CALCULUS OF VARIATIONS: MINIMAL SURFACE OF REVOLUTION

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Abstract. Finding minimal surfaces of revolution is a classical problem solved by calculus of variations. We will first present a classical catenoid solution using calculus of variation, and we will then discuss the conditions of existence by considering the maximum separation between two rings. Finally, we will extend to solving constant-mean-curvature surface of revolution using Lagrange multipliers and calculus of variations, investigating the constants involved in different Delaunay surfaces.

Contents

1. Introduction

The calculus of variations is a field of mathematical analysis that seeks to find the path, curve, surface, etc. that minimizes or maximizes a given function. It involves finding a function $u(x)$ that produces extreme values in a **functional**—i.e. a definite integral involving the function and its derivative:

$$\mathcal{F}(u) = \int_a^b f(x, u(x), u'(x))dx.$$

The brachistochrone problem is often considered the birth of calculus of variations: to find the curve of fastest decent from a point A to a lower point B. Other problems like the hanging cable problem, the isoperimetric problem, and the minimal surfaces of revolution problem also employ calculus of variations. This paper will focus on investigating *Minimal Surface of Revolution*—specifically finding a solution to the zero mean curvature problem and extending to constant mean curvature problems. Before diving into the problem, some useful tools will be introduced.

1.1. **The Euler-Lagrange Equation.** Let's first consider the minimization problem in $\mathbb{R}^2$. Consider a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. There are many ways to find a minimum point of f, and one way is by studying the directional derivatives. It is a necessary condition for the minimum (a, b) to have zero directional derivatives in any direction $\mathbf{u}$:

$$\nabla_{\mathbf{u}} f = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = 0, \ \ \forall \mathbf{u} \in \mathbb{R}^2.$$

Now, in higher dimensions, we seek to find a function u that minimizes a functional of the form

$$\mathcal{F}(u) = \int_a^b f(x, u(x), u'(x))dx,$$

where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. We use the same method of finding directional derivatives. The direction can now be an arbitrary function $\phi \in \{w \in C^1([a, b]) : w(a) = w(b) = 0\}$. If $\mathcal{F}$ attains minimum at $u \in C^2[a, b]$, then for any $\epsilon > 0$, we should have

$$\mathcal{F}(u) \leq \mathcal{F}(u + \epsilon\phi).$$

Adding $\epsilon\phi$ to u can be interpreted as slightly deforming u in the direction of ϕ. The term $\epsilon\phi$ is thus called **variation** of the function u. Substituting $y = u + \epsilon\phi$, we obtain a function of ϵ:

$$\Phi(\epsilon) = \mathcal{F}(u + \epsilon\phi).$$

We can rewrite $f = f(x, p, \xi)$, where $p = u + \epsilon\phi$ and $\xi = u' + \epsilon\phi'$. Given $\mathcal{F}$ attains a minimum at u, the first derivative of Φ should equal zero when $\epsilon = 0$ in any direction ϕ:

$$\begin{aligned}\Phi'(0) &= \int_a^b \frac{d\mathcal{F}}{d\epsilon}\bigg|_{\epsilon=0} dx \\ &= \int_a^b [f_p(x, u(x), u'(x))\phi(x) + f_\xi(x, u(x), u'(x))\phi'(x)]dx \\ &= 0.\end{aligned}$$

This is called the weak Euler-Lagrange Equation.

To obtain a stronger version, we assume $u \in C^2([a, b])$ and then integrate by parts:

$$\begin{aligned}&\int_a^b f_p(x, u(x), u'(x))\phi(x)dx + [f_\xi(x, u(x), u'(x))\phi(x)]_b^a \\ &- \int_a^b [\frac{d}{dx} f_\xi(x, u(x), u'(x))]\phi(x)dx \\ &= \int_a^b [f_p(x, u(x), u'(x)) - \frac{d}{dx} f_\xi(x, u(x), u'(x))]\phi(x)dx \\ &= 0\end{aligned}$$

Since this holds for every ϕ, we obtain the strong Euler-Lagrange equation:

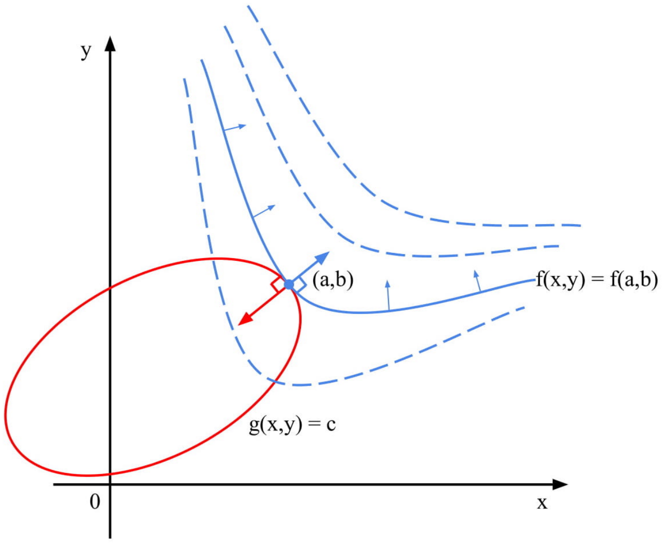
$$f_p(x, u(x), u'(x)) = \frac{d}{dx} f_\xi(x, u(x), u'(x))$$

It is important to note that the Euler-Lagrange equation may not have a solution. This observation becomes useful in solving the minimal surfaces of revolution problem.

