2, \mathcal{E}_2))) = \bigoplus_{k=1}^{n} H^v(\mathcal{L}_1^{(k)}, \mathcal{H}om((p_r^{(k)})^* \mathcal{E}_1, (p_r^{(k)})^* \mathcal{E}_2)).$$

Notice that there is a canonical homomorphism

$$\mathcal{H}om((p_r^{(k)})^* \mathcal{E}_1, (p_r^{(k)})^* \mathcal{E}_2) \to (p_r^{(k)})^* \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2).$$

Thus the required map p_r^* can be constructed by taking cohomology.

Case 2. $\mathcal{L}_1 \neq \mathcal{L}_2$: In this case, we have

$$\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = \bigoplus_{x \in \mathcal{L}_1 \cap \mathcal{L}_2} \mathrm{Hom}(\mathcal{E}_1|_x, \mathcal{E}_2|_x),$$

and

$$\mathrm{Hom}(p_r^*((\mathcal{L}_1, \alpha_1, \mathcal{E}_1)), p_r^*((\mathcal{L}_2, \alpha_2, \mathcal{E}_2))) = \bigoplus_{y \in p_r^{-1}(\mathcal{L}_1) \cap p_r^{-1}(\mathcal{L}_2)} \mathrm{Hom}(\mathcal{E}_1|_{p_r(y)}, \mathcal{E}_2|_{p_r(y)}),$$

where the second equation comes from $(p_r^{(k)})^* \mathcal{E}|_y \cong \mathcal{E}|_{p_r(y)}$, $\forall k \in \{1, 2, ..., n\}$. Now assume that $f \in \mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2))$ has components $f_x \in \mathrm{Hom}(\mathcal{E}_1|_x, \mathcal{E}_2|_x)$ with $x \in \mathcal{L}_1 \cap \mathcal{L}_2$. We define the component $(p_r^* f)_y$ of $p_r^* f$ corresponding to $y \in p_r^{-1} \mathcal{L}_1 \cap p_r^{-1} \mathcal{L}_2$ by

$$(p_r^* f)_y = f_{p_r(y)} \in \mathrm{Hom}(\mathcal{E}_1|_{p_r(y)}, \mathcal{E}_2|_{p_r(y)}).$$

One can check that p_r^* defined above is indeed a functor by proving its compatibility with composition. The proof is similar to that of p_{r*}, so we will omit it here.

Apart from the pull-back and push-forward functors of p_r defined above, we also need the pull-back and push-forward functor of a translation. A translation on E^τ is a map $t : E^\tau \to E^\tau$ of the form $t(x, y) = (x - x_0, y - y_0)$ for some fixed (x_0, y_0) in $\mathbb{R}^2$. We define its pull-back by $t^*(\mathcal{L}, \alpha, \mathcal{E}) \coloneqq (t^{-1}(\mathcal{L}), \alpha, t^* \mathcal{E})$. Since t is an isomorphism, we can define t^* on morphisms in an obvious way. Moreover, one can easily verify that t^* is indeed a functor, i.e., it is compatible with composition. We can define the push-forward functor t_* in a similar way.

Similar to the case of π_r^* and π_{r*}, we have the following lemma about adjointness of p_r^* and p_{r*}:

Lemma 3.9. *Let $p_r : E^{r\tau} \to E^\tau$ be as above and $t : E^\tau \to E^\tau$ be a translation of the form $t(x,y) = (x + \frac{m}{n}, y)$, with $m, n \in \mathbb{Z}$. Define $p = t \circ p_r : E^{r\tau} \to E^\tau$. Let $(\mathcal{L}_1, \alpha_1, \mathcal{E}_1)$ and $(\mathcal{L}_2, \alpha_2, \mathcal{E}_2)$ be objects in $\mathcal{FK}^0(E^\tau)$ and $\mathcal{FK}^0(E^{r\tau})$ respectively. Then we have following functorial isomorphisms:*

$$\mathrm{Hom}(p^*(\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) \cong \mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), p_*(\mathcal{L}_2, \alpha_2, \mathcal{E}_2)),$$

and

$$\mathrm{Hom}(p_*(\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) \cong \mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), p^*(\mathcal{L}_2, \alpha_2, \mathcal{E}_2))$$

The detailed proof of this lemma can be found in Kreussler's paper [3], and we will not repeat it here.

4. The Equivalence

In this section, we will construct a functor ϕ_τ from $D^b(\mathrm{Coh}(E_\tau))$ to $\mathcal{FK}^0(E^\tau)$ and prove the following main theorem:

Main Theorem: There is a functor $\phi_\tau : D^b(\mathrm{Coh}(E_\tau)) \to \mathcal{FK}^0(E^\tau)$ and it is an equivalence of additive categories that is compatible with the shift functors.

First, we have to define ϕ_τ on objects of $D^b(\mathrm{Coh}(E_\tau))$. Recall that any element of $D^b(\mathrm{Coh}(E_\tau))$ is a direct sum of some elements with the form $\mathcal{F}[n]$, where $\mathcal{F}$ is a vector bundle or a skyscraper sheaf. Thus, we only have to define ϕ_τ for objects of the form $\mathcal{F} = \mathcal{F}[0] \in D^b(\mathrm{Coh}(E_\tau))$. Then we can extend the definition of ϕ_τ to objects of the form $\mathcal{F}[n]$ by shift functors, and then to a general object by taking finite direct sums in both categories.

