

ON THE PRINCIPAL EIGENVECTOR OF A GRAPH

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ABSTRACT. The principal ratio of a connected graph G , $\gamma(G)$, is the ratio between the largest and smallest coordinates of the principal eigenvector of the adjacency matrix of G . Over all connected graphs on n vertices, $\gamma(G)$ ranges from 1 to n^{cn} . Moreover, $\gamma(G) = 1$ if and only if G is regular. This indicates that $\gamma(G)$ can be viewed as an irregularity measure of G , as first suggested by Tait and Tobin (El. J. Lin. Alg. 2018). We are interested in how stable this measure is. In particular, we ask how γ changes when there is a small modification to a regular graph G . We show that this ratio is polynomially bounded if we remove an edge belonging to a cycle of bounded length in G , while the ratio can jump from 1 to exponential if we join a pair of vertices at distance 2. We study the connection between the spectral gap of a regular graph and the stability of its principal ratio. A naive bound shows that given a constant multiplicative spectral gap and bounded degree, the ratio remains polynomially bounded if we add or delete an edge. Using results from matrix perturbation theory, we show that given an *additive* spectral gap larger than $2\sqrt{n}$, the ratio stays bounded after adding or deleting an edge.

CONTENTS

1. Introduction	2
2. General preliminaries	3
2.1. Definitions and notation	3
2.2. Results from linear algebra	4
2.3. Results from spectral graph theory	6
2.3.1. Observations about the adjacency operator	6
2.3.2. Existence of the principal eigenvector for a connected graph	6
2.3.3. Bounds on the largest eigenvalue for a graph	7
2.3.4. The largest eigenvalue of subgraphs	8
2.3.5. Graph products	8
3. Preliminary results about the principal eigenvector	9
3.1. Observations and naive bounds on the ratio	9
3.2. Chebyshev polynomials and principal eigenvectors	11
3.2.1. Chebyshev polynomials	11
3.2.2. Applications to principal eigenvectors	11
4. Main results	12
4.1. Adding or removing an edge in bounded distance	12
4.1.1. Adding an edge	12

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4.1.2. Removing an edge	14
4.2. Multiplicative spectral gap and stability of the ratio	15
4.3. Additive spectral gap and stability of the ratio	16
4.3.1. Existence of graphs with large additive spectral gap and large diameter	16
4.3.2. Large additive spectral gap implies bounded ratio	17
4.4. Open questions	20
Acknowledgments	21
References	21

1. INTRODUCTION

It is known that the adjacency matrix A_G of every connected graph G has a simple largest eigenvalue λ_1 , and that λ_1 has an eigenvector with all-positive coordinates, called the principal eigenvector of G , which we denote by \mathbf{q} . Therefore a unique-up-to-scaling all-positive eigenvector can be associated with every connected graph. Then it is natural to study how \mathbf{q} reflects the structure of the graph. All our discussions will be asymptotic as the number of vertices approaches infinity in a family of graph.

Cioabă and Gregory [11] first defined the principal ratio of G , $\gamma(G) = q_{\max}/q_{\min}$, to be the ratio between the largest and smallest coordinates of \mathbf{q} . This ratio is 1 for regular graphs, while it can grow at factorial rate (i.e., $\gamma(G) > n^{cn}$ for some positive constant c) [11]. Since $\gamma(G) \geq 1$ where equality holds if and only if G is regular, it is natural to think of $\gamma(G)$ as a measure of the irregularity of G . This view was suggested by Tait and Tobin [13].

A basic observation is that, given a connected graph G with largest eigenvalue λ_1 and diameter D , the principal ratio satisfies

$$(1.1) \quad \gamma(G) \leq \lambda_1^D.$$

We are interested in the stability of γ , i.e., how a slight change of G influences $\gamma(G)$. In particular, given a d -regular graph G , we ask how $\gamma(G)$ changes from the constant 1 if we add or remove one edge in G . (We call the resulting graphs $G + e$ and $G - e$, respectively.) We always assume the edge we remove will not disconnect G (i.e., e is a non-bridge edge), so that the principal eigenvector of $G - e$ is defined.