1.2. **Lagrangian Multipliers.** In addition to the Euler-Lagrange equation, the Lagrangian multiplier method is useful in solving more complicated optimization problems that are subject to given equality constraints. In this paper, we will focus on single-constraint Lagrangians.

Consider the following problem. Given two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with continuous first derivatives, find points $(a, b) \in \mathbb{R}^2$ that maximize/minimize f with the constraint that $g(a, b) = c$ for some $c \in \mathbb{R}$. Another way of interpreting the problem is to consider the contour line of $g = c$. Along this line, we aim to find the extremas of f; in other words, find the points (a, b) where f does not change to the first order. This can happen in two cases:

(1) f has zero gradient at (a, b), or
(2) The contour line $f = f(a, b)$ is parallel to $g = c$ at (a, b).



Let's focus on the second case. Since the gradient is always perpendicular to the contour line, having two parallel contour lines is equivalent to having parallel gradient at (a, b). Therefore, for some $\lambda \in \mathbb{R}$,

$$\nabla f(a, b) = \lambda \nabla g(a, b). \tag{1.1}$$

This equation holds true in the first case too, when $\lambda = 0$. Therefore, $f(a, b)$ is an extrema if there exists some $\lambda \in \mathbb{R}$ that satisfies the following set of equations:

$$\begin{cases} \nabla f(a, b) = \lambda \nabla g(a, b) \\ g(a, b) = c. \end{cases} \tag{1.2}$$

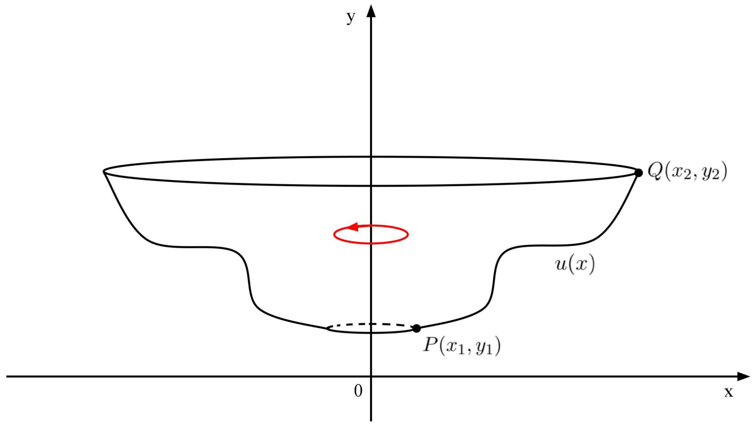
To express (1.2) in a more succinct way, we define the *Lagrangian*,

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c), \tag{1.3}$$

where λ is called the *Lagrangian multiplier*. The contrained extremas of f are thus the critical points of the Lagrangian: $\nabla \mathcal{L} = 0$. This can be easily generalized to finite dimensions with f and g having continuous first partial derivatives. Furthermore, Lagrange multipliers can be generalized to infinite-dimensional problems, although then it is not as obvious.

2. Minimal Surface of Revolution

With all the tools introduced, we are ready to find minimal surfaces of revolution. Given two points P, Q in a half-plane with coordinates (x_1, y_1) and (x_2, y_2), consider a curve $u(x)$ connecting P and Q. The surface of revolution is generated by rotating the curve with respect to y-axis. We aim to find the curve that minimizes the surface area. Another interpretation is to find minimal surfaces connecting two rings of radius x_1 and x_2. A natural physical model is the soap film formed between two wire rings of different radius, separated at a given distance. Depending on P and Q's positions, the problem has either a continuous or discontinuous solution.



2.1. **Continuous solution: Catenoid.** By assuming a continuous solution in C^2, we will be able to use calculus of variations to find the minimizing curve.

Let $u(x) \in C^2([x_1, x_2])$ be a minimizer. The surface area of revolution is obtained by integrating over cylinders of radius x:

$$\begin{aligned}\text{Area} &= \int_{x_1}^{x_2} 2\pi x ds \\ &= 2\pi \int_{x_1}^{x_2} x\sqrt{1+(u'(x))^2}dx.\end{aligned}$$

As the Lagrangian is in the form $f = f(x, \xi) = x\sqrt{1+\xi^2}$, the Euler-Lagrange equation becomes

$$\begin{aligned}&\frac{d}{dx} f_\xi(x, u'(x)) \\ &= \frac{d}{dx}\frac{xu'(x)}{\sqrt{1+(u'(x))^2}} \\ &= 0.\end{aligned} \tag{2.1}$$

To solve it, we integrate both sides of the equation and get $f_\xi(x, u'(x)) = a$ for some constant $a \in \mathbb{R}$. Using algebraic transformations, we obtain $u'(x) = \frac{a}{\sqrt{x^2-a^2}}$. Therefore,

$$u(x) = \int \frac{a}{\sqrt{x^2-a^2}}dx.$$

Substitute $x = a \cdot \cosh(r)$ and $dx = a \cdot \sinh(r)dr$:

$$\begin{aligned}u(x) &= \int \frac{a^2 \sinh(r)}{\sqrt{a^2(\cosh^2(r)-1)}}dr \\ &= \int \frac{a^2 \sinh(r)}{a \sinh(r)}dr \\ &= \int a\, dr \\ &= ar + b \\ &= a \cdot \cosh^{-1}\left(\frac{x}{a}\right) + b.\end{aligned}$$

The above solution is the inverse of a catenary curve: $w(x) = u^{-1}(x) = a \cdot \cosh(\frac{x-b}{a})$ for some $a, b \in \mathbb{R}$. The surface of revolution is called a catenoid.