To define ϕ_τ on any vector bundle or skyscraper sheaf, we first define it on objects of the form $\mathcal{L}(\varphi) \otimes F(V, \exp(N))$, where $\varphi = t^*_{a\tau+b}\varphi_0 \cdot \varphi_0^{n-1}$ and V is a finite dimensional vector space and $N \in \mathrm{End}(V)$ is a cyclic nilpotent endomorphism. Here, we call a nilpotent endomorphism N cyclic, if the corresponding $\mathbb{C}[N]$-module structure on V is cyclic. Moreover, the following lemma tells us that N is cyclic if and only if $\dim\ker N = 1$.

Lemma 4.1. *N is cyclic if and only if* $\dim\ker N = 1$.

Proof. Consider the minimal polynomial P of N. Since N is nilpotent, $P(x) = x^r$ for some r, where r is determined by $N^{r-1} \neq 0 = N^r$. Now, we assume that N is cyclic, so that there exists a generator $w \in V$ such that $V = \mathbb{C}[N] \cdot w$. Now, I claim that $\{w, Nw, ..., N^{r-1}w\}$ is a basis of V. It clearly generates V since $V = \mathbb{C}[N] \cdot w$ and $N^r w = 0$. So we only have to verify that they are linearly independent. Assume that $a_0 w + a_1 Nw + ... + a_{r-1}N^{r-1}w = 0$. Then $Q(N)w = 0$, where the polynomial Q is defined by $Q(x) = a_0 + a_1 x + ... + a_{r-1}x^{r-1}$. Using $V = \mathbb{C}[N] \cdot w$ once again, we know that $Q(N) \cdot V = 0$. So the minimal polynomial P should divide Q, which is a contradiction. Therefore $\{w, Nw, ..., N^{r-1}w\}$ is a basis of V. Consequently $r = \dim(V)$ and $\dim\ker N = 1$.

Conversely, assume that $\dim(V) = n$. Then $\dim\ker N = 1$ tells us that there is only one block in N's Jordan normal form. Thus $N^{n-1} \neq 0 = N^n$. Take $w \in V$ such that $N^{n-1}w \neq 0$, then $w, Nw, ..., N^{n-1}w$ are linearly independent and form a basis of V. Therefore $V = \mathbb{C}[N] \cdot w$ and N is cyclic. □

Back to the definition of ϕ_τ, we define

$$\phi_\tau(L(\varphi) \otimes F(V, \exp(N))) = (\mathcal{L}, \alpha, \mathcal{E}),$$

where $\mathcal{L}$ is defined by $(a+t, (n-1)a+nt)$, α is the unique possible real number satisfying $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$, and $\mathcal{E}$ is a locally free sheaf represented by $M = \exp(-2\pi i b \mathbf{1}_V + N)$. Sometimes, we use the notation $(\mathcal{L}, \alpha, M)$ instead of $(\mathcal{L}, \alpha, \mathcal{E})$.

Next, we will define ϕ_τ for a general vector bundle $\mathcal{F}$ on E_τ. By Proposition 2.22, there exists a positive integer r and a function $\varphi = t^*_{a\tau+b}\varphi_0 \cdot \varphi_0^{n-1}$ such that $\mathcal{F} \cong \pi_{r*}(L(\varphi) \otimes F(V, \exp(N)))$. Then we define

$$\phi_\tau(\mathcal{F}) = \phi_\tau(\pi_{r*}(L(\varphi) \otimes F(V, \exp(N)))) \coloneqq p_{r*}\phi_{r\tau}(L(\varphi) \otimes F(V, \exp(N))).$$

Here, noticing that $L(\varphi) \otimes F(V, \exp(N))$ is a coherent sheaf over $E_{r\tau}$, we can use $\phi_{r\tau}$ to map it to an object in $\mathcal{FK}^0(E^{r\tau})$. After that, we apply the push-forward functor p_{r*} to get an object in $\mathcal{FK}^0(E^\tau)$, which is our definition of $\phi_\tau(\mathcal{F})$.

To finish the definition of ϕ_τ on objects, we also need to define ϕ_τ for a skyscraper sheaf. Following Polishchuk and Zaslow's notation in their paper [7], we use $A = S(a\tau+b, V, N)$ to represent a thickened skyscraper sheaf A. To be specific, for every $z_0 \in \mathbb{C}$ and a indecomposible nilpotent endomorphism $N \in \mathrm{End}(V)$, we have the corresponding coherent sheaf of $\mathbb{C}$ supported at z_0. Namely, $\mathcal{O}_{rz_0} \otimes V/(z - z_0 - \frac{N}{2\pi i})$, where $r = \dim V$ is the smallest positive integer such that $N^r = 0$ (this is because N is indecomposible). We denote by $S(z_0, V, N)$ the direct image of this sheaf on E_τ. Using this notation, we have the following definition for $A = S(a\tau + b, V, N)$:

$$\phi_\tau(A) \coloneqq (\mathcal{L}, \frac{1}{2}, \exp(2\pi i b \mathbf{1}_V + N)).$$

Here, $\mathcal{L}$ is defined by $(-a, t)$.