In Section 4.1, we study the cases where the edge we add to or remove from a regular graph is between vertices of bounded distance. We show that

- $\gamma(G + e)$ can jump to exponential in n when the degree is bounded [Theorem 4.3]. In our example, e connects two vertices at distance 2 in G . By (1.1), boundedness of the degree is necessary here.
- If we remove an edge belonging to a cycle of bounded length in G , $\gamma(G - e)$ is always polynomially bounded regardless of the degree [Theorem 4.11].

We also study the relevance of the spectral gap to the stability of $\gamma(G)$ for regular graphs. In Section 4.2, based on (1.1), we note that

- $\gamma(G \pm e)$ is always polynomially bounded in n when G is a bounded-degree expander graph, i.e., when the degree is bounded and the spectral gap of G is bounded from below [Observation 4.15].

In Section 4.3.2, we put this problem in the more general context of perturbations of matrices. By adapting theorems and proofs from Stewart and Sun's book [5] to our special case, we show that

- If there is an additive spectral gap larger than $2\sqrt{n}$, then $\gamma(G \pm e)$ is bounded [Theorem 4.17].

This result does not follow from (1.1). Indeed, in Section 4.3.1 we construct graphs with degree of order $n^{(2+t)/3}$ and additive spectral gap of order n^t , having diameter of order $n^{(1-t)/3}$ for any constant $0 < t < 1$. Similar applications of matrix perturbation theory in link analysis for networks can be found in [9].

When computing or giving bounds on the coordinates of the principal eigenvector for certain types of graphs, we take advantage of the properties of Chebyshev polynomials, a family of orthogonal polynomials which has found numerous applications in discrete mathematics. Here is an incomplete list of the areas of such applications:

- the matchings polynomial of graphs, by Heilmann and Lieb [1]
- approximate Inclusion–Exclusion, by Linial and Nisan [4]
- analysis of Boolean functions, in bounding the real degree of the OR function, by Nisan and Szegedy [6]
- the diameter of regular and bipartite biregular graphs, by van Dam and Haemers [7]
- counting restricted permutations, by Mansour and Vainshtein [8]
- the mixing rate of non-backtracking random walks, by Alon et al. [10].

In Section 3.2, we state some properties of Chebyshev polynomials and show their connection with the principal eigenvectors of certain graphs.

2. GENERAL PRELIMINARIES

2.1. Definitions and notation.

By a *graph* we mean what is often called a *simple graph* (undirected graph with no self-loops and no parallel edges). G will always denote a connected graph with n vertices. We denote by $V(G)$ and $E(G)$ the set of vertices and edges of G , respectively. We usually identify the set of vertices with the set $[n] = \{1, 2, \dots, n\}$, so the vertices are labeled $1, \dots, n$. We write $i \sim_G j$ if vertices i, j are adjacent in G . We denote by $N_G(j)$ the set of neighbors of j in G . We use $\deg_G(j)$ to denote the degree of vertex j in G . We write \overline{G} for the complement of G . Let $\text{dist}_G(i, j)$ denote the distance between vertices i and j in G . Let $D(G) := \max_{i,j} \text{dist}(i, j)$ denote the diameter of G .

We use $M_n(\mathbb{R})$ to denote the set of $n \times n$ real matrices. We write \mathbf{j} for the all ones vector, and J for the all-ones matrix. We use A_G to denote the adjacency matrix of G . We note that A_G is a real symmetric matrix, so its eigenvalues are real. We write $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ to denote the eigenvalues of A_G . We also denote $\lambda_1(G)$ by λ_G . We write $\mathbf{q}(G)$ for the principal eigenvector of A_G scaled to have l^2 norm 1. Let $q_i(G)$ denote the coordinate corresponding to vertex i in $\mathbf{q}(G)$. We write $q_{\max}(G)$ and $q_{\min}(G)$ for the maximum and minimum coordinates of $\mathbf{q}(G)$, and $v_{\max}(G)$ and $v_{\min}(G)$ for corresponding vertices. Recall that the principal ratio of G is defined as

