TWO-SIDED BAR CONSTRUCTION

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Abstract. This paper consists of three parts. In the first, we introduce the two-sided monoidal bar construction and present various applications in specific categories. We then focus our attention on the monadic bar construction, which generalizes the monoidal bar construction. At the end, we define the bar construction on operads and exhibit the relations between operadic and monadic bar constructions.

Contents

1. Introduction

The two-sided bar construction is an algebraic gadget appearing in various settings and encoding many constructions. In abelian categories, the bar construction yields a resolution, allowing us to compute derived functors; in the category of topological spaces, the bar construction gives the classifying spaces BG of topological monoids G. Since monads are monoids in the category of endofunctors, the bar construction can be applied on monads as well. Specifically, given a category $\mathcal{V}$, a monad C, a C-algebra X and a C-functor F, one can build a simplicial object $B_\bullet(F, C, X)$ in $\mathcal{V}$. With additional effort, one can use the monadic bar construction to "deloop" a space. In many situations, the bar construction $B_\bullet(C, C, X)$ can be regarded as a cofibrant replacement of X. In the last section, we define the operadic bar construction. It is well-known that an operad induces a monad and there is a one-to-one correspondence between algebras over operads and algebras over monads. We make explicit the relationship between the operadic bar construction and the monadic bar construction on monads arising from operads.

1.1. **Simplicial objects.** Let $\mathcal{V}$ be a category. We shall work in the category $\mathcal{SV}$ of simplicial objects in $\mathcal{V}$. Let's recall the definition of $\mathcal{SV}$ and simplicial homotopies.

Definition 1. An object X in $\mathcal{SV}$ is a sequence of objects $\{X_n\}_{n\geq 0}$ in $\mathcal{V}$ equipped with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ in $\mathcal{V}$ such that for $0 \leq i \leq n$,

$$d_i d_j = d_{j-1} d_i \qquad \text{if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i < j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{if } i \leq j$$

A morphism $f : X \to Y$ is a sequence $\{f_n : X_n \to Y_n\}_{n\geq 0}$ of maps in $\mathcal{V}$ such that $d_i f_n = f_{n-1} d_i$ and $s_i f_n = f_{n+1} s_i$. Graphically,

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\ {\scriptstyle s_i}\uparrow & & \uparrow{\scriptstyle s_i} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

Definition 2. A *simplicial homotopy* $h : f \simeq g$ in $\mathcal{SV}$ between morphisms $f, g : X \to Y$ consists of maps $h_i : X_q \to Y_{q+1}$ where $0 \leq i \leq q$ such that

$$d_0 h_0 = f_q$$

$$d_{q+1} h_q = g_q$$

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j, \\ d_j h_{j-1} & \text{if } i = j, \\ h_j d_{i-1} & \text{if } i > j+1, \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j, \\ h_j s_{i-1} & \text{if } i > j. \end{cases}$$

2. The monoidal bar construction

In this section, we introduce the monoidal bar construction in a monoidal category. From now on we fix $(\mathcal{V}, \otimes, \kappa)$ a monoidal category.

2.1. **In a monoidal category.**

Definition 3. A *monoid* (G, μ, η) in $\mathcal{V}$ is an object G in $\mathcal{V}$ together with morphisms $\mu : G \otimes G \to G$ and $\eta : \kappa \to G$ such that the following two diagrams commute:

$$\begin{array}{ccc} G \otimes G \otimes G & \xrightarrow{id \otimes \mu} & G \otimes G \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \otimes G & \xrightarrow[\mu]{} & G \end{array} \qquad \begin{array}{ccccc} \kappa \otimes G & \xrightarrow{\eta \otimes id} & G \otimes G & \xleftarrow{id \otimes \eta} & G \otimes \kappa \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & G & & \end{array}$$

A *morphism of monoids* $f : (G, \mu, \eta) \to (G', \mu', \eta')$ satisfies $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$.

For instance, in $(\boldsymbol{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ a monoid object is a ring, or more generally, in $(\boldsymbol{Mod_R}, \otimes, R)$, a monoid object is an algebra; in $(\boldsymbol{Top}, \times, \{*\})$, a monoid object is called a topological monoid. Topological groups are a special case of topological monoids.

Definition 4. A *left G-object* is an object $N \in \mathcal{V}$ together with a morphism $\lambda : G \otimes N \to N$ in $\mathcal{V}$ such that

$$\begin{array}{ccc} \kappa \otimes N & \xrightarrow{\eta \otimes id} & G \otimes N \\ & \underset{\cong}{\searrow} & \downarrow \lambda \\ & & N \end{array} \qquad \begin{array}{ccc} G \otimes G \otimes N & \xrightarrow{\mu \otimes id} & G \otimes N \\ {\scriptstyle id \otimes \lambda} \downarrow & & \downarrow \lambda \\ G \otimes N & \xrightarrow[\lambda]{} & G \end{array}$$

A *morphism of left G-objects* $f : (N, \lambda) \to (N', \lambda')$ satisfies

$$\begin{array}{ccc} G \otimes N & \xrightarrow{id \otimes f} & G \otimes N' \\ \lambda \downarrow & & \downarrow \lambda' \\ N & \xrightarrow[f]{} & N' \end{array}$$

Similarly, a *right G-object* is an object M in $\mathcal{V}$ together with a map $\rho : M \otimes G \to M$ such that

$$\begin{array}{ccc} M \otimes \kappa & \xrightarrow{id \otimes \eta} & M \otimes G \\ & \underset{\cong}{\searrow} & \downarrow \rho \\ & & M \end{array} \qquad \begin{array}{ccc} M \otimes G \otimes G & \xrightarrow{\rho \otimes id} & M \otimes G \\ {\scriptstyle id \otimes \mu} \downarrow & & \downarrow \rho \\ M \otimes G & \xrightarrow[\rho]{} & G \end{array}$$

A *morphism of right G-objects* $f : (M, \rho) \to (M', \rho')$ satisfies

$$\begin{array}{ccc} M \otimes G & \xrightarrow{f \otimes id} & M' \otimes G \\ \rho \downarrow & & \downarrow \rho' \\ M & \xrightarrow[f]{} & M' \end{array}$$

The notions of left and right G-objects generalize the notions of modules over rings. This means in $(\boldsymbol{Ab}, \otimes, \mathbb{Z})$, the left and right objects over a ring regarded as a monoid are exactly the same structures as the classical notions of modules over a ring. In a category of chain complexes equipped with tensor product, a monoid is a dg-algebra and a left(right) module is a left(right) dg-module. We shall recall the definition of the category of chain complexes later.

Notation. Denote by $\mathcal{B}(\mathcal{V})$ the category of triples (M, G, N) consisting of a monoid G, a right G-object M and a left G-object N in the monoidal category $\mathcal{V}$.

Construction 5. Construct the bar construction functor

$$B_\bullet : \mathcal{B}(\mathcal{V}) \to \mathcal{SV}.$$

The q-dimensional component of $B_\bullet(M, G, N)$ is $M \otimes G^{\otimes q} \otimes N$. The face maps multiply two adjacent elements – either G with G, or M with G, or G with N; the degeneracy maps insert the monoidal unit κ. Explicitly, for $i = 0, 1, \ldots, q$,

$$\begin{aligned} d_i : \quad & B_q(M, G, N) \to B_{q-1}(M, G, N) \\ & (g_0 \otimes g_1 \otimes \ldots \otimes g_q \otimes g_{q+1}) \mapsto g_0 \otimes g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_{q+1} \end{aligned}$$

$$\begin{aligned} s_i : \quad & B_q(M, G, N) \to B_{q+1}(M, G, N) \\ & (g_0 \otimes g_1 \otimes \ldots \otimes g_n \otimes g_{q+1}) \mapsto g_0 \otimes g_1 \otimes \ldots \otimes g_i \otimes \kappa \otimes g_{i+1} \otimes \ldots \otimes g_{q+1} \end{aligned}$$

where $g_0 \in M,\ g_1, \ldots, g_q \in G,\ g_{q+1} \in N$. The simplicial object $B_\bullet(M, G, N)$ is called the *monoidal bar construction.* The triple of morphism $(f^M : M \to M', f^G : G \to G', f^N : N \to N')$ is sent to the sequence of maps $\{f_q\}_{q \geq 0}$ where

$$\begin{aligned} f_q : \quad & B_q(M, G, N) \to B_q(M', G', N') \\ & g_0 \otimes g_1 \otimes \ldots \otimes g_q \otimes g_{q+1} \mapsto f^M(g_0) \otimes f^G(g_1) \otimes \ldots \otimes f^G(g_q) \otimes f^N(g_{q+1}). \end{aligned}$$

Remark. The term "bar construction" comes from the use of vertical bars for tensor products. In the pre-LaTeX age, it was difficult to typeset the symbol $\otimes$ and an element in $M \otimes G^{\otimes n} \otimes N$ was written as $g_0\ [g_1|g_2| \ldots |g_n]\ g_{n+1}$.

2.2. In an abelian category.

Fix $\mathcal{A}$ an abelian category. In an abelian category, simplicial objects can be regarded as chain complexes under the Dold-Kan correspondence. In particular, this correspondence allows us to rewrite the simplicial objects obtained through the monoidal bar construction as chain complexes. We shall prove such chain complexes yield resolutions of monoids in $\mathcal{A}$, which one can use to compute derived functors. Recall by definition a category is abelian if

- it has a zero object 0, that is both initial and terminal,
- it has all binary products and binary coproducts,
- it has all kernels and cokernels, defined respectively to be the equalizer and coequalizer of a map $f : A \to B$ with the zero map $A \to 0 \to B$, and
- all monomorphims and epimorphisms arise as kernels or cokernels, respectively.
- finite products and finite coproducts coincide, called biproducts or direct sum.