Now, to get a functor, we also have to define ϕ_τ for morphisms.

Notice that we have shift functors in both categories and that we can take finite direct sums, thus we only have to define

$$\phi_\tau : \mathrm{Hom}_{D^b(\mathrm{Coh}(E_\tau))}(A_1, A_2[n]) \to \mathrm{Hom}_{\mathcal{FK}^0(E^\tau)}(\phi_\tau(A_1), \phi_\tau(A_2)[n]).$$

Since both sides vanish if $n \notin \{0, 1\}$, we only have to define the map in case $n = 0$ and $n = 1$. Moreover, by using Serre duality

$$\mathrm{Hom}(A_1, A_2[1]) \cong \mathrm{Hom}(A_2, A_1)^*$$

and

$$\mathrm{Hom}(\phi_\tau(A_1), \phi_\tau(A_2)[1]) \cong \mathrm{Hom}(\phi_\tau(A_2), \phi_\tau(A_1))^*$$

in both sides, we only have to deal with the case when $n = 0$.

Step 1. We first define ϕ_τ for $A_i = L(\varphi_i) \otimes F(V_i, \exp(N_i))$, and we assume that $\phi_\tau(A_i) = (\mathcal{L}_i, \alpha_i, M_i)$.

Case 1. We further assume that $\mathcal{L}_1 \neq \mathcal{L}_2$. Then we have

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathrm{Coh}(E_\tau))}(A_1, A_2) &= \mathrm{Hom}(L(\varphi_1) \otimes F(V_1, \exp(N_1)), L(\varphi_2) \otimes F(V_2, \exp(N_2))) \\ &= H^0(E_\tau, L(\varphi_2\varphi_1^{-1}) \otimes F(V_1^* \otimes V_2, \exp(N_2 - N_1^*))) \\ &\cong H^0(E_\tau, L(\varphi_2\varphi_1^{-1})) \otimes V_1^* \otimes V_2.\end{aligned}$$

Here, the last isomorphism $\mathcal{V}_{\varphi_2\varphi_1^{-1}, N_2 - N_1^*}$ is given in Proposition 2.23, and we use $N_2 - N_1^*$ to denote the endomorphism $\mathbf{1}_{V_1^*} \otimes N_2 - N_1^* \otimes \mathbf{1}_{V_2^*}$ of $V_1^* \otimes V_2$.

The degree of the line bundle $L(\varphi_2\varphi_1^{-1})$ is $n_2 - n_1$. Therefore, when $n_1 > n_2$, the degree is negative and there are no holomorphic sections, i.e., $\mathrm{Hom}(A_1, A_2) = 0$. Moreover, $\mathrm{Hom}(\phi_\tau(A_1), \phi_\tau(A_2)$ also vanishes because $\alpha_1 > \alpha_2$. If $n_1 = n_2$, notice that the only degree 0 line bundle that admits non-zero holomorphic sections is the trivial bundle (or the structure sheaf), thus $\mathrm{Hom}(A_1, A_2) = 0$ or $L(\varphi_1) \cong L(\varphi_2)$. If it is the first case, then both morphism spaces are zero. If it is the second case, then the problem reduces to homomorphisms of vector spaces. If $n_1 < n_2$, then

$$\mathrm{Hom}_{D^b(\mathrm{Coh}(E_\tau))}(A_1, A_2) \cong H^0(E_\tau, L(\varphi_2\varphi_1^{-1})) \otimes V_1^* \otimes V_2.$$

Moreover, one can compute that

$$\varphi_2\varphi_1^{-1} = t^*_{a_1\tau+b_1}\varphi_0 \cdot t^*_{a_2\tau+b_2}\varphi_0^{-1} \cdot \varphi_0^{n_2-n_1} = t^*_{a_{12}\tau+b_{12}}(\varphi_0^{n_2-n_1}),$$

where

$$a_{12} = \frac{a_2 - a_1}{n_2 - n_1} \text{ and } b_{12} = \frac{b_2 - b_1}{n_2 - n_1}.$$

Therefore, we have the standard basis of theta functions on $H^0(L(\varphi_2\varphi_1^{-1}))$:

$$\begin{aligned}&t^*_{a_{12}\tau+b_{12}}\theta\left[\frac{k}{n_2 - n_1}, 0\right]((n_2 - n_1)\tau, (n_2 - n_1)z) \\ &\qquad = \theta\left[\frac{k}{n_2 - n_1}, 0\right]((n_2 - n_1)\tau, (n_2 - n_1)(z + a_{12}\tau + b_{12})),\end{aligned}$$

$k \in \mathbb{Z}/(n_2 - n_1)\mathbb{Z}$. We use f_k to denote this function. On the other hand, the points of $\mathcal{L}_1 \cap \mathcal{L}_2$ can easily be found from ϕ_τ to be

$$e_k = \left(\frac{k + a_2 - a_1}{n_2 - n_1}, \frac{n_1 k + n_1 a_2 - n_2 a_1}{n_2 - n_1}\right), \ k \in \mathbb{Z}/(n_2 - n_1)\mathbb{Z}.$$