$$(2.1) \quad \gamma(G) := \frac{q_{\max}(G)}{q_{\min}(G)}.$$

We write L_G for the Laplacian of G , defined as the $n \times n$ matrix

$$L_G = \text{diag}(\deg(1), \deg(2), \dots, \deg(n)) - A_G.$$

L_G is positive semidefinite. The principal eigenvalue of L_G is defined to be the eigenvalue corresponding to the eigenvector \mathbf{j} ; its value is zero. We write $\delta(G)$ to denote the smallest non-principal eigenvalue of L_G . We note that $\delta(G) = 0$ if and only if G is disconnected. $\delta(G)$ is the *algebraic connectivity* of the graph G , as first defined by Fiedler [2].

For a d -regular graph G , let $f_A(t)$ be the characteristic polynomial of its adjacency matrix A_G . The characteristic polynomial of the Laplacian L_G is

$$(2.2) \quad f_L(t) = f_A(d - t).$$

It follows that

$$(2.3) \quad \delta(G) = d - \lambda_2(G).$$

We refer to the right-hand side as the *additive spectral gap* of G . We refer to

$$(2.4) \quad \frac{\delta(G)}{d} = 1 - \frac{\lambda_2}{d}$$

as the *multiplicative spectral gap* of G . We use this terminology for regular graphs only.

In all notation, we omit the graph G when it is clear from context.

Let C_n denote the cycle with n vertices. Let P_r denote the path with r vertices; it has $r - 1$ edges. Let K_s denote the clique with s vertices; it has $\binom{s}{2}$ edges. Following the notation used in previous papers on this subject, we use $P_r \cdot K_s$ to denote the graph obtained by merging the vertex at one end of P_r with one vertex in K_s . So $P_r \cdot K_s$ has $n = r + s - 1$ vertices, $r - 1 + \binom{s}{2}$ edges, and diameter r . This has been called a *kite graph* or a *lollipop graph*. We will call it a kite graph.

By a *family of graphs*, we mean an infinite set of non-isomorphic finite graphs.

Let $f(n) \geq 1$. We say the rate of growth of $f(n)$ is polynomially bounded if for all sufficiently large n , $f(n) < n^c$ for some constant c . We say $f(n)$ is exponential if for all sufficiently large n , $f(n) \geq a^n$ for some constant $a > 1$. We say $f(n)$ has factorial growth if for all sufficiently large n , $f(n) \geq n^{cn}$ for some positive constant c .

Given a family \mathcal{G} of graphs, we label the graphs as G_1, \dots, G_i, \dots , and let n_1, \dots, n_i, \dots be the corresponding number of vertices. We say $\gamma(\mathcal{G})$ is polynomially bounded in n if there is some constant c such that for all sufficiently large i , $\gamma(G_i) < n_i^c$. We say $\gamma(\mathcal{G})$ grows exponentially in n if there is some constant $a > 1$ such that for all sufficiently large i , $G_i > a^{n_i}$.

2.2. Results from linear algebra.

In this section we introduce results from linear algebra that we will use for later proofs. Orthonormality in \mathbb{R}^n refers to the standard dot product. Given a matrix A , we write a_{ij} for the entry on the i -th row and in the j -th column of A , and we write $A = (a_{ij})$.

Definition 2.5. For an $m \times n$ real matrix M , the operator norm induced by l^2 vector norm ($\|\cdot\|$) is

$$\|M\| = \sup_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}} \frac{\|M\mathbf{x}\|}{\|\mathbf{x}\|}.$$

Fact 2.6. In addition to being subadditive, the operator norm is also submultiplicative, i.e., $\|AB\| \leq \|A\|\|B\|$ for $A, B \in M_n(\mathbb{R})$.