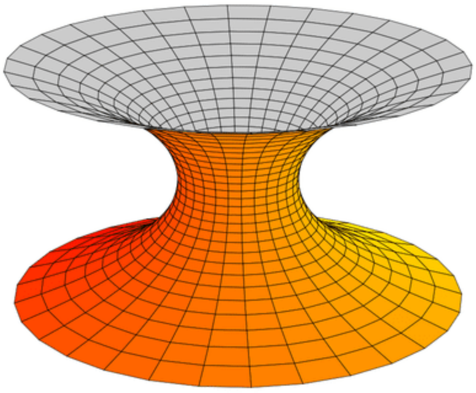
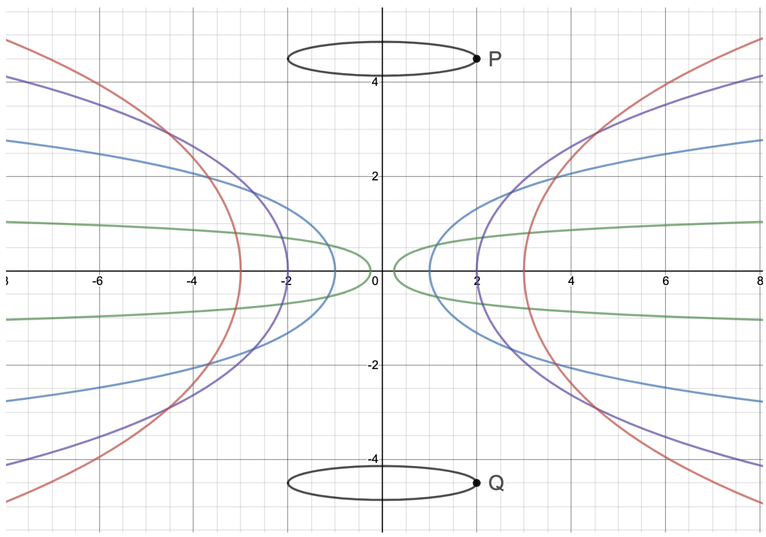


FIGURE 1. Graph from Karakoc, Selcuk. (2017). On Minimum Homotopy Areas. 10.13140/RG.2.2.36102.78405.

To get a solution that satisfies the boundary conditions, we need to pick a and b such that the following system of equations are satisfied:

$$\begin{cases} x_1 = a \cdot \cosh(\frac{y_1 - b}{a}) \\ x_2 = a \cdot \cosh(\frac{y_2 - b}{a}). \end{cases}$$

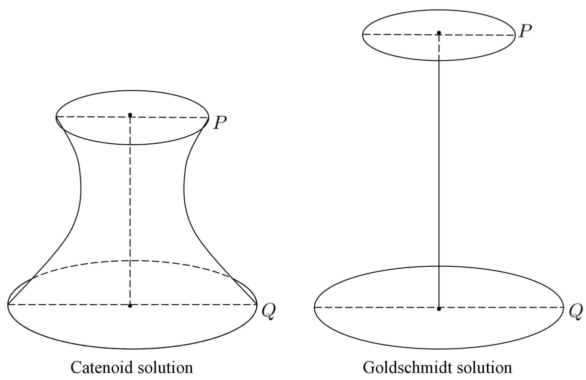
However, it is obvious for some boundary conditions, the above solution does not exist. The figure below shows the family of $\cosh^{-1}$ functions centered around the x-axis. Clearly, two small rings separated far apart about the x-axis, as shown in the figure, cannot be connected by catenary solution.



2.2. **Discontinuous solution: Goldschmidt Solution.** When the previous method fails, it is impossible to find a smooth connected surface of revolution connecting the two rings. But we can find a disconnected solution that consists of two disks within each ring. This is called Goldschmidt solution—a discontinuous function that is non-zero at the boundary points and zero everywhere else. The function revolves into two disks with P and Q on their edges, and a line connecting the centers. In the soap film model, Goldschmidt solution is manifested by two separate soap films inside the rings. We will focus on investigating what boundary conditions give rise to such discontinuous solution.

Note that the Goldschmidt solution always exists as a local minimum, but it only becomes the absolute minimum when

(1) the catenary solution does not exist, or
(2) the catenary solution gives an area greater than Goldschmidt's two disks.



Given the radius of two rings, we aim to find the range of separation where the catenary solution is the absolute minimum, and automatically the complement is the Goldschmidt solution range.