These axioms imply that the hom-sets in an abelian category has the structure of an abelian group. One can imagine $\mathcal{A} = Ab$, or more generally, $\mathcal{A} = Mod_R$. We briefly recall the structure of *the category $Ch(\mathcal{A})$ of chain complexes in $\mathcal{A}$*, which will play a central rule in this section.

Definition 6. A *chain complex* A in $\mathcal{A}$ is described as

$$A := \ldots \xleftarrow{d_{-1}} A_{-1} \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} \ldots$$

where $A_i \in Ob(\mathcal{A})$ and $d_i \circ d_{i+1} = 0$. The degree of $a \in A_n$, denoted by $|a|$, is n. A *map of chain complexes* $f : (A, d_A) \to (B, d_B)$ of degree $|f| = r$ is a family of maps $f_n : A_n \to B_{n+r}$ such that $d_B \circ f = (-1)^r f \circ d_A$.

The monoidal structure of $Ch(\mathcal{A})$ is given by tensor product of chain complexes. Explicitly, given (A, d_A) and (B, d_B), their tensor product is the chain complex defined as

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j$$

with differential defined as[1]

$$d_{A\otimes B}(a\otimes b) := d_A(a)\otimes b + (-1)^{|a|} a\otimes d_B(b).$$

The tensor unit is the chain complex concentrated in degree 0 on the unit of $\mathcal{A}$. Furthermore, the switching map $\tau : a\otimes b \mapsto (-1)^{|a||b|} b\otimes a$ endows $Ch(\mathcal{A})$ with a symmetric structure.

Let $Ch_{\geq 0}(\mathcal{A})$ denote the *category of non-negatively graded chain complexes in* $\mathcal{A}$. One can verify $Ch_{\geq 0}(\mathcal{A})$ is also a symmetric monoidal category under tensor product $\otimes$ and switching map τ.

Let $A_\bullet$ be a simplicial object in an abelian category $\mathcal{A}$. There are several natural ways to construct a non-negatively graded chain complex from $A_\bullet$.

Definition 7. The *unnormalized chain complex* $A_* \in Ch_{\geq 0}(\mathcal{A})$ is defined as

$$A_* := \left(A_n, \partial_n := \sum_{i=0}^{n} (-1)^i d_i\right)_{n\geq 0}$$

where d_i-s are the face maps in $A_\bullet$.

The simplicial identities imply that A_* is indeed a chain complex. The map $A_\bullet \mapsto A_*$ yields a functor from $\mathcal{SA}$ to $Ch_{\geq 0}(\mathcal{A})$.

Consider the subcomplex $DA_* \subset A_*$ generated by the degeneracies; namely,

$$DA_n := \sum_{i=1}^{n-1} Im(s_i)$$

By simplicial identities, we can check

$$\partial s_j = \sum_{i=0}^{n} (-1)^i d_i s_j = \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_i,$$

which means DA_* is stable under ∂ and is in fact a subcomplex.

Definition 8. The *normalized chain complex* CA_* of $A_\bullet$ is defined as

$$CA_* := A_*/DA_*.$$

Definition 9. The *Moore complex*[2] NA_* of $A_\bullet$ is defined by $NA_0 := A_0$ and

$$NA_n := \bigcap_{i=0}^{n-1} ker(d_i : A_n \to A_{n-1})$$

with the differential $(-1)^n d_n : NA_n \to NA_{n-1}$.

[1] In a category with graded objects, there are signs involved in the formulas. One of the useful sign convention is the Koszul sign rule, which is as follows. For any maps $f : A \to A'$ and $g : B \to B'$ between graded objects, define

$$(f\otimes g)(a\otimes b) := (-1)^{|g||a|} f(a)\otimes g(b).$$

Namely, you get signs according to the degree and according to moving the second map across terms.

[2] The Moore complex is defined for all simplicial modules, not necessarily abelian.

By the simplicial identity $d_{n-1}d_n = d_{n-1}d_{n-1}$, one can verify NA_* is indeed a complex.

As one would expect, these three complexes are intimately related. Lemma 8.3.7 in [7] proves that the normalized complex CA_* and the Moore complex NA_* are isomorphic; the following theorem makes explicit the relations between simplicial objects and the associated complexes. We refer to Theorem 8.4.1 in [7] for the proofs and details.

Theorem 10. *(Dold-Kan Correspondence) For any abelian category $\mathcal{A}$, the Moore complex functor $A_\bullet \mapsto NA_*$ is an equivalence of categories between $S\mathcal{A}$ and $Ch_{\geq 0}(\mathcal{A})$. Moreover, A_*, CA_* and NA_* are all naturally homotopically equivalent.*

Applying the correspondence to the bar construction $B_\bullet(M, A, N)$, we get a chain complex which contains the same homological information:

$$\mathcal{B}(\mathcal{A}) \xrightarrow[\text{Construction}]{\text{Bar}} \mathcal{S}\mathcal{A} \xrightarrow{Dold-Kan} Ch_{\geq 0}(\mathcal{A}).$$

2.2.1. *In a category of modules.*

In this section, we focus on the category of R-modules and prove that the bar construction yields a resolution of R-algebras.

Let $\mathcal{A} = (\mathbf{Mod}_R, \otimes, R)$ be the category of modules over a commutative ring R. It is clear that $\mathcal{A}$ is an abelian monoidal category. A monoid A in $\mathcal{A}$ is a unital associative R-algebra and left and right A-objects are left and right modules over A in the classical sense. Let $B_*(M, A, N)$ denote the unnormalized chain complex associated to the bar construction $B_\bullet(M, A, N)$.

Theorem 11. *If A and N are R-flat, $B_*(A, A, N)$ yields a resolution of N by A-flat left A-modules. Similarly, if A and M are R-flat, $B_*(M, A, A)$ yields a resolution of M by A-flat right A-modules.*

Proof. We often denote $\otimes = \otimes_R$. By construction,

$$B_*(A, A, N) = 0 \leftarrow A \otimes N \xleftarrow{\partial_1} A^{\otimes 2} \otimes N \xleftarrow{\partial_2} \dots,$$

where $A^{\otimes q+1} \otimes N$ is the degree q component and $\partial_q : A^{\otimes q+1} \otimes N \to A^{\otimes q} \otimes N$. For $q \geq 1$, $A^{\otimes q} \otimes N$ is a left A-module, where the A-module structure is given by multiplication on the first factor, i.e.

$$a \cdot (a_0 \otimes \dots \otimes a_q \otimes n) = \mu(a \otimes a_0) \otimes \dots \otimes a_q \otimes n$$

This means $B_*(A, A, N)$ is a complex of left A-modules. The augmented complex is given by the left action $\lambda : A \otimes N \to N$ of A on N:

$$\bar{B}_*(A, A, N) = 0 \leftarrow N \xleftarrow{\lambda} A \otimes N \xleftarrow{\partial_1} A^{\otimes 2} \otimes N \xleftarrow{\partial_2} \dots$$

To prove $B_*(A, A, N)$ is a resolution of N, it suffices to find a contracting homotopy[3] of $\bar{B}_*(A, A, N)$. For $q \geq 0$, define a sequence of maps $\{s_q\}_{q \geq 0}$ where

$$\begin{array}{rccc} s_q : & A^{\otimes q} \otimes N & \to & A^{\otimes q+1} \otimes N \\ & a_0 \otimes \dots \otimes a_{q-1} \otimes n & \mapsto & 1 \otimes a_0 \otimes \dots \otimes a_{q-1} \otimes n \end{array}$$

[3] By definition, $\{s_q : C_{q-1} \to C_q\}$ is a contracting homotopy for a complex $(C_\bullet, d)$ if $1_C = sd + ds$.

For $q \geq 1$, take $\boldsymbol{a} := a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1} \otimes n \in A^{\otimes q} \otimes N$ and compute

$$\begin{aligned}(\partial_q s_q + s_{q-1}\partial_{q-1})(\boldsymbol{a}) &= \partial_q(1 \otimes \boldsymbol{a}) + 1 \otimes \partial_{q-1}(\boldsymbol{a}) \\ &= \sum_{i=0}^{q}(-1)^i d_i(1 \otimes \boldsymbol{a}) + 1 \otimes \partial_{q-1}(\boldsymbol{a}) \\ &= \left(\boldsymbol{a} + \sum_{i=1}^{q}(-1)^i d_i(1 \otimes \boldsymbol{a})\right) + \partial_{q-1}(\boldsymbol{a}) \\ &= \boldsymbol{a} + (-1) \cdot \partial_{q-1}(\boldsymbol{a}) + \partial_{q-1}(\boldsymbol{a}) \\ &= \boldsymbol{a}\end{aligned}$$

This implies for $q \geq 1$, $\partial_q s_q + s_{q-1}\partial_{q-1} = id$. Regard μ as ∂_0; it is immediate that $\partial_0 s_0 = \mu s_0 = id$. Therefore, $\partial s + s\partial = id$ and $\{s_q\}_{q\geq 0}$ is the contracting homotopy of $\bar{B}_*(A, A, N)$.

In addition, assume A and N are R-flat. Equivalently, this means $(-) \otimes_R A$ and $(-) \otimes_R N$ are (left) exact. Since

$$(-) \otimes_A (A \otimes_R \ldots \otimes_R A \otimes_R N) \cong (-) \otimes_R A \otimes_R \ldots \otimes_R A \otimes_R N,$$

the left A-module $A \otimes_R \ldots \otimes_R A \otimes_R N$ is A-flat. Therefore, $B_*(A, A, N)$ is a A-flat resolution, which completes the first part of the proofs.