Now we can define the map ϕ_τ by mapping f_k to e_k up to a constant. To be specific, let $T \in V_1^* \otimes V_2$, then we define

$$\phi_\tau(\mathcal{V}(f_k \otimes T)) = \exp(-\pi i \tau a_{12}^2(n_2 - n_1)) \exp[a_{12}(N_2 - N_1^* - 2\pi i(n_2 - n_1)b_{12})]T \cdot e_k.$$

Case 2. Now we deal with the case where $\mathcal{L}_1 = \mathcal{L}_2$. Under this assumption, we know that $n_1 = n_2$ and $a_1 = a_2$. Therefore, $L(\varphi_2\varphi_1^{-1})$ is of degree zero. Because the trivial bundle is the only line bundle with nontrivial sections, we have $H^0(L(\varphi_2\varphi_1^{-1})) = 0$ or $\varphi_1 = \varphi_2$. If $\varphi_1 \neq \varphi_2$, then $b_1 \neq b_2$ and there does not exist a common eigenvalue of $M_1 = \exp(-2\pi i b_1 \mathbf{1}_{V_1} + N_1)$ and $M_2 = \exp(-2\pi i b_2 \mathbf{1}_{V_2} + N_2)$ because of the lemma bellow. This tells us that $\mathrm{Hom}((\mathcal{L}_1, \alpha_1, M_1)(\mathcal{L}_2, \alpha_2, M_2)) = 0$, so the spaces of morphisms are zero on both sides.

Lemma 4.2. *Let N be a nilpotent linear morphism of V and b be a real number. Then $M = \exp(-2\pi i b \mathbf{1}_V + N)$ only has the eigenvalue $\exp(-2\pi i b)$.*

Proof. Since N is nilpotent, N only has the eigenvalue 0. So in N's Jordan normal form J, its diagonal entries are all 0. Thus, $\exp(J)$ is a upper triangular matrix with all its diagonal entries equal to 1, which tells us that $\exp(J)$ only has the eigenvalue 1. Since N and J are similar, so are $\exp(N)$ and $\exp(J)$, hence $\exp(N)$ only has the eigenvalue 1. Then it is easy to see that $\exp(-2\pi i b \mathbf{1}_V + N) = \exp(-2\pi i b) \cdot \exp(N)$ only has the eigenvalue $\exp(-2\pi i b)$. $\square$

If $\varphi_1 = \varphi_2$, then the operators M_i have the same eigenvalue $\exp(-2\pi i b_1)$. This tells us that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{FK}^0(E^\tau)}(\phi_\tau(A_1), \phi_\tau(A_2)) &= H^0(\mathcal{L}_1, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) \\
&= \{f : V_1 \to V_2 | M_1 \circ f = f \circ M_2\} \\
&= \{f : V_1 \to V_2 | \exp(N_1) \circ f = f \circ \exp(N_2)\} \\
&= \mathrm{Hom}(F(V_1, \exp(N_1)), F(V_2, \exp(N_2))) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{Coh}(E_\tau))}(A_1, A_2)
\end{aligned}$$

Then we define ϕ_τ to be this isomorphism .

Step 2. Now we extend the definition of ϕ_τ to morphisms between locally free sheaves. Assume that $\mathcal{F}$ and $\mathcal{G}$ are two locally free sheaves over E_τ, then by Proposition 2.22 again, we know that $\mathcal{F} \cong \pi_{r_1 *}\mathcal{E}_1$ and $\mathcal{G} \cong \pi_{r_2 *}\mathcal{E}_2$, where r_1 and r_2 are two positive integers and $\mathcal{E}_i = L(\varphi_i) \otimes F(V_i, \exp(N_i))$ are two vector bundles (locally free sheaves) on $E_{r_i \tau}$. Then we consider the cartesian product

$$\begin{tikzcd}
E \arrow[r, "\widetilde{\pi}_{r_2}"] \arrow[d, "\widetilde{\pi}_{r_1}"'] & E_{r_2\tau} \arrow[d, "\pi_{r_2}"] \\
E_{r_1\tau} \arrow[r, "\pi_{r_1}"] & E_\tau
\end{tikzcd} .$$

That is to say $E := E_{r_1\tau} \times_{E_\tau} E_{r_2\tau}$, and we denote the projections by $\widetilde{\pi}_{r_i} : E \to E_{r_i\tau}$. When $\gcd(r_1, r_2) = 1$, $E \cong E_{r_1 r_2 \tau}$ is an elliptic curve. In general, E is a disjoint union of several elliptic curves. Concretely, we assume that $d = \gcd(r_1, r_2)$. Then $E \cong E_{r\tau} \times \mathbb{Z}/d\mathbb{Z}$ is a disjoint union of d elliptic curves, where $r := \frac{r_1 r_2}{d}$. The restriction of the map $\widetilde{\pi}_{r_i}$ to the v-th connected component is denoted to be $\pi_{r_i, v}$. It is the composition of the isogeny $\pi_{\frac{r_{3-i}}{d}} : E_{r\tau} \times \{\nu\} \to E_{r_i\tau}$ with the translation by $\nu\tau$ on $E_{r_1\tau}$ and with the identity on $E_{r_2\tau}$. We can also use translations by $s_i\tau$ on $E_{r_i\tau}$ for any pair of integers (s_1, s_2) satisfying $s_1 - s_2 \equiv \nu \mod d$. The choice of the pair does not affect our conclusion because they only differ by a translation on $E_{r\tau}$ by $s\tau$ for some $s \in \mathbb{Z}$.