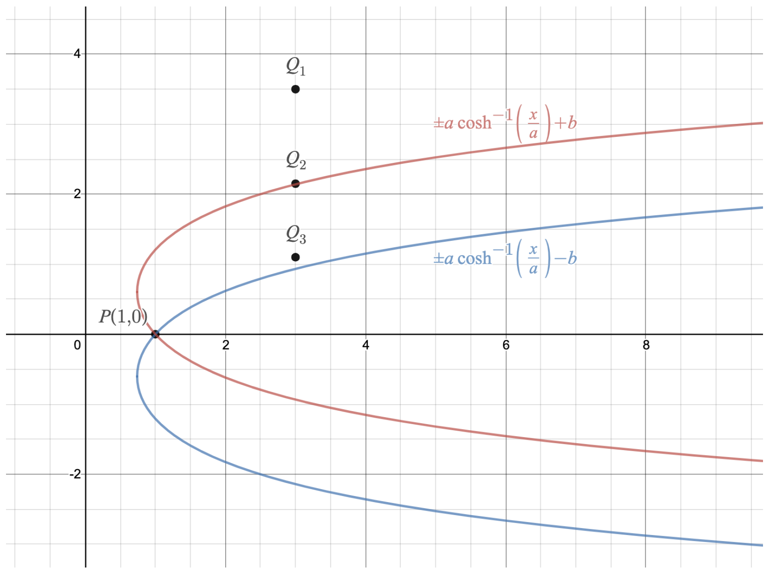
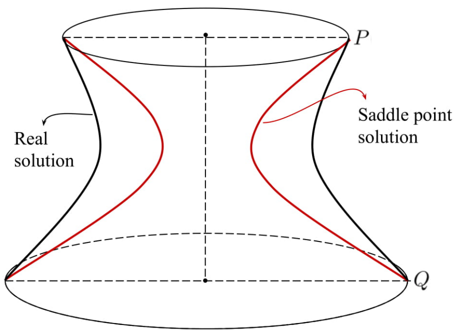


FIGURE 2. Graph of two possible catenaries passing through P. Q_1 has no catenary solution, Q_2 has exactly one, and Q_3 has two.

2.2.1. *First case.* For any P, Q, by rotating and scaling, we can assume P is at $(1,0)$ and Q is at (x_1, y_1) where $x_1, y_1 \geq 0$. Then, we have the following set of equations:

$$\begin{cases} 0 = \pm a \cdot \cosh^{-1}(\frac{1}{a}) + b \\ y_1 = \pm a \cdot \cosh^{-1}(\frac{x_1}{a}) + b \end{cases} \tag{2.2}$$

Note, to be consistent with the original setup, we are representing catenary with $\cosh^{-1}$ functions, so we need both the positive and negative $\cosh^{-1}$ to complete the full curve. Depending on the position of (x_1, y_1), solving (2.2) can give two, one, or zero solution for a. In the case of two solutions, one will lead to the absolute minimum area catenary, while the



other represents a saddle point solution. In the case of zero solutions, the catenary solution fails.

We aim to find the boundary values of y_1, given the value of x_1, such that a catenary solution *exists* between P and Q. It is clear that zero is the minimum value of y_1, in which case $u(x)$ is simply a segment of the x-axis connecting P and Q, and the surface of revolution is a horizontal washer. Finding the maximum value of y_1 is much more challenging and interesting. Translated into the soap film model, we are looking for the maximum separation between two rings of radius 1 and x_1 so that a connecting soap film can form.

Simplifying (2.2), we obtain $b = \pm a \cdot \cosh^{-1}(\frac{1}{a})$ and $y_1 = \pm a \cdot \cosh^{-1}(\frac{x_1}{a}) \pm a \cdot \cosh^{-1}(\frac{1}{a})$. Note $0 < a \leq 1$ needs to be true for $\cosh^{-1}(\frac{1}{a})$ to exist, so the maximum value of y_1 must be in the form

$$y_1 = a \cdot \cosh^{-1}(\frac{x_1}{a}) + a \cdot \cosh^{-1}(\frac{1}{a}) \tag{2.3}$$

For y_1 to be the maximum, taking the derivative of (2.3) in respect to a must result in zero:

$$\cosh^{-1}(\frac{x_1}{a}) + \cosh^{-1}(\frac{1}{a}) - \frac{x_1}{\sqrt{{x_1}^2 - a^2}} - \frac{1}{\sqrt{1 - a^2}} = 0 \tag{2.4}$$

This equation is difficult to solve, but by graphing

$$g(a) = \cosh^{-1}(\frac{x_1}{a}) + \cosh^{-1}(\frac{1}{a}) - \frac{x_1}{\sqrt{{x_1}^2 - a^2}} - \frac{1}{\sqrt{1 - a^2}},$$

it is clear that there exists a single root a_0 that will maximize y_1 (see Figure 3). To prove the uniqueness algebraically, we first realize g only exists in $(0, 1)$. Then, we need to show:

(1) $g(a)$ is decreasing,
(2) $g(a)$ approaches positive infinity as a approaches 0, and
(3) $g(a)$ approaches negative infinity as a approaches 1.

Proof. (1) For any $\alpha, \beta \in (0, 1)$, if $\alpha < \beta$, since $\cosh^{-1}(x)$ is a strictly increasing function, we have

$$\cosh^{-1}(\frac{x_1}{\alpha}) + \cosh^{-1}(\frac{1}{\alpha}) > \cosh^{-1}(\frac{x_1}{\beta}) + \cosh^{-1}(\frac{1}{\beta}).$$

Furthermore, it is clear that

$$-\frac{x_1}{\sqrt{{x_1}^2 - \alpha^2}} - \frac{1}{\sqrt{1 - \alpha^2}} > -\frac{x_1}{\sqrt{{x_1}^2 - \beta^2}} - \frac{1}{\sqrt{1 - \beta^2}}.$$

Thus, $g(\alpha) > g(\beta)$, and g is decreasing.