Consider

$$B_*(M, A, A) := 0 \leftarrow M \otimes A \xleftarrow{\partial_1} M \otimes A^{\otimes 2} \xleftarrow{\partial_2} \ldots.$$

This is a complex of right A-modules. The augmented complex is

$$\bar{B}_*(M, A, A) = 0 \leftarrow M \xleftarrow{\rho} M \otimes A \xleftarrow{\partial_1} M \otimes A^{\otimes 2} \xleftarrow{\partial_2} M \otimes A^{\otimes 3} \xleftarrow{\partial_3} \ldots,$$

where $\rho : M \otimes A \to M$ is the right action of A on M. For $q \geq 0$, define a sequence of maps $\{s_q : M \otimes A^{\otimes q} \to M \otimes A^{\otimes q+1}\}_q$ where

$$s_q(m \otimes a_0 \otimes \ldots \otimes a_{q-1}) = m \otimes a_0 \otimes \ldots \otimes a_{q-1} \otimes (-1)^q.$$

Take $\boldsymbol{a} := m \otimes a_0 \otimes \ldots \otimes a_{q-1} \in M \otimes A^{\otimes q}$. Compute

$$\begin{aligned}(\partial_q s_q + s_{q-1}\partial_{q-1})(\boldsymbol{a}) &= (\partial_q s_q)(\boldsymbol{a}) + (s_{q-1}\partial_{q-1})(\boldsymbol{a}) \\ &= \partial_q(\boldsymbol{a} \otimes (-1)^q) + (\partial_{q-1}(\boldsymbol{a})) \otimes (-1)^{q-1} \\ &= \sum_{i=0}^{n}(-1)^i d_i(\boldsymbol{a} \otimes (-1)^q) + \left(\sum_{i=0}^{q-1}(-1)^i d_i(\boldsymbol{a})\right) \otimes (-1)^{q-1} \\ &= (-1)^q d_q((\boldsymbol{a} \otimes (-1)^q) \\ &= (-1)^{2q}\boldsymbol{a} \otimes 1 \\ &= \boldsymbol{a}.\end{aligned}$$

This means for $q \geq 1$, $\partial_q s_q + s_{q-1}\partial_{q-1} = id$. Regarding ρ as ∂_0, we deduce $\partial s + s\partial = id$. Therefore, $\{s_q\}_q$ is the contracting homotopy of $\bar{B}_*(M, A, A)$. Now, suppose A and M are R-flat. One can verify $M \otimes A^{\otimes q}$ is A-flat for all q. Hence $B_*(M, A, A)$ is a A-flat resolution of M. □

Recall that the functors $Tor_*^A(M, -)$ and $Tor_*^A(-, N)$ can be computed using flat resolutions of either variable (See Lemma 3.2.8 in [7]). Theorem 11 implies

$$Tor_*^A(M, N) \cong H_*(B_*(M, A, A) \otimes_A N) \cong H_*(M \otimes_A B_*(A, A, N)).$$

Furthermore, observe

$$M \otimes_A \left(\underbrace{A \otimes \ldots \otimes A}_{(q+1)\text{-times}} \otimes N \right) \cong M \otimes \underbrace{A \otimes \ldots \otimes A}_{q\text{-times}} \otimes N \cong \left(M \otimes \underbrace{A \otimes \ldots \otimes A}_{(q+1)\text{-times}} \right) \otimes_A N$$

and then

$$M \otimes_A B_*(A, A, N) \cong B_*(M, A, N) \cong B_*(M, A, A) \otimes_A N.$$

This means the homology of $B_*(M, A, N)$ gives the functor Tor:

$$Tor_*^A(M, N) \cong H_*(B_*(M, A, N)).$$

2.2.2. *In a category of chain complexes.*

In this section, we shall consider the case of $Ch(\mathcal{A})$ where $\mathcal{A}$ is abelian. We first develop the notion of bicomplexes, or double complexes, and then define the symmetric monoidal category $bCh(\mathcal{A})$ of bicomplexes in $\mathcal{A}$.

Definition 12. A *bicomplex* in $\mathcal{A}$ is a family of objects $A = \{A_{p,q}\}$ in $\mathcal{A}$ with horizontal and vertical differentials

$$d^h : A_{p,q} \to A_{p-1,q} \qquad d^v : A_{p,q} \to A_{p,q-1}$$

satisfying $d^v \circ d^h + d^h \circ d^v = 0$. For $a \in A_{p,q}$, the horizontal degree is $|a|_h = p$; the vertical degree is $|a|_v = q$; the total degree is $|a| = |a|_h + |a|_v = p + q$.

A *morphism of bicomplexes* $f : A \to B$ is a family of maps $f_{p,q} : A_{p,q} \to B_{p,q}$ that commute with the differentials, i.e. $d^h \circ f = f \circ d^h$ and $d^v \circ f = f \circ d^v$.

We often picture the bicomplex $A = \{A_{p,q}\}$ as a lattice in which every square anticommutes.

$$\begin{array}{ccccccccc}
 & & \ldots & & \ldots & & \ldots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longleftarrow & A_{p-1,q+1} & \xleftarrow{d^h} & A_{p,q+1} & \xleftarrow{d^h} & A_{p+1,q+1} & \longleftarrow & \ldots \\
 & & \downarrow d^v & & \downarrow d^v & & \downarrow d^v & & \\
\ldots & \longleftarrow & A_{p-1,q} & \xleftarrow{d^h} & A_{p,q} & \xleftarrow{d^h} & A_{p+1,q} & \longleftarrow & \ldots \\
 & & \downarrow d^v & & \downarrow d^v & & \downarrow d^v & & \\
\ldots & \longleftarrow & A_{p-1,q-1} & \xleftarrow{d^h} & A_{p,q-1} & \xleftarrow{d^h} & A_{p+1,q-1} & \longleftarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \ldots & & \ldots & & \ldots & &
\end{array}$$

Each row $A_{*,q}$ and each column $A_{p,*}$ is a chain complex. However, because of anticommutativity, $d^v_{*,q} : A_{*,q} \to A_{*,q-1}$ is not a morphism in $Ch(\mathcal{A})$. We can obtain a morphism of chain complexes by introducing signs as follows

$$(d^v_{*,q})' := (-1)^{|x|_h} d^v_{*,q} = (-1)^* d^v_{*,q} : X_{*,q} \to X_{*,q-1}.$$

Replacing all the d^v with $(d^v)'$, we get another lattice in which every square commutes.

With these definitions, we can construct the category $bCh(\mathcal{A})$ of bicomplexes in $\mathcal{A}$. The monoidal structure is given by the tensor product:

$$(A \otimes B)_{p,q} = \bigoplus_{\substack{m+s=p \\ n+t=q}} A_{m,n} \otimes B_{s,t}$$

with the following horizontal and vertical differentials

$$d^h(a\otimes b) = d^h(a)\otimes b + (-1)^{|a|} a\otimes d^h(b)$$
$$d^v(a\otimes b) = d^v(a)\otimes b + (-1)^{|a|} a\otimes d^v(b)$$

The switching map $\tau : a\otimes b \mapsto (-1)^{|a||b|} a\otimes b$ endows $bCh(\mathcal{X})$ with a symmetric structure.

Definition 13. Given a bicomplex A, the *total complex* $Tot(A)$ is defined as

$$Tot(A) = \left(\bigoplus_{p+q=n} A_{p,q}, d_{Tot} := d^v + d^h\right)_n .$$

The anticommutivity condition implies that $(d_{Tot})^2 = 0$. The total complex induces a functor $Tot : bCh(\mathcal{A}) \to Ch(\mathcal{A})$.

Now, we shall develop the relationship between the bar construction on triples in $Ch(\mathcal{A})$ and chain complexes. Recall an element

$$((M, d_M), (A, d_A), (N, d_N)) \in \mathcal{B}(Ch(\mathcal{A}))$$

consists of a dg-algebra A, a right dg-module M and a left dg-module N. The bar construction gives the following simplicial object

$$B_\bullet(M, A, N) = M\otimes N \quad \leftleftarrows \quad M\otimes A\otimes N \ldots .$$

Let $B_*(M, A, N)$ denote the corresponding unnormalized complex:

$$B_*(M, A, N) := M\otimes N \xleftarrow{\partial_1} M\otimes A\otimes N \xleftarrow{\partial_2} \ldots \xleftarrow{\partial_p} M\otimes A^{\otimes p}\otimes N \xleftarrow{\partial_{p+1}} \ldots .$$

where ∂ is the alternating sum of the face maps. Write the elementary tensors in degree p as

$$m\otimes a_1\otimes \ldots \otimes a_p\otimes n.$$

Since (A, d_A), (M, d_M) and (N, d_N) all have internal differentials, we get an induced differential

$$(d^v_{*,q})' : M\otimes (A_q)^{\otimes *}\otimes N \to M\otimes (A_{q-1})^{\otimes *}\otimes N$$

given by

$$\begin{aligned}(d^v_{*,q})'(m\otimes a_1\otimes \ldots\otimes a_p\otimes n) &= d_M(m)\otimes a_1\otimes\ldots\otimes a_p\otimes n \\ &+ \sum_{i=1}^{p} (-1)^{\epsilon_i} m\otimes a_1\otimes\ldots\otimes d_A(a_i)\otimes\ldots\otimes a_p\otimes n \\ &+ (-1)^{|m|+|a_1|+\ldots+|a_p|} m\otimes a_1\otimes\ldots\otimes a_p\otimes d_N(n)\end{aligned}$$

where $\epsilon_i = |m| + |a_1| + \ldots + |a_{i-1}|$. Observe that $(d^v_{*,q})'$ is exactly the iterated tensor product of chain complexes; the sign $(-1)^{\epsilon_i}$ is a direct consequence of the Koszul sign rule. However, one can check, with the differential $(d^v_{*,q})'$, the square below is commutative.