Using Corollary 2.21, which tells us the adjointness properties of π_* and π^*, we have the following canonical isomorphism

$$\begin{aligned}
\mathrm{Hom}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}(\pi_{r_1 *}\mathcal{E}_1, \pi_{r_2 *}\mathcal{E}_2) \\
&\cong \mathrm{Hom}(\widetilde{\pi}^*_{r_1}\mathcal{E}_1, \widetilde{\pi}^*_{r_2}\mathcal{E}_2) \\
&= \bigoplus_{\nu=1}^{d} \mathrm{Hom}(\pi^*_{r_1,\nu}\mathcal{E}_1, \pi^*_{r_2,\nu}\mathcal{E}_2)
\end{aligned}$$

On the other hand, we can do the similar construction in the symplectic side. To be specific, we also consider the following Cartesian diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\widetilde{p}_{r_2}"] \arrow[d, "\widetilde{p}_{r_1}"'] & E^{r_2\tau} \arrow[d, "p_{r_2}"] \\ E^{r_1\tau} \arrow[r, "p_{r_1}"] & E^{\tau} \end{tikzcd}$$

Here, $\widetilde{E} \coloneqq E^{r_1\tau} \times_{E^\tau} E^{r_2\tau}$ and p_{r_i} are the corresponding projections. Similar to the case of coherent sheaves, we know that $\widetilde{E} = E^{r\tau} \times \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(r_1, r_2)$ and $r = \frac{r_1 r_2}{d}$. We denote $p_{r_i,\nu}$ to be the map $\widetilde{p}_{r_i}$ stricted to the ν-th connected components. Similar to $\pi_{r_i,\nu}$, $p_{r_i,\nu}$ is the composition of the map $p_{\frac{r_{3-i}}{d}} : E^{r\tau} \times \{\nu\} \to E^{r_i\tau}$ with a translation of the form $(x, y) \mapsto (x-n, y)$, where n is determined by corresponding translations on the elliptic curve $E_{r_i\tau}$.

Since p_* and p^* are also adjoint, there exists a functorial isomorphism

$$\begin{aligned} &\mathrm{Hom}(p_{r_1*}(\mathcal{L}_1, \alpha_1, \mathcal{E}_1), p_{r_2*}(\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) \\ &\qquad\qquad \cong \bigoplus_{\nu=1}^{d} \mathrm{Hom}(p^*_{r_1,\nu}(\mathcal{L}_1, \alpha_1, \mathcal{E}_1), p^*_{r_2,\nu}(\mathcal{L}_2, \alpha_2, \mathcal{E}_2)). \end{aligned}$$

Combining these isomorphisms, we can define ϕ_τ by the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\pi_{r_1*}\mathcal{E}_1, \pi_{r_2*}\mathcal{E}_2) \arrow[r, "\cong"] \arrow[dddd, "\phi_\tau"] & \oplus \mathrm{Hom}(\pi^*_{r_1,\nu}\mathcal{E}_1, \pi^*_{r_2,\nu}\mathcal{E}_2) \arrow[d, "\oplus\phi_{r\tau}"] \\ & \oplus \mathrm{Hom}(\phi_{r\tau}(\pi^*_{r_1,\nu}\mathcal{E}_1), \phi_{r\tau}(\pi^*_{r_2,\nu}\mathcal{E}_2)) \arrow[d, "\cong"] \\ & \oplus \mathrm{Hom}(p^*_{r_1,\nu}\phi_{r_1\tau}(\mathcal{E}_1), p^*_{r_2,\nu}\phi_{r_2\tau}(\mathcal{E}_2)) \\ & \\ \mathrm{Hom}(\phi_\tau(\pi_{r_1*}\mathcal{E}_1), \phi_\tau(\pi_{r_2*}\mathcal{E}_2)) \arrow[r, "="] & \mathrm{Hom}(p_{r_1*}\phi_{r_1\tau}(\mathcal{E}_1), p_{r_2*}\phi_{r_2\tau}(\mathcal{E}_2)). \arrow[uu, "\cong"'] \end{tikzcd}$$

Here, we use the isomorphism $\phi_{r\tau}(\pi^*(\mathcal{E})) \cong p^*(\phi_\tau(\mathcal{E}))$, where $\mathcal{E} \cong L(\varphi) \otimes F(V, \exp(N))$ and π is an isogeny, and the compatibility of translations with ϕ. And notice that $\pi^*_{r_1,\nu}\mathcal{E}_i$ still have the form of $L(\varphi') \otimes F(V', \exp(N'))$, so we can apply $\phi_{r\tau}$ to the morphism space between then as in STEP 1.