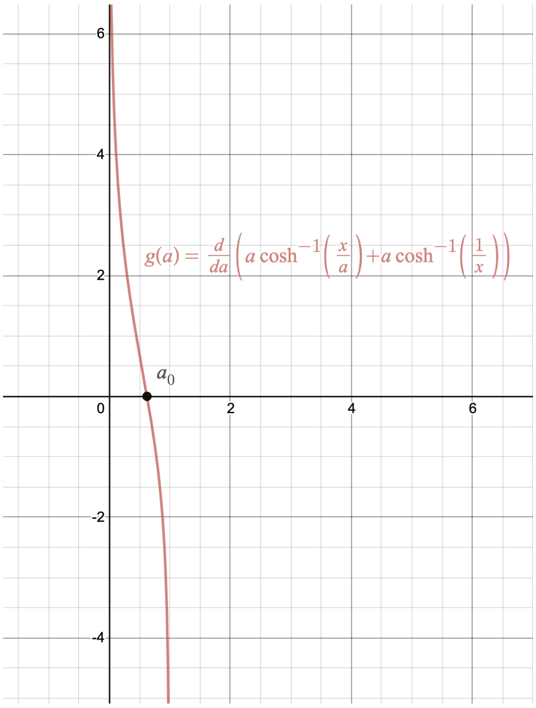


FIGURE 3. $g(a)$ with a_0 as the root that will maximize y_1

(2) When a approaches 0, $\cosh^{-1}(\frac{x_1}{a}) + \cosh^{-1}(\frac{1}{a})$ approaches positive infinity, and $-\frac{x_1}{\sqrt{x_1{}^2-a^2}} - \frac{1}{\sqrt{1-a^2}}$ approaches $-\frac{x_1}{\sqrt{x_1{}^2}} - 1 = -2$. Thus, $g(a)$ approaches positive infinity as $a \to 0$.

(3) When a approaches 1, $\cosh^{-1}(\frac{x_1}{a})+\cosh^{-1}(\frac{1}{a})$ approaches $2\cosh^{-1}(x_1)$, and $-\frac{x_1}{\sqrt{x_1{}^2-a^2}}$ $-\frac{1}{\sqrt{1-a^2}}$ approaches negative infinity. Thus, $g(a)$ approaches negative infinity as $a \to 1$.

As we proved that g is decreasing, has at least one point positive, and one point negative in its domain $(0,1)$, g has an unique root. $\square$

Having proven the uniqueness, we can easily find the maximum separation y_1 by plugging the root a_0 into (2.3).

2.2.2. *Second case.* Although we found a catenary solution between P and Q, the catenoid it forms may not have a smaller area than Goldschimdt's two disks. To study the area more easily, we will rotate the previous setup and use x-axis as the rotation axis, representing the catenary with cosh functions.

Starting with $P(0,1)$ and $Q(x_2, y_2)$ where $x_2, y_2 > 0$, we aim to find the separation x_2 where the catenoid has a smaller area than two disks. The following set of equations should be satisfied:

$$\begin{cases} 1 = a \cdot \cosh(\frac{0-b}{a}) \\ y_2 = a \cdot \cosh(\frac{x_2-b}{a}). \end{cases} \tag{2.5}$$