$$\begin{array}{ccc} M\otimes (A_q)^{\otimes p-1}\otimes N & \xleftarrow{\partial_p} & M\otimes (A_q)^{\otimes p}\otimes N \\ \big\downarrow (d^v_{*,q})' & & \big\downarrow (d^v_{*,q})' \\ M\otimes (A_{q-1})^{\otimes p-1}\otimes N & \xleftarrow{\partial_p} & M\otimes (A_{q-1})^{\otimes p}\otimes N \end{array}$$

In order to obtain a bicomplex, let the vertical differential be

$$d^v_{*,q} := (-1)^*(d^v_{*,q})' : M \otimes (A_q)^{\otimes *} \otimes N \to M \otimes (A_{q-1})^{\otimes *} \otimes N.$$

Therefore, the unnormalized complex $B_*(M, A, N)$ can be regarded as a bicomplex with horizontal differential $\partial_\bullet$ and vertical differential d^v. It can be pictured as

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \\
M \otimes N & \xleftarrow{\partial_1} & M \otimes A_q \otimes N & \xleftarrow{\partial_2} & M \otimes (A_q)^{\otimes 2} \otimes N & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v_{1,q}} & & \downarrow{\scriptstyle d^v_{2,q}} & & \\
M \otimes N & \xleftarrow{\partial_1} & M \otimes A_{q-1} \otimes N & \xleftarrow{\partial_2} & M \otimes (A_{q-1})^{\otimes 2} \otimes N & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & &
\end{array}$$

We can then take the total complex

$$(Tot(B_*(M, A, N)), d := d^v + \partial_\bullet).$$

Explicitly, we can compute

$$\begin{aligned}
d(m \otimes n) &= \partial(m \otimes n) + d^v(m \otimes n) \\
&= d^v(m \otimes n) \\
&= d_M(m) \otimes n + (-1)^{|m|} m \otimes d_N(n)
\end{aligned}$$

$$\begin{aligned}
d(m \otimes a \otimes n) &= \partial(m \otimes a \otimes n) + d^v(m \otimes a \otimes n) \\
&= [(ma) \otimes n - m \otimes (an)] + \\
&\quad (-1)^1 \Big[d_M(m) \otimes a \otimes n + (-1)^{|m|} m \otimes d_A(a) \otimes n \\
&\qquad + (-1)^{|m|+|a|} m \otimes a \otimes d_A(n) \Big].
\end{aligned}$$

Summarize these constructions in the following theorem.

Theorem 14. *The bar construction in $Ch(\mathcal{A})$ can be identified with a chain complex via the following composition of functors*

$$\mathcal{B}(Ch(\mathcal{A})) \xrightarrow[\text{Construction}]{\text{Bar}} \left\{ \begin{array}{c} \text{Simplicial} \\ \text{objects in } Ch(\mathcal{A}) \end{array} \right\} \xrightarrow{Dold-Kan} bCh(\mathcal{A}) \xrightarrow{Tot} Ch(\mathcal{A})$$

2.3. **In the category of topological spaces.** Let **Top** be the category of topological spaces. Let G be a topological monoid and let $*$ be the trivial right and left G-objects. Define BG as the geometric realization of the bar construction $B_\bullet(*, G, *)$:

$$BG := |B_\bullet(*, G, *)|$$

For a more exhaustive description of the geometric realization of simplicial spaces, we refer the reader to Section 11 in [4]. A map $G \to H$ of topological monoids induces a map $G^n \times \Delta^n \to H^n \times \Delta^n$ and then a map $BG \to BH$. Furthermore, the identity map $G \to G$ induces the identity $BG \to BG$ and a composition $G \to H \to$

K induces the composition $BG \to BH \to BK$. Therefore, B defines a functor from topological monoids to topological spaces.

We shall state some results on BG. We refer to Section 7 of [5] for the proofs.

Theorem 15. *If G is a topological group, BG can be identified with the classifying space for principle G-bundles and $EG := |B_\bullet(*, G, G)|$ can be identified with the total space of the universal bundle over BG.*

Theorem 16. *If G is grouplike and has nondegenerate base point, there is a natural weak homotopy equivalence $G \to \Omega BG$, given by*

$$(g, t) \mapsto (g, t, 1 - t) \in G \times \Delta^1$$

BG is called a "delooping" of G.

3. The monadic bar construction

In this section, we extend the notion of bar construction to the monadic setting and show that the monoidal bar construction defined in section 2 is a special case of the monadic bar construction.

Definition 17. A monad (C, μ, η) in a category $\mathcal{V}$ consists of

- an endofunctor $C : \mathcal{V} \to \mathcal{V}$ and
- a multiplication natural transformation $\mu : C^2 \to C$ and
- a unit natural transformation $\eta : 1_{\mathcal{V}} \to C$,

so that the following diagrams commute:

$$\begin{array}{ccc} CCC & \xrightarrow{C\mu} & CC \\ {\scriptstyle \mu C}\downarrow & & \downarrow{\scriptstyle \mu} \\ CC & \xrightarrow[\mu]{} & C \end{array} \qquad\qquad \begin{array}{ccccc} C & \xrightarrow{\eta C} & CC & \xleftarrow{C\eta} & C \\ & {\scriptstyle 1_C}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_C} & \\ & & C & & \end{array}$$

A monad C in $\mathcal{V}$ is essentially a monoid in the monoidal category of endofunctors on $\mathcal{V}$. For a monad (C, μ, η), we can define C-functors which generalize the notions of right objects over monoids in the categorical setting.

Definition 18. A *C-functor* (F, ρ) in a category $\mathcal{W}$ is a functor $F : \mathcal{V} \to \mathcal{W}$ with a natural transformation of functors $\rho : FC \to F$ such that the following diagrams commute:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FC \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \rho} \\ & & F \end{array} \qquad\qquad \begin{array}{ccc} FCC & \xrightarrow{F\mu} & FC \\ \downarrow{\scriptstyle \rho C} & & \downarrow{\scriptstyle \rho} \\ FC & \xrightarrow[\rho]{} & C \end{array}$$

A *morphism of C-functors* $\pi : (F, \rho) \to (F', \rho')$ is a natural transformation $\pi : F \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{\pi C} & F'C \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ F & \xrightarrow[\pi]{} & F' \end{array}$$

Analogously, we can define C-algebras which generalize the notion of left objects over monoids.

Definition 19. A *C-algebra* (X,λ) over C is an object $X \in \mathcal{V}$ equipped with a natural transformation $\lambda : CX \to X$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & CX \\ & \searrow^{\cong} & \downarrow{\scriptstyle\lambda} \\ & & X \end{array} \qquad \begin{array}{ccc} CCX & \xrightarrow{\mu} & CX \\ \downarrow{\scriptstyle C\lambda} & & \downarrow{\scriptstyle\lambda} \\ CX & \xrightarrow[\lambda]{} & X \end{array}$$

A *morphism of C-algebras* $f : (X,\lambda) \to (X',\lambda')$ is a map $f : X \to X'$ in $\mathcal{V}$ such that the following diagram is commutative:

$$\begin{array}{ccc} CX & \xrightarrow{Cf} & CX' \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Now construct the category $\mathcal{B}(\mathcal{V},\mathcal{W})$. The objects are triples

$$((F,\rho),(C,\mu,\eta),(X,\lambda)),$$

abbreviated (F,C,X), where C is a monad in $\mathcal{V}$, F is a C-functor in $\mathcal{W}$ and X is a C-algebra. A morphism $(\pi,\psi,f) : (F,C,X) \to (F',C',X')$ in $\mathcal{B}(\mathcal{V},\mathcal{W})$ is a triple consisting of

- a morphism of monads $\psi : C \to C'$ in $\mathcal{V}$
- a morphism of C-functors $\pi : (F,\rho) \to (F',\rho' \circ F'\psi)$ in $\mathcal{W}$ and
- a morphism of C-algebras $f : (X,\lambda) \to (X',\lambda' \circ \psi)$.

Construction 20. Construct the monadic bar construction functor

$$\mathcal{B}(\mathcal{V},\mathcal{W}) \xrightarrow{B_\bullet} \mathcal{SV}.$$

The structure of $B_\bullet(F,C,X)$ can be given explicitly as follows. The n-dimensional component is

$$B_n(F,C,X) = FC^nX.$$

The $n+1$ face maps $d : FC^nX \to FC^{n-1}X$ are

$$d_i = \begin{cases} \rho C^{n-1}X & i = 0 \\ FC^{i-1}\mu C^{n-i-1}X & 0 < i < n \\ FC^{n-1}\lambda & i = n \end{cases}$$

and the $n+1$ degeneracy maps $s : FC^nX \to FC^{n+1}X$ are

$$s_i = FC^i\eta C^{n-i}X$$

where $0 \le i \le n$. Consider $B_\bullet(\pi,\psi,f)$. The n-dimensional component is

$$B_q(\pi,\psi,f) = \pi\psi^q f : FT^qX \to F'(T')^qX'$$

such that

$$\begin{array}{ccc} FC^qX & \xrightarrow{FC^qf} & FC^qX' \\ \downarrow{\scriptstyle\pi\psi^q} & \searrow^{\pi\psi^q f} & \downarrow{\scriptstyle\pi\psi^q} \\ F'(C')^qX & \xrightarrow[F'(C')^qf]{} & F'(C')^qX' \end{array}$$

The following theorem exhibits the monadic bar construction $B_\bullet(C, C, X)$ as a "simplicial resolution" of a C-algebra X (cf. Theorem 11).