Fact 2.7. For a symmetric real matrix M , $\|M\| = \max_{1 \leq i \leq n} |\lambda_i|$. Moreover, if M is non-negative, $\max_{1 \leq i \leq n} |\lambda_i|$ is attained by λ_1 . In particular, for the adjacency matrix A_G of a graph G ,

$$(2.8) \quad \|A_G\| = \lambda_1(G).$$

Theorem 2.9 (Spectral theorem for real symmetric matrices). *If M is an $n \times n$ real symmetric matrix (i.e., $M = M^T$), then M has an orthonormal eigenbasis over \mathbb{R} . In particular, all eigenvalues of M are real.*

Definition 2.10 (Fractional powers of positive semidefinite real symmetric matrices). Let $M \in M_n(\mathbb{R})$ be symmetric and positive semidefinite. Then we can write M as $M = Q\Lambda Q^T$ where Q is an orthogonal matrix, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, and $\lambda_i \geq 0$. For $a \in \mathbb{R}$, we define

$$M^a := Q \text{diag}(\lambda_1^a, \dots, \lambda_n^a) Q^T.$$

This definition is sound. (It does not depend on the particular choice of Q .) It follows that for $a, b \in \mathbb{R}$, $M^a \cdot M^b = M^{a+b}$.

In the rest of Section 2.2, $A = (a_{ij})$ will always denote a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Definition 2.11. A *multiset* is a set in which elements are allowed to have multiple instances. We denote a multiset by double braces. For example, $\{\{a, a, a, b, c, c\}\}$ is a multiset. We also write $\{\{a, a, a, b, c, c\}\}$ as $\{\{a^3, b, c^2\}\}$.

Definition 2.12. The *spectrum* of an $n \times n$ matrix M is the multiset of its eigenvalues. We denote it by $\text{spec}(M)$.

Fact 2.13 (Spectrum of polynomials of a matrix). If g is a polynomial, then

$$\text{spec}(g(A)) = \{\{g(\lambda_1), g(\lambda_2), \dots, g(\lambda_n)\}\}.$$

Definition 2.14. The *Rayleigh quotient* of the matrix A is the function

$$R_A(\mathbf{y}) = \frac{\mathbf{y}^T A \mathbf{y}}{\mathbf{y}^T \mathbf{y}} = \frac{\sum_{1 \leq i, j \leq n} a_{ij} y_i y_j}{\sum_{i=1}^n y_i^2},$$

defined for $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq 0$.

Observation 2.15. If \mathbf{y} is an eigenvector to eigenvalue λ_i , then $R_A(\mathbf{y}) = \lambda_i$.

Theorem 2.16 (Rayleigh's principle).

$$\lambda_1 = \max_{\mathbf{y}} R_A(\mathbf{y}).$$

$$\lambda_n = \max_{\mathbf{y}} R_A(\mathbf{y}).$$

Moreover, $R_A(\mathbf{y}) = \lambda_1$ if and only if \mathbf{y} is an eigenvector to λ_1 .

Corollary 2.17. Given vector $\mathbf{y} = (y_1, \dots, y_n)^T$, we let $|\mathbf{y}|$ be the vector $(|y_1|, \dots, |y_n|)^T$. If \mathbf{y} is an eigenvector to eigenvalue λ_1 , then $|\mathbf{y}|$ is also an eigenvector to λ_1 .

Definition 2.18. Let $B = (b_{ij})$ be an $m \times n$ matrix and let M be a $p \times q$ matrix. The *Kronecker product* of B and M , $B \otimes M$, is the $mp \times nq$ matrix

$$\begin{pmatrix} b_{11}M & b_{12}M & \cdots & b_{1n}M \\ b_{21}M & b_{22}M & \cdots & b_{2n}M \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}M & b_{m2}M & \cdots & b_{mn}M \end{pmatrix}.$$

Fact 2.19. Let $B \in M_n(\mathbb{R})$ and $M \in M_m(\mathbb{R})$. Let the eigenvalues of B be $\lambda_1 \geq \dots \geq \lambda_n$ and let the eigenvalues of M be $\mu_1 \geq \dots \geq \mu_m$. Then

$$\text{spec}(B \otimes M) = \{\{\lambda_i \mu_j, 1 \leq i \leq n, 1 \leq j \leq m\}\}.$$

2.3. Results from spectral graph theory.

In this section we introduce well-known results from spectral graph theory that will be of use later.

2.3.1. Observations about the adjacency operator.

In Sections 2.3.1 through 2.3.3, we fix the graph G and write A for A_G .