Since the catenary is in the form $u(x) = a \cdot \cosh(\frac{x-b}{a})$, we have

$$\sqrt{1 + (u'(x))^2} = \sqrt{1 + \sinh^2(\frac{x-b}{a})} = \frac{1}{a}u(x).$$

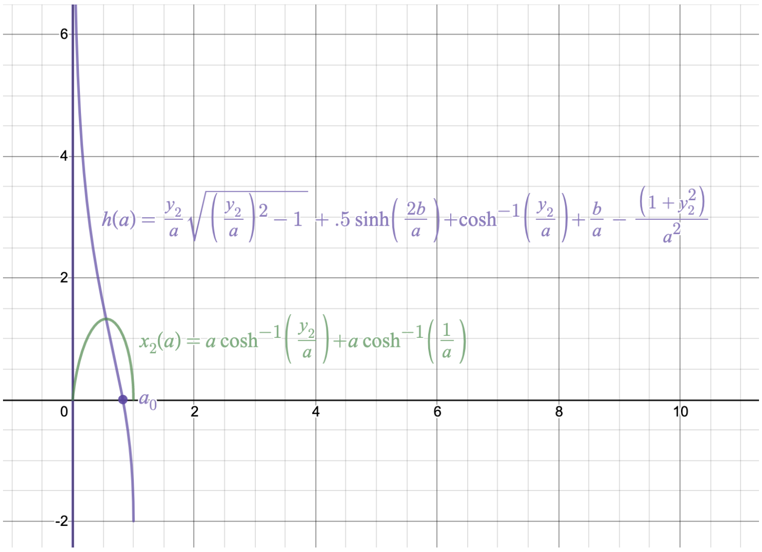


FIGURE 4. $h(a)$ with a_0 as the root, and $x_2(a)$

Then, the area of the catenoid becomes

$$\begin{aligned}
\text{Area} &= \int_0^{x_2} 2\pi u(x)\sqrt{1+(u'(x))^2}dx \\
&= \frac{2\pi}{a}\int_0^{x_2} (u(x))^2 dx \\
&= 2\pi a \int_0^{x_2} \cosh^2\left(\frac{x-b}{a}\right)dx \\
&= \pi a \int_0^{x_2} \left(\cosh\left(\frac{2x-2b}{a}\right)+1\right)dx \\
&= \pi a \left[\frac{a}{2}\sinh\left(\frac{2x-2b}{a}\right)+x\right]_0^{x_2} \\
&= \frac{\pi a^2}{2}\left[\sinh\left(\frac{2x_2-2b}{a}\right)+\sinh\left(\frac{2b}{a}\right)+\frac{2x_2}{a}\right] \\
&= \frac{\pi a^2}{2}\left[2\cosh\left(\frac{x_2-b}{a}\right)\sinh\left(\frac{x_2-b}{a}\right)+\sinh\left(\frac{2b}{a}\right)+\frac{2x_2}{a}\right].
\end{aligned}$$

Equating the area to that of two disks, we obtain

$$\frac{\pi a^2}{2}\left[2\cosh\left(\frac{x_2-b}{a}\right)\sinh\left(\frac{x_2-b}{a}\right)+\sinh\left(\frac{2b}{a}\right)+\frac{2x_2}{a}\right] = \pi(1+{y_2}^2). \tag{2.6}$$

Substituting $\frac{y_2}{a} = \cosh(\frac{x_2-b}{a})$, (2.6) can be simplified to

$$a^2\left[\frac{y_2}{a}\sqrt{\frac{y_2}{a}^2-1}+\frac{1}{2}\sinh\left(\frac{2b}{a}\right)+\cosh^{-1}(\frac{y_2}{a})+\frac{b}{a}\right] = 1+{y_2}^2.$$

Given $a \neq 0$, we transform the equation for the last time:

$$\frac{y_2}{a}\sqrt{\frac{y_2}{a}^2-1}+\frac{1}{2}\sinh\left(\frac{2b}{a}\right)+\cosh^{-1}(\frac{y_2}{a})+\frac{b}{a}-\frac{1+{y_2}^2}{a^2} = 0. \tag{2.7}$$

Notice from (2.5), we know $b = a\cosh^{-1}(\frac{1}{a})$. To solve for a, we graph the function $h(a) = \frac{y_2}{a}\sqrt{\frac{y_2}{a}^2-1}+\frac{1}{2}\sinh\left(\frac{2b}{a}\right)+\cosh^{-1}(\frac{y_2}{a})+\frac{b}{a}-\frac{1+{y_2}^2}{a^2}$ and find the root a_0. The uniqueness of this root can be proven similarly to $g(a)$'s root in the previous case. Since h represents the area of the catenoid minus that of two disks, the graph implies when $a < a_0$, the absolute minimum surface is the Goldschmidt disks, and when $a \geq a_0$ it is the catenoid. In addition, if we graph the resulting x_2 with respect to a using (2.5):

$$x_2 = a\cosh^{-1}(\frac{y_2}{a}) + a\cosh^{-1}(\frac{1}{a}), \tag{2.8}$$

it is strictly decreasing when $a \geq a_0$. Thus, plugging a_0 into (2.8), we will obtain a maximum value of x_2 where the catenary solution not only exists, but also produces the absolute minimum surface area. When the separation is greater than x_2, Goldschmidt's two disks emerge as the solution.

2.2.3. *Rings of Equal Size.* Working with the general case with arbitrary radius, it is hard to produce a concrete solution. One can get a more definite answer by assuming the two rings are of the same size.

Firstly, to check when a catenary solution exists, set $x_1 = 1$ and plug it into (2.4):

$$\cosh^{-1}(\frac{1}{a}) = \frac{1}{a}\frac{1}{\sqrt{\frac{1}{a^2} - 1}}.$$

Letting $\cosh(r) = \frac{1}{a}$, this becomes

$$r = \cosh(r)\frac{1}{\sinh(r)} = \tanh(r),$$

which will give the approximations $r = 1.2$ and $a = 0.552$. Plugging this back into (2.3), we have $y_1 = 1.325$. Thus, for two rings of radius R, the maximum separation is $1.325R$ for a catenary solution to exist.

Secondly, to check if the catenoid has minimum surface area, set $y_2 = 1$ and plug it into (2.7):

$$\frac{1}{a}\sqrt{(\frac{1}{a})^2 - 1} + \frac{1}{2}\sinh(2\cosh^{-1}(\frac{1}{a})) + 2\cosh^{-1}(\frac{1}{a}) - \frac{2}{a^2} = 0.$$

Letting $\cosh(s) = \frac{1}{a}$, this becomes

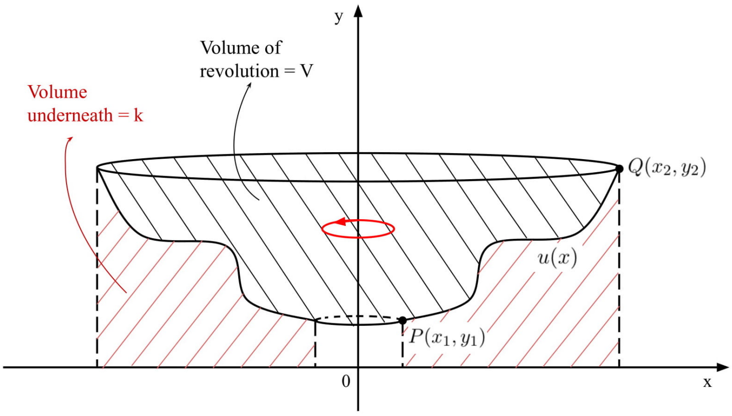
$$\begin{aligned} \cosh(s)\sinh(s) + \frac{1}{2}\sinh(2s) + 2s - 2\cosh^2(s) &= 0 \\ \sinh(2s) - \cosh(2s) - 1 + 2s &= 0 \\ (\sinh(s) - \cosh(s))^2 &= 1 - 2s. \end{aligned}$$