STEP 3. Now we have to deal with the case where A_1 or A_2 is a torsion sheaf. By Serre Duality, we know that $\mathrm{Hom}(A_1, A_2) \cong \mathrm{Ext}^1(A_2, A_1) = 0$ when A_1 is a torsion sheaf and A_2 is locally free. Meanwhile, one can obtain $\alpha_1 = \frac{1}{2} > \alpha_2$ by definition, and thus $\mathrm{Hom}((\mathcal{L}_1, \alpha_1, \mathcal{E}_1), (\mathcal{L}_2, \alpha_2, \mathcal{E}_2)) = 0$ and everything fits nicely. The only case that remains is when $A_2 = S(a_2\tau + b_2, V_2, N_2)$ is a torsion sheaf. Now, we have two cases to consider: A_1 is a locally free sheaf or A_1 is a torsion sheaf.

Case 1. In this case, we assume that A_1 is a locally free sheaf.

Case 1.1. Since every locally free sheaf is isomorphic to the push-forward of a vector bundle of the form $L(\varphi_1)\otimes F(V_1,\exp(N_1))$, we first consider the case where $A_1 = L(\varphi_1)\otimes F(V_1,\exp(N_1))$. Since A_2 has only one non-zero stalk, V_2 at $a_2\tau+b_2$, we have

$$\mathrm{Hom}(A_1,A_2)=\mathrm{Hom}(V_1,V_2)\cong V_1^*\otimes V_2.$$

On the other hand, $\mathcal{L}_1\cap\mathcal{L}_2$ has only one point in this case, thus

$$\mathrm{Hom}(\phi_\tau(A_1),\phi_\tau(A_2))=\mathrm{Hom}(V_1,V_2)\cong V_1^*\otimes V_2.$$

And the isomorphism $\phi_\tau : V_1^*\otimes V_2\to V_1^*\otimes V_2$ is defined by

$$\exp[-\pi i\tau(na_2^2+2a_1a_2)-2\pi i(a_2b_1+a_1b_2+na_2b_2)]\cdot \\ \exp[-(a_1+na_2)\cdot \mathbf{1}_{V_1^*}\otimes N_2+a_2\cdot {}^tN_1\otimes \mathbf{1}_{V_2}].$$

Case 1.2. Now we assume that A_1 is an arbitrary locally free sheaf. Then $A_1\cong \pi_{r*}\mathcal{E}_1$ for some isogeny π_r and $\mathcal{E}_1 = L(\varphi_1)\otimes F(V_1,\exp(N_1))$. We define ϕ_τ by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_{r*}\mathcal{E}_1,A_2) \arrow[ddd, "\phi_\tau"] & \mathrm{Hom}(\mathcal{E}_1,\pi_r^*A_2) \arrow[l, "\cong"'] \arrow[d, "\cong"', "\phi_{r\tau}"] \\
 & \mathrm{Hom}(\phi_{r\tau}(\mathcal{E}_1),\phi_{r\tau}(\pi_r^*A_2)) \arrow[d, "\cong"'] \\
 & \mathrm{Hom}(\phi_{r\tau}(\mathcal{E}_1),p_r^*\phi_\tau(A_2)) \arrow[d, "\cong"'] \\
\mathrm{Hom}(\phi_\tau(\pi_{r*}\mathcal{E}_1),\phi_\tau(A_2)) & \mathrm{Hom}(p_{r*}\phi_{r\tau}(\mathcal{E}_1),\phi_\tau(A_2)). \arrow[l, "="]
\end{tikzcd}$$

Here, we use the isomorphism $\phi(\pi_r^*A_2)\cong p_r^*\phi(A_2)$, which can be easily verified from the definitions. Notice that $\mathcal{E}_1$ has the form of $L(\varphi_1)\otimes F(V_1,\exp(N_1))$, so we can apply $\phi_{r\tau}$ to the morphism space $\mathrm{Hom}(\mathcal{E}_1,\pi_r^*A_2)$ as in *Case 1.1.*

Case 2. Now we discuss the second case where $A_1 = S(a_1\tau+b_1,V_2,N_2)$ is also a torsion sheaf. If A_1 and A_2 have different support, then $\mathrm{Hom}(A_1,A_2)=\mathrm{Ext}^1(A_1,A_2)=0$. On the symplectic side, $\phi(A_i)=(\mathcal{L}_i,\frac{1}{2},M_i)$ where $\mathcal{L}_i=(-a_i,t)$. Therefore, if $a_1\neq a_2$, then $\mathcal{L}_1\cap\mathcal{L}_2=\emptyset$ and $\mathrm{Hom}(\phi_\tau(A_1),\phi_\tau(A_2))=0$. If $a_1=a_2$ but $b_1\neq b_2$, then M_1 and M_2 do not have common eigenvalues. Therefore, we have

$$H^0(\mathcal{L}_1,\mathcal{H}om(M_1,M_2))=H^1(\mathcal{L}_1,\mathcal{H}om(M_1,M_2))=0,$$

and thus

$$\mathrm{Hom}(\phi_\tau(A_1),\phi_\tau(A_2))=0.$$

Finally, if A_1 and A_2 have the same support, i.e., $a_1=a_2$ and $b_1=b_2$, then

$$\begin{aligned}
\mathrm{Hom}(A_1,A_2) &= \mathrm{Hom}_{\mathcal{O}_{E^\tau,a_1\tau+b_1}}((V_1,N_1),(V_2,N_2))\\
&=\{f\in\mathrm{Hom}(V_1,V_2)\mid f\circ N_1=N_2\circ f\}\\
&=\{f\in\mathrm{Hom}(V_1,V_2)\mid f\circ M_1=M_2\circ f\}\\
&=H^0(\mathcal{L}_1,\mathcal{H}om(\mathcal{E}_1,\mathcal{E}_2))\\
&=\mathrm{Hom}(\phi_\tau(A_1),\phi_\tau(A_2)).
\end{aligned}$$