Theorem 21. *$X_\bullet$ is a strong deformation retract of $B_\bullet(C, C, X)$ in $\mathcal{SV}$.*

Proof. To $X \in \mathcal{V}$, we can associate a simplicial object $X_\bullet \in \mathcal{SV}$ such that $X_q = X$ for all q and face and degeneracy maps are all identity. Define

$$\tau_*(\eta) = \{\tau_q(\eta) := s_0^q \circ \eta\}_{q \geq 0} : X_\bullet \to B_\bullet(C, C, X)$$
$$\epsilon_*(\lambda) = \{\epsilon_q(\lambda) := \lambda \circ d_0^q\}_{q \geq 0} : B_\bullet(C, C, X) \to X_\bullet$$

We claim that $\epsilon_*(\lambda) \circ \tau_*(\eta) = id$. Compute

$$[\epsilon_*(\lambda) \circ \tau_*(\eta)](X_0) = [(\lambda \circ d_0^0) \circ (s_0^q \circ \eta)](X_0) = [\lambda \circ \eta](X_0).$$

By Definition 19 of C-algebras, the composition above is the identity. Since the face and degeneracy maps in $X_\bullet$ are all identity, the composition is identity on any X_q.

Define the simplicial homotopy $h_i : B_q(C, C, X) \to B_{q+1}(C, C, X)$ by

$$h_i = s_0^i \eta d_0^i : C^{q+1}X \to C^{q+2}X,$$

where η is regard as a map $C^{q+1-i}X \to C^{q+2-i}X$. One can verify that h is a homotopy from the identity map on $B_\bullet(C, C, X)$ to $\tau_*(\eta)\epsilon_*(\lambda)$ by checking all the identities in Definition 2. It remains to check $h_i \circ \tau_q(\eta) = \tau_{q+1}(\eta)$ for i. Compute

$$\begin{aligned}
h_i \circ \tau_q(\eta) &= (s_0^i \circ \eta \circ d_0^i) \circ (s_0^q \circ \eta) \\
&= s_0^i \circ \eta \circ (d_0^i \circ s_0^q) \circ \eta \\
&= s_0^i \circ \eta \circ (s_0^{q-i}) \circ \eta && \text{by simplicial identity } d_i \circ s_i = 1. \\
&= s_0^i \circ s_0 \circ s_0^{q-i} \circ \eta && \text{since } \eta \text{ can be identified with } s_0 \\
&= s_0^q \circ \eta \\
&= \tau_{q+1}(\eta).
\end{aligned}$$

□

The monadic bar construction can be used to "deloop" a space. This is not surprising because the classifying space BG is a "delooping" of the topological group G (cf. Theorem 16).

Theorem 22. *Let X be a algebra over the monad $\Omega^n\Sigma^n$. Then*

$$X \leftarrow B(\Omega^n\Sigma^n, \Omega^n\Sigma^n, X) \to \Omega^n B(\Sigma^n, \Omega^n\Sigma^n, X)$$

displays a weak homotopy equivalence between X and $\Omega^n B(\Sigma^n, \Omega^n\Sigma^n, X)$.

Theorem 13.1 in [4] proves a more general case, showing that this chain of homotopy equivalence holds for any monad C arising from an operad.

3.1. **Relations to the monoidal bar construction.** The monoidal bar construction can be viewed as a special case of monadic bar construction.

To any monoid (G, μ, η) in a symmetric category $(\mathcal{V}, \otimes, \kappa)$, we associate a monad in $\mathcal{C}$ which, by a slight abuse of notation, is still denoted by (G, μ, η):

$$\begin{aligned}
G &: \mathcal{V} \to \mathcal{V}, && V \mapsto G \otimes V \\
\mu &: GG \to G, && G \otimes G \otimes V \mapsto \mu(G \otimes G) \otimes V \\
\eta &: 1_{\mathcal{V}} \to G, && \kappa \otimes V \mapsto \eta(\kappa) \otimes V
\end{aligned}$$

The diagrams in Definition 3 ensure that G is a monad in $\mathcal{V}$. It follows a left G-object is precisely a G-algebra and a right G-object is precisely a G-functor. This means we can identify $\mathcal{B}(\mathcal{V})$ with $\mathcal{B}(\mathcal{V},\mathcal{V})$ and the following diagram commutes:

$$\begin{array}{ccc}
\begin{array}{c}\mathcal{B}(\mathcal{V})\\ (M,G,N)\end{array} & \xrightarrow{\text{Monoidal bar construction}} & \begin{array}{c}\mathcal{S}\mathcal{V}\\ B_\bullet(M,G,N)\end{array} \\
\cong\downarrow & & \downarrow\cong \\
\begin{array}{c}\mathcal{B}(\mathcal{V},\mathcal{V})\\ (M,G,N)\end{array} & \xrightarrow[\text{Monadic bar construction}]{} & \begin{array}{c}\mathcal{S}\mathcal{V}\\ \mathcal{B}_\bullet(M,G,N)\end{array}
\end{array}$$

4. The operadic bar construction

We now focus the attention on bar constructions on operads. Let $(\mathcal{V},\otimes,\kappa)$ be a symmetric monoidal category.

4.1. Definitions of an operad.

Definition 23. [6] An *operad* $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\eta : \kappa \to \mathcal{C}(1)$, a right action of the symmetric group Σ_j on $\mathcal{C}(j)$ for each j and a product map

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j)$$

for $k \geq 1$ and $j_s \geq 0$, where $\Sigma j_s = j$. The product map γ is required to be associative, unital and equivariant in the following senses.

(a) The following associativity diagrams commute, where $\Sigma j_s = j$ and $\Sigma i_t = i$; we let $g_s = j_1 + \ldots + j_s$ and $h_s = i_{g_{s-1}+1} + \ldots + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \left(\bigotimes_{s=1}^{k} \mathcal{C}(j_s)\right) \otimes \left(\bigotimes_{r=1}^{j} \mathcal{C}(i_r)\right) & \xrightarrow{\gamma\otimes id} & \mathcal{C}(j) \otimes \left(\bigotimes_{r=1}^{j} \mathcal{C}(i_r)\right) \\
 & & \downarrow\gamma \\
\cong\downarrow & & \mathcal{C}(i) \\
 & & \uparrow\gamma \\
\mathcal{C}(k) \otimes \left[\bigotimes_{s=1}^{k} \left(\mathcal{C}(j_s) \otimes \left(\bigotimes_{q=1}^{j_s} \mathcal{C}(i_{g_{s-1}+q})\right)\right)\right] & \xrightarrow[id\otimes(\otimes_s\gamma)]{} & \mathcal{C}(k) \otimes \left(\bigotimes_{s=1}^{k} \mathcal{C}(h_s)\right)
\end{array}$$

(b) The following unit diagrams commute:

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes (\kappa)^k & \xrightarrow{\cong} & \mathcal{C}(k) \\
id\otimes\eta^k \downarrow & \nearrow_{\gamma} & \\
\mathcal{C}(k) \otimes \mathcal{C}(1)^k & &
\end{array}
\qquad
\begin{array}{ccc}
\kappa \otimes \mathcal{C}(k) & \xrightarrow{\cong} & \mathcal{C}(j) \\
\eta\otimes id \downarrow & \nearrow_{\gamma} & \\
\mathcal{C}(1) \otimes \mathcal{C}(j) & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$, the permutation $\sigma(j_1,\ldots,j_k) \in \Sigma_j$ permute k blocks of letters as σ permutes

k letters and $\tau_1 \oplus \ldots \oplus \tau_k \in \Sigma_j$ is the block sum:

$$\begin{CD}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \mathcal{C}(j_k) @>{\sigma\otimes\sigma^{-1}}>> \mathcal{C}(k) \otimes \mathcal{C}(j_{\sigma(1)}) \otimes \ldots \otimes \mathcal{C}(j_{\sigma(k)}) \\
@V{\gamma}VV @VV{\gamma}V \\
\mathcal{C}(j) @>>{\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})}> \mathcal{C}(j)
\end{CD}$$

and

$$\begin{CD}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \mathcal{C}(j_k) @>{id\otimes\tau_1\otimes\ldots\otimes\tau_k}>> \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \mathcal{C}(j_k) \\
@V{\gamma}VV @VV{\gamma}V \\
\mathcal{C}(j) @>>{\tau_1\oplus\ldots\oplus\tau_k}> \mathcal{C}(j)
\end{CD}$$

The $\mathcal{C}(j)$ are to be thought of as objects of parameters for "j-ary operations" that accept j inputs and produce one output. Thinking of elements as operations, we think of $\gamma(c \otimes d_1 \otimes \ldots \otimes d_k)$ as the composite of the operation c with the $\otimes$-product of the operations d_s.

Definition 24. Let $\mathcal{C}$ be an operad. A $\mathcal{C}$-*algebra* is an object X together with maps

$$\theta : \mathcal{C}(j) \otimes X^{\otimes j} \to X$$

for $j \geq 0$ that are associative, unital and equivariant in the following senses.