We first note how the adjacency operator A of a graph G acts on vectors.

Observation 2.20. Given $\mathbf{y} = (y_1, \dots, y_n)$, the i th coordinate of $A\mathbf{y}$ is given by

$$(A\mathbf{y})_i = \sum_{j:i \sim j} y_j.$$

Corollary 2.21. If $\mathbf{y} = (y_1, \dots, y_n)$ is an eigenvector of A to eigenvalue ρ , then

$$\rho y_i = \sum_{j:j \sim i} y_j.$$

Recall that \mathbf{j} denotes the all-ones vector.

Corollary 2.22. \mathbf{j} is an eigenvector of A if and only if G is regular.

Proof. Suppose G is d -regular. Then $A\mathbf{j} = (\deg(1), \deg(2), \dots, \deg(n))^T = d\mathbf{j}$. Suppose $A\mathbf{j} = d\mathbf{j}$. Then for any $i \in n$, $|N(i)| = (d \cdot 1)/1 = d$. \square

2.3.2. Existence of the principal eigenvector for a connected graph.

In this section we establish the existence of the principal eigenvector for a connected graph. This follows from the Perron–Frobenius theorem for irreducible matrices, though given that the adjacency matrix is real and symmetric, it can be proved in much simpler ways using Rayleigh’s principle.

Proposition 2.23. If G is a connected graph and \mathbf{y} is an eigenvector to eigenvalue λ_1 , then \mathbf{y} has no zero coordinates.

Proof. It suffices to prove that $|\mathbf{y}|$ has no zero entries. By Corollary 2.17, $|\mathbf{y}|$ is an eigenvector to eigenvalue λ_1 . Suppose $|y_i| = 0$. Then by Corollary 2.21, $\sum_{j:j \sim i} |y_j| = 0$. Therefore $|y_j| = 0$ for all $j \sim i$. Since G is connected, by repeating this argument we have $\mathbf{y} = \mathbf{0}$, but by assumption $\mathbf{y} \neq \mathbf{0}$. \square

Proposition 2.24. If G is a connected graph and \mathbf{y} is an eigenvector to eigenvalue λ_1 , then the coordinates of \mathbf{y} are either all positive or all negative.

Proof. Suppose there are entries of opposite signs in \mathbf{y} . Since G is connected and \mathbf{y} has no zero entries, there has to be edges between vertices of positive entry and vertices of negative entry. Since the entries of A_G corresponding to these edges are 1, we can strictly increase $R_{A_G}(\mathbf{y})$ by changing the sign of all negative entries into positive. But this contradicts Theorem 2.16 (Rayleigh’s principle). \square

Corollary 2.25. For a connected graph G , λ_1 is simple.

Proof. Two all-positive or all-negative vectors cannot be orthogonal. \square

Theorem 2.26. For every connected graph G , the largest eigenvalue λ_1 is simple, and it has a unique-up-to-scaling all-positive eigenvector (the principal eigenvector).

Proof. Proposition 2.24 together with Corollary 2.17 proves the existence of an all-positive eigenvector to λ_1 . Since λ_1 is simple, this eigenvector is unique up to scaling. \square

Corollary 2.27. *If G is connected and \mathbf{y} is an eigenvector to eigenvalue λ with all-positive coordinates, then λ is the largest eigenvalue of A .*

Proof. Since λ_1 is simple, any eigenvector \mathbf{y} to an eigenvalue other than λ_1 must be orthogonal to the principal eigenvector. \square

Corollary 2.28. *For a connected d -regular graph G , $\lambda_1 = d$.*

Proof. Follows from Corollary 2.22. \square

Corollary 2.29. *For the clique on n vertices, $\lambda_1 = n - 1$.*

Observation 2.30. Corollary 2.28 also proves (2.3).

2.3.3. *Bounds on the largest eigenvalue for a graph.*

Let Δ denote $\max_{1 \leq i \leq n} \deg(i)$, the maximum degree of G .

Fact 2.31. For every graph G , $\lambda_1 \leq \Delta$. For connected graphs, equality holds if and only if G is regular.