We have finished the definition of the functor ϕ_τ. The next step is to verify that ϕ_τ is indeed a functor, i.e., we have to show the compatibility of ϕ with compositions or to prove the commutativity of the following diagram

$$\begin{tikzcd} \mathrm{Hom}(A_1, A_2[k]) \otimes \mathrm{Hom}(A_2[k], A_3[l]) \arrow[r] \arrow[d, "\phi_\tau\otimes\phi_\tau"] & \mathrm{Hom}(A_1, A_3[l]) \arrow[d, "\phi_\tau"] \\ \mathrm{Hom}(\phi_\tau(A_1), \phi_\tau(A_2)[k]) \otimes \mathrm{Hom}(\phi_\tau(A_2)[k], \phi_\tau(A_3)[l]) \arrow[r] & \mathrm{Hom}(\phi_\tau(A_1), \phi_\tau(A_3)[l]). \end{tikzcd}$$

To have non-zero morphism spaces in the diagram, we have to require that $0 \leq k \leq l \leq 1$. We denote $\phi(A_i) = (\mathcal{L}_i, \alpha_i, M_i)$. If $l = 1$, then using the canonical isomorphism $\mathrm{Hom}(V_1 \otimes V_2^*, V_3^*) \cong \mathrm{Hom}(V_3 \otimes V_1, V_2)$ and the isomorphism $\mathrm{Hom}(A, B[1]) \cong \mathrm{Hom}(B, A)^*$ in both categories, we know that the diagram above is equivalent to the following diagram

$$\begin{tikzcd} \mathrm{Hom}(A_3, A_1) \otimes \mathrm{Hom}(A_1, A_2) \arrow[r] \arrow[d, "\phi_\tau\otimes\phi_\tau"] & \mathrm{Hom}(A_3, A_2) \arrow[d, "\phi_\tau"] \\ \mathrm{Hom}(\phi_\tau(A_3), \phi_\tau(A_1)) \otimes \mathrm{Hom}(\phi_\tau(A_1), \phi_\tau(A_2)) \arrow[r] & \mathrm{Hom}(\phi_\tau(A_3), \phi_\tau(A_2)). \end{tikzcd}$$

Therefore, we only have to deal with the case where $k = l = 0$.

The detailed proof of this case can be found in section 4 of Kreussler's paper [3], and we will omit it here.

Now, we have constructed a functor $\phi_\tau : D^b(\mathrm{Coh}(E_\tau)) \to \mathcal{FK}^0(E^\tau)$ that is, by definition, additive, fully faithful and compatible with shift functors. To prove our main theorem that ϕ_τ is an equivalence, we only need to prove that any indecomposible object in $\mathcal{FK}^0(E^\tau)$ is isomorphic to an object of the form $\phi_\tau(A)$, where A is a vector bundle or a skyscraper sheaf on E_τ. Let $(\mathcal{L}, \alpha, M)$ be an indecomposible object in $\mathcal{FK}^0(E^\tau)$. Then recall that $(\mathcal{L}, \alpha, M_1) \oplus (\mathcal{L}, \alpha, M_2) = (\mathcal{L}, \alpha, M_1 \oplus M_2)$, thus $(\mathcal{L}, \alpha, M)$ is indecomposible implies that M is indecomposible. Therefore, there exists only one Jordan block in M's Jordan normal form. Moreover, since we only consider locally free sheaves whose monodromy only has eigenvalues of modulus one, the diagonal entries of M's Jordan form should be the same complex number with modulus one. Therefore, we can describe M, up to conjugation, as

$$M = \exp(-2\pi i b + N) \in \mathrm{GL}(V),$$

where b is a real number and N is a cyclic nilpotent endomorphism of V. Because ϕ_τ is compatible with the shift functors, we can assume that $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$. If $\alpha = \frac{1}{2}$, then the line $\mathcal{L}$ is perpendicular to the x-axis, and we denote $a \in (-1, 0]$ to be the x-intercept of a line in $\mathbb{R}^2$ that represents $\mathcal{L}$. One can easily verify that $\phi_\tau(S(-a\tau - b, V, N)) = (\mathcal{L}, \alpha, M)$. If $\alpha < \frac{1}{2}$, we first fix a pair of relatively prime nonnegative integers (r, n) such that $\frac{n}{r}$ is the slope of the line passing through the origin and $\exp(i\pi\alpha)$, i.e., $r + in$ is a real multiple of $\exp(i\pi\alpha)$. Next, we can determine a real number a by requiring $\frac{ra}{n} \in [0, \frac{1}{n})$ to be the smallest nonnegative x-intercept of $\mathcal{L}$. Then we define $\varphi = t^*_{ra\tau+b}\varphi_0 \cdot \varphi_0^{n-1}$. And one can easily verify that $\phi_\tau(\pi_{r*}(L_{r\tau}(\varphi) \otimes F_{r\tau}(V, \exp(N)))) = (\mathcal{L}, \alpha, M)$. In conclusion, any object in $\mathcal{FK}^0(E^\tau)$ is isomorphic to an object in the image of ϕ_τ, and ϕ_τ is indeed an equivalence from $D^b(\mathrm{Coh}(E_\tau))$ to $\mathcal{FK}^0(E^\tau)$.