(a) The following associativity diagrams commute, where $j = \sum j_s$.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \otimes X^{\otimes j} & \xrightarrow{\gamma\otimes id} & \mathcal{C}(j) \otimes X^{\otimes j} \\
 & & \downarrow \theta \\
\cong \downarrow & & X \\
 & & \uparrow \theta \\
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes X^{\otimes j_1} \otimes \ldots \otimes \mathcal{C}(j_k) \otimes X^{\otimes j_k} & \xrightarrow[id\otimes\theta^k]{} & \mathcal{C}(k) \otimes X^{\otimes k}
\end{array}$$

(b) The following unit diagram commutes:

$$\begin{array}{ccc}
\kappa \otimes X & \xrightarrow{\cong} & X \\
\eta\otimes id \downarrow & \nearrow \theta & \\
\mathcal{C}(1) \otimes X & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_j$:

$$\begin{array}{ccc}
\mathcal{C}(j) \otimes X^{\otimes j} & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{C}(j) \otimes X^{\otimes j} \\
 & {}_{\gamma}\searrow \quad \swarrow_{\gamma} & \\
 & X &
\end{array}$$

In order to obtain an operadic bar construction, we shall use an equivalent definition of an operad due to Kelly[3]. We first introduce the notion of Σ-sequences upon which this equivalent notion of operad is based.

Let Σ be the category whose objects are the finite sets $\boldsymbol{n} = \{1, \ldots, n\}$ where $\mathbf{0}$ is the empty set and morphisms are the symmetric group S_n.

Definition 25. A Σ-sequence in $\mathcal{V}$ is a functor[4] $\mathcal{C} : \Sigma^{op} \to \mathcal{V}$ with $\mathcal{C}(0) = \kappa$. The morphisms between Σ-sequences are natural transformations.

Since $\mathcal{C}$ is contravariant, Σ_j gives a right action on $\mathcal{C}(j)$. In other words, given $\tau, \sigma \in \Sigma_j$ and $f \in \mathcal{C}(j)$,

$$\mathcal{C}(\tau\sigma)(f) = \mathcal{C}(\sigma)\mathcal{C}(\tau)(f)$$

Notation. Let $\Sigma[\mathcal{V}]$ denote the category of functors $\Sigma \to \mathcal{V}$ and $\Sigma^{op}[\mathcal{V}]$ denote the category of functor $\Sigma^{op} \to \mathcal{V}$.

Construction 26. An object X in $\mathcal{V}$ induces a covariant functor

$$X^* : \Sigma \to \mathcal{V}$$
$$\boldsymbol{n} \mapsto X^{\otimes n}$$

Proof. Every element $\sigma \in \Sigma_j$ has a left action on $X^{\otimes j}$ given by permutation of factors in the tensor product, i.e.

$$\mathcal{C}(\sigma) : \boldsymbol{x} := x_1 \otimes \ldots \otimes x_j \mapsto x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(j)}$$

We shall verify $\mathcal{C}(\sigma)$ is a left action. Let $\boldsymbol{y} := y_1 \otimes \ldots \otimes y_j = \mathcal{C}(\sigma)(\boldsymbol{x})$. Take $\tau \in \Sigma_j$. Compute

$$\begin{aligned}\mathcal{C}(\tau)\mathcal{C}(\sigma)(\boldsymbol{x}) = \mathcal{C}(\tau)(\boldsymbol{y}) &= y_{\tau^{-1}(1)} \otimes \ldots \otimes y_{\tau^{-1}(j)} \\ &= x_{\sigma^{-1}(\tau^{-1}(1))} \otimes \ldots \otimes x_{\sigma^{-1}(\tau^{-1}(j))} = \mathcal{C}(\tau\sigma)(\boldsymbol{x}).\end{aligned}$$

Therefore, X^* is indeed a covariant functor. □

Recall the tensor product of $\mathcal{C} \in \Sigma^{op}[\mathcal{V}]$ and $\mathcal{D} \in \Sigma[\mathcal{V}]$ is

$$\mathcal{C} \otimes_\Sigma \mathcal{D} = \coprod_{k \geq 0} \mathcal{C}(k) \otimes_{\Sigma_k} \mathcal{D}(k).$$

In particular, given $\mathcal{C} \in \Sigma^{op}[\mathcal{V}]$ and $X^* \in \Sigma[\mathcal{V}]$,

$$\mathcal{C} \otimes_\Sigma X^* = \coprod_{k \geq 0} \mathcal{C}(k) \otimes_{\Sigma_k} X^{\otimes k}.$$

Theorem 27. *The category* $(\Sigma^{op}[\mathcal{V}], \otimes_{Day}, \mathcal{I}_0)$ *is a symmetric monoidal structure where* $\mathcal{I}_0(n) = \begin{cases} \kappa & n = 0 \\ \emptyset & n > 0 \end{cases}$. *Given* $\mathcal{C}, \mathcal{D} \in \Sigma^{op}[\mathcal{V}]$, *the Day convolution can be described explicitly as:*

$$(\mathcal{C} \otimes_{Day} \mathcal{D})(j) = \coprod_{j_1 + j_2 = j} (\mathcal{C}(j_1) \otimes \mathcal{D}(j_2)) \otimes_{\Sigma_{j_1} \times \Sigma_{j_2}} \Sigma_j.$$

The theorem above allows us to apply Construction 26 for $\mathcal{C} \in \Sigma^{op}[\mathcal{V}]$. In other words, the Day tensor power $\mathcal{C}^{\otimes_{Day} n}$ induces a covariant functor

$$\mathcal{C}^* : \Sigma \to \Sigma^{op}[\mathcal{V}]$$

Note that $\mathcal{C}^*$ is itself a covariant functor $\Sigma \to \Sigma^{op}[\mathcal{V}]$. Since it takes values in $\Sigma^{op}[\mathcal{V}]$, $\mathcal{C}^*$ is also a contravariant functor $\Sigma^{op} \to \mathcal{V}$ and induces

$$\mathcal{C}^* : \Sigma \times \Sigma^{op} \to \mathcal{V}$$

[4]Equivalently, $\mathcal{C}$ is a contravariant functor $\Sigma \to \mathcal{V}$.

whose (k, j) component is

$$\mathcal{C}^*(k,j) = \coprod_{j_1+\ldots+j_k=j} (\mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k)) \otimes_{\Sigma_{j_1}\times\ldots\times\Sigma_{j_k}} \Sigma_j.$$

Here $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$ can be embedded as a subgroup of Σ_j via block sums of permutations. For the covariant action of Σ, the symmetric group Σ_k permutes $j_1, \ldots, j_k$ in the coproduct. For the contravariant action of Σ, Σ_j has a right action on the last factor Σ_j.

Theorem 28. *Given $\mathcal{C} \in \Sigma^{op}[\mathcal{V}]$ and $X^* \in \Sigma[\mathcal{V}]$, the following two covariant functors are isomorphic*

$$(\mathcal{C} \otimes_\Sigma X^*)^* \cong \mathcal{C}^* \otimes_\Sigma X^*.$$

Proof. This theorem means that the $*$ construction commutes with $-\otimes_\Sigma X^*$. Therefore, it suffices to check that

$$- \otimes_\Sigma X^* : (\Sigma^{op}[\mathcal{V}], \otimes_{Day}, \mathcal{I}_0) \to (\mathcal{V}, \otimes, \kappa)$$

is symmetric monoidal. Take $\mathcal{C}, \mathcal{D} \in \Sigma^{op}[\mathcal{V}]$.

$$\begin{aligned}
&(\mathcal{C} \otimes_{Day} \mathcal{D}) \otimes_\Sigma X^* \\
&= \coprod_{k\geq 0} \left(\coprod_{j_1+j_2=k} (\mathcal{C}(j_1) \otimes \mathcal{D}(j_2)) \otimes_{\Sigma_{j_1}\times\Sigma_{j_2}} \Sigma_k \right) \otimes_{\Sigma_k} X^{\otimes k} \\
&= \coprod_{k\geq 0} \coprod_{j_1+j_2=k} (\mathcal{C}(j_1) \otimes \mathcal{D}(j_2)) \otimes_{\Sigma_{j_1}\times\Sigma_{j_2}} X^{\otimes k} \\
&\cong \coprod_{j_1,j_2\geq 0} (\mathcal{C}(j_1) \otimes \mathcal{D}(j_2)) \otimes_{\Sigma_{j_1}\times\Sigma_{j_2}} X^{\otimes j_1+j_2} \\
&\cong \coprod_{j_1,j_2\geq 0} \big(\mathcal{C}(j_1) \otimes_{\Sigma_{j_1}} X^{\otimes j_1}\big) \otimes \big(\mathcal{D}(j_2) \otimes_{\Sigma_{j_2}} X^{\otimes j_2}\big) \\
&\cong \left(\coprod_{j_1\geq 0} \mathcal{C}(j_1) \otimes_{\Sigma_{j_1}} X^{\otimes j_1} \right) \otimes \left(\coprod_{j_2\geq 0} \mathcal{D}(j_2) \otimes_{\Sigma_{j_2}} X^{\otimes j_2} \right) \\
&= (\mathcal{C} \otimes_\Sigma X^*) \otimes (\mathcal{D} \otimes_\Sigma X^*)
\end{aligned}$$

Furthermore, $\mathcal{I}_0 \otimes_\Sigma X^* \cong \kappa$. □

With these definition and theorems, we are able to define the $\boxtimes$ product of Σ-sequences, which endows $\Sigma^{op}[\mathcal{V}]$ with another monoidal structure.