The above equation gives $s = 0.639$ and thus $a = 0.826$. Using (2.5), we have $x_2 = 1.05$. We can now conclude, for two rings of radius R, when the separation is smaller than $1.05R$, we obtain a catenoid solution; when it is greater than $1.05R$, we obtain Goldschmidt's two disks. More specifically, when the separation is between $1.05R$ and $1.325R$, a catenoid exists but has greater surface area, while when it is greater than $1.325R$, a continuous solution does not exist.

3. Constant-Mean-Curvature Surface of Revolution

The previous section finds minimal surfaces of revolution without constraints. This is equivalent as finding a *zero mean curvature* surface. In this section, we will extend to *constant mean curvature* and find minimal surfaces of revolution subject to a given volume constraint.

As in the previous setup, given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in a half-plane, we want to find the curve $u(x)$ connecting P and Q that minimizes the surface area of revolution, with the volume enclosed equal to some constant $V \in \mathbb{R}$. It is clear that the volume underneath



the surface of revolution also equals to some constant $k \in \mathbb{R}$:

$$\text{Volume} = \int_{x_1}^{x_2} 2\pi x u(x) dx = k.$$

The surface area of revolution is the same as in the previous section:

$$\text{Area} = 2\pi \int_{x_1}^{x_2} x\sqrt{1 + (u'(x))^2} dx.$$

Now, since this is an optimization problem with single constraint, we can apply the Lagrangian mulitplier method to find $u(x)$ and $\lambda \in \mathbb{R}$ by solving the following set of equations:

$$\begin{cases} \nabla[2\pi \int_{x_1}^{x_2} x\sqrt{1 + (u'(x))^2} dx] = \lambda \nabla[2\pi \int_{x_1}^{x_2} x u(x) dx] \\ 2\pi \int_{x_1}^{x_2} x u(x) dx = k. \end{cases}$$

This constant λ is called the *mean curvature*. The first equation tells us that

$$\int_{x_1}^{x_2} x\sqrt{1 + (u'(x) + \epsilon\phi'(x))^2} dx = \lambda \int_{x_1}^{x_2} x(u(x) + \epsilon\phi(x)) dx,$$

for any ϕ. Taking the derivative with respect to ϵ and then integrating by parts gives

$$\frac{\partial}{\partial x}\left(\frac{x u'(x)}{\sqrt{1 + (u'(x))^2}}\right) = \lambda x. \tag{3.1}$$

Note how this constant mean curvature equation is similar to the zero mean curvature equation (2.1) from section 2. We can simplify (3.1) with the same method of integrating both sides:

$$\frac{x u'(x)}{\sqrt{1 + (u'(x))^2}} = \lambda' x^2 + c', \tag{3.2}$$

where $\lambda' = \frac{1}{2}\lambda$. We will attempt to solve this equation by considering different values of c'.

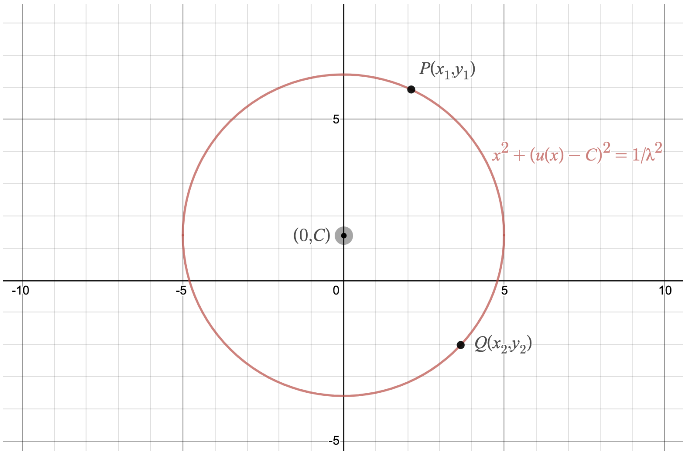
3.1. **Special Case: Sphere.** We first consider a simple case: when $c' = 0$. As x in (3.2) can now be cancelled, we will get $u'(x) = \frac{\pm\lambda' x}{\sqrt{1-(\lambda' x)^2}}$, which leads to

$$u(x) = \int \frac{\pm\lambda' x}{\sqrt{1-(\lambda' x)^2}} dx.$$

By substituting $r = 1 - (\lambda' x)^2$ and $dr = -2(\lambda')^2 dx$, we can conclude

$$\begin{aligned} u(x) &= \pm\frac{1}{2\lambda'} \int \frac{1}{\sqrt{r}} dr \\ &= \pm\frac{1}{\lambda'}\sqrt{1-(\lambda' x)^2} + C. \end{aligned}$$