Proof. Let \mathbf{y} be an eigenvector to λ_1 , and let y_i be the maximum coordinate, then

$$\lambda_1 y_i = \sum_{j:i \sim j} y_j \leq \Delta y_i.$$

Therefore $\lambda_1 \leq \Delta$. Suppose G is d -regular, then $\Delta = d = \lambda_1$ by Corollary 2.28. On the other hand, suppose G is connected and $\lambda_1 = \Delta$. Then $\Delta y_i = \sum_{j:i \sim j} y_j$. Therefore vertex i has degree Δ and the neighbors of i have coordinates as large as i . Repeat this argument for the neighbors of i . Since the graph is connected, we have that every vertex has degree Δ and the same coordinate as y_i . \square

We denote the arithmetic and quadratic mean of the degrees of vertices by

$$(2.32) \quad d_{\text{avg}} := \frac{\sum_{i=1}^n \deg(i)}{n}$$

and

$$(2.33) \quad d_{\text{qavg}} := \sqrt{\frac{\sum_{i=1}^n \deg(i)^2}{n}}.$$

It is well-known that the quadratic mean of a multiset of numbers is not less than the arithmetic mean.

Fact 2.34. For every graph G , $\lambda_1 \geq d_{\text{avg}}$.

Proof. By Theorem 2.16 (Rayleigh's principle),

$$\lambda_1(A) = \max_{\mathbf{y} \neq \mathbf{0}} R_A(\mathbf{y}) \geq R_A(\mathbf{j}) = \frac{\mathbf{j}^T A \mathbf{j}}{\mathbf{j}^T \mathbf{j}} = \frac{\sum_{i=1}^n \deg(i)}{n}. \quad \square$$

We can improve this to a stronger bound.

Fact 2.35. For every G , $\lambda_1 \geq d_{\text{qavg}}$.

Proof. Let $\text{spec}(G) = \{\{\lambda_1, \dots, \lambda_n\}\}$, where $\lambda_1 \geq \dots \geq \lambda_n$. By Fact 2.13,

$$\text{spec}(A^2) = \{\{\lambda_1^2, \dots, \lambda_n^2\}\}.$$

Let $\mathbf{y} = (y_1, \dots, y_n)^T$ be an eigenvector of λ_n , then by Observation 2.15, $\frac{\mathbf{y}^T G \mathbf{y}}{\mathbf{y}^T \mathbf{y}} = \lambda_n$. Since A is non-negative,

$$\frac{|\mathbf{y}|^T A |\mathbf{y}|}{|\mathbf{y}|^T |\mathbf{y}|} = \frac{\sum_{i=1}^n \sum_{j=1}^n |y_i| a_{ij} |y_j|}{\mathbf{y}^T \mathbf{y}} \geq \frac{|\sum_{i=1}^n \sum_{j=1}^n y_i a_{ij} y_j|}{\mathbf{y}^T \mathbf{y}} = \left| \frac{\mathbf{y}^T A \mathbf{y}}{\mathbf{y}^T \mathbf{y}} \right| = |\lambda_n|.$$

Then by Theorem 2.16 (Rayleigh's principle), $\lambda_1 \geq |\lambda_n|$. Therefore λ_1^2 is the largest eigenvalue of A^2 . Again by Rayleigh's principle,

$$\lambda_1^2 \geq \frac{\mathbf{j}^T A^2 \mathbf{j}}{\mathbf{j}^T \mathbf{j}} = \frac{\mathbf{j}^T A^T A \mathbf{j}}{\mathbf{j}^T \mathbf{j}} = \frac{(A \mathbf{j})^T A \mathbf{j}}{\mathbf{j}^T \mathbf{j}} = \frac{\sum_{i=1}^n \deg(i)^2}{n}.$$

Thus

$$\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n \deg(i)^2}{n}}. \quad \square$$

2.3.4. The largest eigenvalue of subgraphs.

Fact 2.36. If H is a proper subgraph of a connected graph G , then $\lambda_1(G) > \lambda_1(H)$.