Definition 29. Take $\mathcal{C}, \mathcal{D} \in \Sigma^{op}[\mathcal{V}]$. Define

$$\mathcal{C} \boxtimes \mathcal{D} = \mathcal{C} \otimes_\Sigma \mathcal{D}^*.$$

Explicitly, the j^{th} component is

$$\begin{aligned}
(\mathcal{C} \boxtimes \mathcal{D})(j) &= \coprod_{k\geq 0} \left(\mathcal{C}(k) \otimes_{\Sigma_k} \coprod_{j_1+\ldots+j_k=j} (\mathcal{D}(j_1) \otimes \ldots \otimes \mathcal{D}(j_k)) \otimes_{\Sigma_{j_1}\times\ldots\times\Sigma_{j_k}} \Sigma_j. \right) \\
&= \coprod_{j_1+\ldots+j_k=j} \mathcal{C}(k) \otimes_{\Sigma_k} (\mathcal{D}(j_1) \otimes \ldots \otimes \mathcal{D}(j_k) \otimes_{\Sigma_{j_1}\times\ldots\Sigma_{j_k}} \Sigma_j)
\end{aligned}$$

The unit of the $\boxtimes$ product is $\mathcal{I}_1 \in \Sigma^{op}[\mathcal{V}]$ given by

$$\mathcal{I}_1(\boldsymbol{n}) = \begin{cases} \kappa, & n = 0, 1; \\ \emptyset, & n > 1. \end{cases}$$

Σ_k has a right action on the second factor given by permuting the j_i's in product $\mathcal{D}(j_1) \otimes \ldots \otimes \mathcal{D}(j_k)$ and in $\Sigma_{j_1} \times \ldots \Sigma_{j_k}$.

Theorem 30. *Operads in $\mathcal{V}$ are monoids in the category $(\Sigma^{op}[\mathcal{V}], \boxtimes, \mathcal{I}_1)$.*

Proof. We refer to [3] for the proof that this definition of an operad agrees with Definition 23. □

4.2. **Operadic triples.** In this section, we shall give an explicit description of the category of operadic triples, denoted by $\mathcal{O}(\Sigma, \mathcal{V})$. It is an analogue of the monadic triples $\mathcal{B}(\Sigma, \mathcal{V})$.

For $C \in \Sigma^{op}[\mathcal{V}]$ and $X^* \in \Sigma[\mathcal{V}]$, define

$$\mathcal{C} \odot X^* := (\mathcal{C} \otimes_\Sigma X^*)^* \in \Sigma[\mathcal{V}].$$

Lemma 31. *For $\mathcal{C}, \mathcal{D} \in \Sigma^{op}[\mathcal{V}]$ and $X^* \in \Sigma[\mathcal{V}]$.*

$$(\mathcal{C} \boxtimes \mathcal{D}) \odot X^* \cong \mathcal{C} \odot (\mathcal{D} \odot X^*).$$

Proof. Using Theorem 28 and Definition 29, we can deduce

$$\begin{aligned} (\mathcal{C} \boxtimes \mathcal{D}) \odot X^* &\cong ((\mathcal{C} \boxtimes \mathcal{D}) \otimes_\Sigma X^*)^* \\ &\cong ((\mathcal{C} \otimes_\Sigma \mathcal{D}^*) \otimes_\Sigma X^*)^* \\ &\cong (\mathcal{C} \otimes_\Sigma (\mathcal{D}^* \otimes_\Sigma X^*))^* \\ &\cong (\mathcal{C} \otimes_\Sigma (\mathcal{D} \otimes_\Sigma X^*)^*)^* \\ &\cong \mathcal{C} \odot (\mathcal{D} \odot X^*) \end{aligned}$$

□

A little more explicitly, the isomorphism is given by

$$\begin{aligned} &\left[\Big(\mathcal{C}(k) \otimes_{\Sigma_k} (\mathcal{D}(j_1) \otimes \ldots \otimes \mathcal{D}(j_k) \otimes_{\Sigma_{j_1} \times \ldots \Sigma_{j_k}} \Sigma_j)\Big) \otimes_{\Sigma_j} X^{\otimes j}\right]^* \\ \cong &\left[\mathcal{C}(k) \otimes_{\Sigma_k} \mathcal{D}(j_1) \otimes \ldots \mathcal{D}(j_k) \otimes_{\Sigma_{j_1} \times \ldots \Sigma_{j_k}} X^{\otimes (j_1 + \ldots + j_k)}\right]^* \\ \cong &\left[\mathcal{C}(k) \otimes_{\Sigma_k} \Big((\mathcal{D}(j_1) \otimes_{\Sigma_{j_1}} X^{\otimes j_1}) \otimes \ldots \otimes (\mathcal{D}(j_k) \otimes_{\Sigma_{j_k}} X^{\otimes j_k})\Big)\right]^* \end{aligned}$$

Theorem 32. *Suppose $X \in \mathcal{V}$ is a $\mathcal{C}$-algebra with*

$$\theta : \mathcal{C} \otimes_\Sigma X^* = \coprod_j \mathcal{C}(j) \otimes_\Sigma X^{\otimes j} \to X.$$

It induces a functor $X^* \in \Sigma[\mathcal{V}]$ equipped with a natural transformation of covariant functors

$$\theta^* : \mathcal{C} \odot X^* = \Big(\mathcal{C} \otimes_\Sigma X^*\Big)^* \to X^*$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{I}_1 \odot X^* & \xrightarrow{\eta \odot id} & \mathcal{C} \odot X^* \\ & \searrow^{\cong} & \downarrow{\scriptstyle \theta^*} \\ & & X^* \end{array} \qquad \begin{array}{ccc} (\mathcal{C} \boxtimes \mathcal{C}) \odot X^* & \xrightarrow{\gamma \odot id} & \mathcal{C} \odot X^* \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \theta^*} \\ & & X^* \\ & & \uparrow{\scriptstyle \theta^*} \\ \mathcal{C} \odot (\mathcal{C} \odot X^*) & \xrightarrow[id \odot \theta^*]{} & \mathcal{C} \odot X^* \end{array}$$

where $\eta : \kappa \to \mathcal{C}(1)$ is the unit map of $\mathcal{C}$ and $\gamma : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ is the composition map of $\mathcal{C}$ (cf. Definition 23).

Proof. Observe

$$\mathcal{I}_1 \otimes_\Sigma X^* = \mathcal{I}_1(\mathbf{1}) \otimes_{\Sigma_1} X^{\otimes 1} = \kappa \otimes X.$$

Then the first diagram commutes because the unital condition of θ (cf. Definition 24(b)) implies

$$\begin{array}{ccc} (\kappa \otimes X)^* & \xrightarrow{\eta \odot id} & (\mathcal{C}(1) \otimes X)^* \\ & \searrow^{\cong} & \downarrow{\scriptstyle \theta^*} \\ & & X^* \end{array}$$

is commutative. Recall Definition 24(a) reads

$$\begin{array}{ccc} \mathcal{C}(k) \otimes_{\Sigma_k} \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \otimes_{\Sigma_j} X^{\otimes j} & \xrightarrow{\gamma \otimes id} & \mathcal{C}(j) \otimes_{\Sigma_j} X^{\otimes j} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \theta} \\ & & X \\ & & \uparrow{\scriptstyle \theta} \\ \mathcal{C}(k) \otimes_{\Sigma_k} \left(\mathcal{C}(j_1) \otimes_{\Sigma_{j_1}} X^{\otimes j_1}\right) \otimes \ldots \otimes \left(\mathcal{C}(j_k) \otimes_{\Sigma_{j_k}} X^{\otimes j_k}\right) & \xrightarrow[id \otimes \theta^k]{} & \mathcal{C}(k) \otimes_{\Sigma_k} X^{\otimes k} \end{array}$$

The induced diagram of covariant functors also commutes:

$$\begin{array}{ccc} \left[\mathcal{C}(k) \otimes_{\Sigma_k} \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \otimes_{\Sigma_j} X^{\otimes j}\right]^* & \xrightarrow{\gamma \odot id} & \left[\mathcal{C}(j) \otimes_{\Sigma_j} X^{\otimes j}\right]^* \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \theta^*} \\ & & X^* \\ & & \uparrow{\scriptstyle \theta^*} \\ \left[\mathcal{C}(k) \otimes_{\Sigma_k} \left(\mathcal{C}(j_1) \otimes_{\Sigma_{j_1}} X^{\otimes j_1}\right) \otimes \ldots \otimes \left(\mathcal{C}(j_k) \otimes_{\Sigma_{j_k}} X^{\otimes j_k}\right)\right]^* & \xrightarrow[id \odot \theta^k]{} & \left[\mathcal{C}(k) \otimes_{\Sigma_k} X^{\otimes k}\right]^* \end{array}$$

Observe that the element in the upper left corner belongs to $(\mathcal{C} \boxtimes \mathcal{C}) \odot X^*$ and the element in the bottom left corner belongs to $\mathcal{C} \odot (\mathcal{C} \odot X^*)$. Therefore, the second diagram in the theorem commutes. □

Definition 33. Let (X, θ) be $\mathcal{C}$-algebra. A left $\mathcal{C}$-object is the induced covariant functor X^* equipped with θ^*.

It is noteworthy to point out the left $\mathcal{C}$-object X^* defined above is not a left object in the sense of Definition 4. In this case, X^* is in $\Sigma[\mathcal{V}]$, not $\Sigma^{op}[\mathcal{V}]$ and θ^* is a natural transformation of covariant functors.