Appendix A. Minimal, Calibrated, and Special Lagrangian Submanifolds

A detailed discussion of these three kinds of submanifolds can be found in Port's paper [8]. Moreover, D. Lotay's paper [5] also serves as a decent reference. We will be satisfied with presenting the main theorems without proof here.

First, we define the notion of minimal submanifolds of a Riemannian manifold.

Let M be a Riemannian manifold. The Riemannian metric of M restricts to any submanifold of it. Let S be a submanifold of M. Then the tangent bundle of M splits into two orthonormal parts when restricted to S: $M|_S = TS \oplus NS$. Now, we can define the Levi-Civita connections ∇^M for M and ∇^S for S. Then we define the *second fundamental form* B by $B(X,Y) = \nabla^M_X Y - \nabla^S_X Y$. Then B is a symmetric bilinear form that takes value in NS. The *mean curvature* H is then given by the average of the eigenvalues of the second fundamental form, i.e., $H = \frac{1}{n}\operatorname{Trace}(B)$, where n is the dimension of S. Using the mean curvature, we can define the notion of minimal submanifolds.

Definition A.1. Let M be a Riemann manifold and S be a submanifold of M. Then S is called a *minimal submanifold* of M if its mean curvature $H = 0$.

Remark A.2. Geometrically, one can prove that a minimal manifold is a submanifold such that any small deformation of its embedding does not change its volume. This can be seen by calculating the first variation formula. The detailed proof can be found in Port's paper, so we will omit it here.

Next, we define the notion of calibrated submanifolds of a Riemannian manifold.

Definition A.3. Let M be a Riemannian manifold. The Riemannian metric induces a volume form vol_V on any subspace $V \subset T_xM$ and any $x \in M$. Then a k-form η is called a *calibration* on M if it is closed and $\eta|_V = \lambda \cdot \mathrm{vol}_V$ for some $\lambda \leq 1$ for any oriented k-dimensional subspace $V \subset T_xM$ and any $x \in M$. A submanifold $N \hookrightarrow M$ is *calibrated* with respect to calibration η (or *η-calibrated*) if $\eta|_{T_xN} = \mathrm{vol}_V$ for all $x \in N$.

It turns out that a calibrated submanifold (with respect to any calibration) is always a minimal submanifold. To be specific, we have the following proposition.

Proposition A.4. *Let M be a Riemannian manifold and η be a calibration on M. Let N be a compact calibrated submanifold of M with respect to η. Then N is volume minimizing in its homology class.*

Proof. Let N' be a submanifold of M such that it belongs to the same homology class of N. Then we have

$$\int_N \mathrm{vol}_N = \int_N \eta = \int_{N'} \eta \leq \int_{N'} \mathrm{vol}_{N'} .$$

Here, the first equation holds because N is a calibrated submanifold, and the inequality holds because η is a calibration on M. □

Remark A.5. Since any small variation of N results in a submanifold lying in the same homology class of N, the proposition above tells us that a calibrated submanifold is always volume-minimizing, and thus is a minimal manifold defined above.

Finally, we define the notion of special Lagrangian submanifolds of a Calabi-Yau manifold.

Definition A.6. Let $\widetilde{M}$ be a Calabi-Yau manifold with the Calabi-Yau form $\Omega = \Omega_1 + i\Omega_2$. Than L is a special Lagrangian submanifold of $\widetilde{M}$ if L is Lagrangian and $\Omega_2|_L = 0$.

The main goal of this section is the following proposition.

Proposition A.7. *Let $\widetilde{M}$ be a Calabi-Yau manifold with its Calabi-Yau form $\Omega = \Omega_1 + i\Omega_2$. Then the special Lagrangian submanifolds of $\widetilde{M}$ are the Ω_1-calibrated submanifolds, and thus they are minimal submanifolds of $\widetilde{M}$.*

The proof of this proposition can also be found in Port's paper [8], and we will not repeat it here.

In this paper, we are only concerned with the case where $\widetilde{M}$ is a torus. In this case, all 1-dimensional minimal submanifolds of $\widetilde{M}$ are just geodesics, i.e., the image of lines in $\mathbb{R}^2$ under the map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \cong M$. Moreover, to define a closed submanifold of M, the slope of the line must be rational. We can identify the slope with a pair of coprime intergers (p, q), and the slope is p/q. Apart from the slope, in order to fix the line, we need to know its interception point with the y-axis (or x-axis if q=0).

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