This is the equation of a circle in the form $x^2 + (u(x) - C)^2 = \frac{1}{(\lambda')^2}$, with radius $\frac{1}{\lambda'}$ and center at $(0, C)$. The surface of revolution around the y-axis is thus a sphere, or part of



a sphere, depending on whether P and Q both lie on the y-axis. To get a solution that satisfies the boundary conditions, we need to pick λ' and C such that the following set of equations holds true:

$$\begin{cases} \text{Volume} = \frac{4\pi}{3\lambda'^3} = V \\ {x_1}^2 + (y_1 - C)^2 = \frac{1}{(\lambda')^2} \\ {x_2}^2 + (y_2 - C)^2 = \frac{1}{(\lambda')^2}. \end{cases}$$

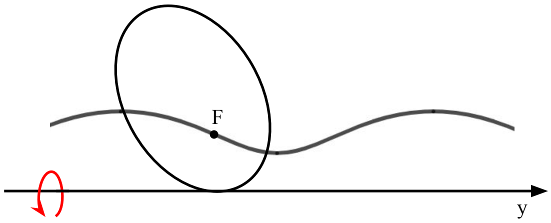
Note due to the way we set up $u(x)$ as a *function* connecting P and Q, we cannot always obtain a single-function solution as circles are not the graph of a function

3.2. **General Case: Delaunay Surfaces.** In the general case, when $c' \neq 0$, we will instead obtain an unsolvable integral for $u(x)$:

$$u(x) = \int \frac{\pm(\lambda' x + c')}{\sqrt{x^2 - (\lambda' x^2 + c')^2}} dx. \tag{3.3}$$

However, we can use computer to estimate its shape with different values of λ' and c'. The constant-mean-curvature surface of revolution formed is called a Delaunay surface. In 1841, Charles-Eugène Delaunay discovered that $u(x)$ can be expressed as roulette of conics—the trace of a conic's focus as the conic rolls on the axis of revolution without sliding. The resulting Delaunay surfaces are the sphere, cylinder, unduloid, and nodoid—respectively the revolution of hyperbolic, circular, elliptical, and hyperbolic roulettes. Moreover, the mean curvature zero catenary can be expressed as a parabolic roulette.

As we understand what constants give rise to sphere and catenoid, what values of λ' and



c' lead to unduloid, nodoid, and cylinder? To distinguish the three, we examine the first derivative of $u(x)$:

$$u'(x) = \frac{\pm(\lambda' x + c')}{\sqrt{x^2 - (\lambda' x^2 + c')^2}}. \tag{3.4}$$

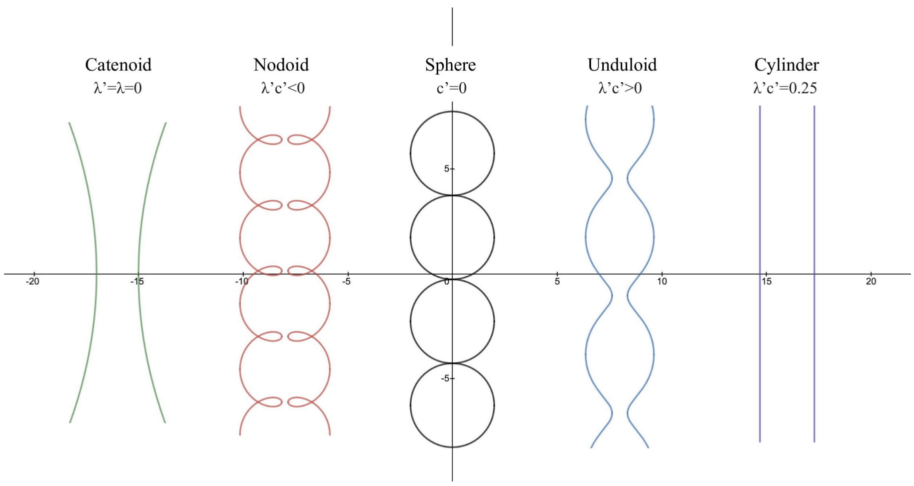
First of all, $u'(x)$ only exists in a small domain, call it (a, b). We can find a and b by solving for when the denominator of $u'(x)$ becomes zero:

$$x^2 - (\lambda' x^2 + c')^2 = 0, \tag{3.5}$$

which will lead to

$$a^2 = \frac{(1 - 2\lambda' c') + \sqrt{1 - 4\lambda' c'}}{2(\lambda')^2},$$
$$b^2 = \frac{(1 - 2\lambda' c') - \sqrt{1 - 4\lambda' c'}}{2(\lambda')^2}.$$

Thus, $u'(x)$ exists in (a, b) where it has non-zero denominator. Observing the shape of each Delaunay curve, we discover that for unduloid curve, $u'(x)$ approaches infinity with the same sign at a and b; for nodoid, $u'(x)$ approaches infinity with different signs at a and b; for cylinder, the derivative does not exists, or say $a = b$. Thus, we obtain cylinder when $\lambda' c' = \frac{1}{4}$. To distinguish unduloid and nodoid, we observe $u'(a)$ approximately has the same sign as c' because $\lambda' a$ is small, and $u'(b)$ has the same sign as λ' when $\lambda' b$ overcomes c'. Therefore, when $\lambda' c' > 0$, the surface of revolution is an unduloid; when $\lambda' c' < 0$, it is a nodoid.



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