Definition 34. A right $\mathcal{C}$-object (M,ρ) is a functor $M \in \Sigma^{op}[\mathcal{V}]$ equipped with a right action $\rho : M \boxtimes \mathcal{C} \to M$, i.e.

$$\rho : \coprod_{k,j_1,\ldots,j_k} M(k) \otimes_{\Sigma_k} \Big(\mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k)\Big) \otimes_{\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}} \Sigma_j \to M(j)$$

such that the following diagrams commute:

$$\begin{array}{ccc} M \boxtimes \mathcal{I}_1 & \xrightarrow{id \boxtimes \eta} & M \boxtimes \mathcal{C} \\ & \searrow^{\cong} & \downarrow \rho \\ & & M \end{array} \qquad \begin{array}{ccc} M \boxtimes \mathcal{C} \boxtimes \mathcal{C} & \xrightarrow{\rho \boxtimes id} & M \boxtimes \mathcal{C} \\ {\scriptstyle id \boxtimes \gamma} \downarrow & & \downarrow \rho \\ M \boxtimes \mathcal{C} & \xrightarrow[\rho]{} & M \end{array}$$

where $\eta : \kappa \to \mathcal{C}(1)$ is the unit map and $\gamma : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ is the composition map of $\mathcal{C}$ (cf. 23).

Notation. Denote by $\mathcal{O}(\Sigma, \mathcal{V})$ the category of operadic triples. The objects are triples

$$((M,\rho), (\mathcal{C}, \gamma, \eta), (X^*, \theta^*))$$

abbreviated $(M, \mathcal{C}, X^*)$ where $\mathcal{C}$ is an operad regarded as a monoid in $(\Sigma^{op}(\mathcal{V}), \boxtimes, \mathcal{I}_1)$, X^* is a left $\mathcal{C}$ object and M is a right $\mathcal{C}$ object. The morphisms are natural transformations of either covariant or contravariant functors.

4.3. **Bar construction.** In this section, we shall define the operadic bar construction,

$$B_\bullet : \mathcal{O}(\Sigma, \mathcal{V}) \to \mathcal{S}\mathcal{V},$$

which is a functor from the category of operadic triples to the category of simplicial objects in $\mathcal{V}$. Note that since X^* here is a covariant functor, the operadic bar construction $B_\bullet(M, \mathcal{C}, X^*)$ is not the same as the monoidal bar construction given in Construction 5.

Given an operadic triple $(M, \mathcal{C}, X^*)$, construct $B_\bullet(M, \mathcal{C}, X^*)$ as follows. The q-dimensional component of $B_\bullet(M, \mathcal{C}, X^*)$ is

$$B_q(M, \mathcal{C}, X^*) = M \otimes_\Sigma (\mathcal{C} \odot (.... \odot (\mathcal{C} \odot (\mathcal{C} \odot X^*))...))$$

where $\mathcal{C}$ appears q-times, or equivalently, by Lemma 31

$$B_q(M, \mathcal{C}, X^*) = (...((M \boxtimes \mathcal{C}) \boxtimes \mathcal{C}) \boxtimes \ldots \boxtimes \mathcal{C}) \otimes_\Sigma X^*$$

The $q+1$ face maps $d_\bullet : B_q(M, \mathcal{C}, X^*) \to B_{q-1}(M, \mathcal{C}, X^*)$ are

$$d_i : B_q(M, \mathcal{C}, X^*) \mapsto \begin{cases} (\rho(M \boxtimes \mathcal{C}) \boxtimes \mathcal{C}^{\boxtimes q-1}) \otimes_\Sigma X^* & i = 0 \\ M \boxtimes \mathcal{C}^{\boxtimes i-1} \boxtimes \gamma(\mathcal{C} \boxtimes \mathcal{C}) \boxtimes \mathcal{C}^{\boxtimes q-i-1} \otimes_\Sigma X^* & 0 < i < q \\ M \otimes_\Sigma (\mathcal{C}^{\odot q-1} \odot \xi(\mathcal{C} \odot X^*)) & i = q. \end{cases}$$

The $q+1$ degeneracy maps $s_i : B_q(M, \mathcal{C}, X^*) \to B_{q-1}(M, \mathcal{C}, X^*)$ insert a new copy of $\mathcal{C}$ and set that coordinate to be $\mathcal{I}_1$.

Theorem 35. *The operadic bar construction $B_\bullet(M, \mathcal{C}, X^*)$ is in $\mathcal{S}\mathcal{V}$.*

Proof. Firstly, observe that all the components are in $\mathcal{V}$. It suffices to check that the face maps $d_i : B_q(M, \mathcal{C}, X^*) \to B_{q-1}(M, \mathcal{C}, X^*)$ satisfy the simplicial identity:

$$d_i \circ d_j = d_{j-1} \circ d_i$$

where $0 \leq i < j \leq q$. When $0 < i < j < q$, the maps d_i, d_j and d_{j-1} are all given by γ. The identity holds since γ is associative. When $i < j-1$, the identity holds because d_i and d_{j-1} don't interact with each other. It remains to check the following two extreme cases. Suppose $i = 0$ and $j = 1$. Without loss of generality, assume $q = 2$. Compute directly

$$(d_0 \circ d_1)(((M \boxtimes \mathcal{C}) \boxtimes \mathcal{C}) \otimes_\Sigma X^*) = \rho(M \boxtimes \gamma(\mathcal{C} \boxtimes \mathcal{C})) \otimes_\Sigma X^*$$
$$(d_0 \circ d_0)(((M \boxtimes \mathcal{C}) \boxtimes \mathcal{C}) \otimes_\Sigma X^*) = \rho(\rho(M \boxtimes \mathcal{C}) \boxtimes \mathcal{C}) \otimes_\Sigma X^*$$

The maps give the same elements exactly because the right action ρ satisfies that $\rho \circ (\rho \boxtimes id) = \rho \circ (id \boxtimes \gamma)$. Suppose $i = q-1$ and $j = q$. Again, it sufficient to check the case of $q = 2$. Compute

$$(d_1 \circ d_1)(M \otimes_\Sigma ((\mathcal{C} \boxtimes \mathcal{C}) \odot X^*)) = M \otimes_\Sigma \theta^*(\gamma(\mathcal{C} \boxtimes \mathcal{C}) \odot X^*)$$
$$(d_1 \circ d_2)(M \otimes_\Sigma (\mathcal{C} \odot (\mathcal{C} \odot X^*))) = M \otimes_\Sigma \theta^*(\mathcal{C} \odot \theta^*(\mathcal{C} \odot X^*))$$

The two elements on the right hand sides agree because θ^* satisfies $\theta^* \circ (\gamma \odot id) = \theta^* \circ (id \odot \theta^*)$. □

4.4. **Relations to the monadic bar construction.** Recall a monadic triple $(F, C, X) \in \mathcal{B}(\mathcal{V}, \mathcal{W})$ consists of a monad C, a C-functor $F : \mathcal{V} \to \mathcal{W}$ and a C-algebra X. We shall build a functor

$$\varphi : \mathcal{O}(\Sigma, \mathcal{V}) \to \mathcal{B}(\mathcal{V}, \mathcal{V})$$

Theorem 36. *The monad C in $\mathcal{V}$ associated to an operad $\mathcal{C}$ is*

$$C : \mathcal{V} \to \mathcal{V}$$
$$Y \mapsto \mathcal{C} \otimes_\Sigma Y^* = \coprod_{j \geq 0} \mathcal{C}(j) \otimes_{\Sigma_j} Y^j.$$

Proof. It is immediate that $CY \in \mathcal{V}$. It remains to check C indeed is a monad. See Construction 2.4 in [4] for the proofs. □

Given (X^*, θ^*) a left $\mathcal{C}$-object, the associated C-algebra is the underlying $\mathcal{C}$-algebra (X, θ). Let $(M, \rho : M \boxtimes C \to M)$ be a right $\mathcal{C}$-object. We often denote by F_M the C-functor arising from M. Define

$$F_M := M \otimes_\Sigma (-)^* : \mathcal{V} \to \mathcal{V}$$
$$Y \mapsto M \otimes_\Sigma Y^*$$

The monad C acts on F on the right by

$$F_M C = (M \boxtimes C) \otimes_\Sigma (-)^* : \mathcal{V} \to \mathcal{V}$$
$$Y \mapsto (M \boxtimes \mathcal{C}) \otimes_\Sigma Y^* = M \otimes_\Sigma \mathcal{C}^* \otimes_\Sigma Y^*$$

The right action $F_M C \to F_M$ can be described as

$$F_M C(Y) = (M \boxtimes \mathcal{C}) \otimes_\Sigma Y^* \xrightarrow{\rho \otimes id} M \otimes_\Sigma Y^* = F_M(Y).$$

Theorem 37. *$B_\bullet(F_M, C, X)$ is isomorphic to $B_\bullet(M, \mathcal{C}, X^*)$ as simplicial objects in $\mathcal{V}$.*

Proof. By construction,

$$F_M C^q = (M \boxtimes C^{\boxtimes q}) \otimes_\Sigma (-)^*.$$

Therefore,

$$F_M C^q X = (M \boxtimes C^{\boxtimes q}) \otimes_\Sigma X^* = B_q(M, \mathcal{C}, X^*).$$

□

With these constructions, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\begin{array}{c}\mathcal{O}(\Sigma, \mathcal{V}) \\ (M, \mathcal{C}, X^*)\end{array} & \xrightarrow{\text{Monoidal bar construction}} & \begin{array}{c}\mathcal{S}\Sigma[\mathcal{V}] \\ B_\bullet(M, \mathcal{C}, X^*)\end{array} \\
\Big\downarrow \varphi & & \Big\downarrow \cong \\
\begin{array}{c}\mathcal{B}(\mathcal{V}, \mathcal{V}) \\ (F_M, C, X)\end{array} & \xrightarrow[\text{Monadic bar construction}]{} & \begin{array}{c}\mathcal{S}\mathcal{V} \\ B_\bullet(F_M, C, X)\end{array}
\end{array}$$

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