Proof. Let \mathbf{y} be a non-negative eigenvector of A_H to eigenvalue $\lambda_1(H)$, which exists by Corollary 2.17. Let G have n vertices. We define $\tilde{\mathbf{y}}$ as the vector with n coordinates obtained by adding zero coordinates to \mathbf{y} where vertices are deleted. If $\tilde{\mathbf{y}}$ has zero coordinates, then by Proposition 2.23, $\tilde{\mathbf{y}}$ is not an eigenvector to $\lambda_1(G)$. Therefore by Rayleigh's principle, $\lambda_1(G) > R_{A_G}(\tilde{\mathbf{y}}) \geq R_{A_H}(\mathbf{y}) = \lambda_1(H)$. If $\tilde{\mathbf{y}}$ does not have zero coordinates, then $\tilde{\mathbf{y}} = \mathbf{y}$. For H to be a proper subgraph of G , at least one edge is deleted. Therefore

$$\lambda_1(G) \geq \frac{\tilde{\mathbf{y}}^T A_G \tilde{\mathbf{y}}}{\tilde{\mathbf{y}}^T \tilde{\mathbf{y}}} = \frac{\tilde{\mathbf{y}}^T A_G \tilde{\mathbf{y}}}{\mathbf{y}^T \mathbf{y}} > \frac{\mathbf{y}^T A_H \mathbf{y}}{\mathbf{y}^T \mathbf{y}} = \lambda_1(H). \quad \square$$

2.3.5. Graph products.

Notation 2.37. Given a graph G and $U \subseteq V(G)$, we denote the induced subgraph of G on the set U by $G[U]$.

Definition 2.38. For graphs $H = (W, F)$ and $G = (V, E)$, the *Cartesian product* of H and G , denoted by $H \square G$, is the graph with the set $W \times V$ as vertices, and $(w_1, v_1) \sim (w_2, v_2)$ if and only if $w_1 = w_2$ and $v_1 \sim_G v_2$, or $v_1 = v_2$ and $w_1 \sim_H w_2$. For each $v \in V$, we call $(H \square G)[W \times \{v\}]$ the horizontal layer corresponding to v . For each $w \in W$, we call $(H \square G)[\{w\} \times V]$ the vertical layer corresponding to w . The horizontal layers are copies of H and the vertical layers are copies of G .

Definition 2.39. For graphs $H = (W, F)$ and $G = (V, E)$, the *lexicographic product* of H and G , denoted by $H \circ G$, is the graph with the set $W \times V$ as vertices, and $(w_1, v_1) \sim (w_2, v_2)$ if and only if either $w_1 \sim_H w_2$ or $w_1 = w_2$ and $v_1 \sim_G v_2$. For each $w \in W$ we call $(H \circ G)[\{w\} \times V]$ the vertical layer corresponding to w .

Recall that J denotes the all-ones matrix.

Observation 2.40. The adjacency matrix of $G \circ H$ is $A_G \otimes J_{|V(H)|} + I_{|V(G)|} \otimes A_H$.

3. PRELIMINARY RESULTS ABOUT THE PRINCIPAL EIGENVECTOR

As previously introduced, G will always denote a connected graph. We write λ for λ_1 , the largest eigenvalue of the adjacency matrix of G . We use \mathbf{q} to mean the principal eigenvector of the adjacency matrix, the all-positive eigenvector to λ_1 . We assume \mathbf{q} is scaled to have l^2 norm 1 unless otherwise stated.

3.1. Observations and naive bounds on the ratio.

First, we note that \mathbf{q} reflects the symmetries of G .

Notation 3.1. Given a permutation π on a set X and an element $a \in X$, we write $\pi(a)$ for the image of a under π . We use M^π to mean the row permutation matrix of π , where $M_{i,j}^\pi = 1$ if $\pi(j) = i$ and $M_{i,j}^\pi = 0$ otherwise. Given a vector \mathbf{y} , we write $\pi(\mathbf{y})$ to denote $M^\pi \mathbf{y}$.

Definition 3.2. Given a permutation group S on a set X , the orbit of $a \in X$ under S is

$$\mathcal{O}_S(a) := \{\pi(a) \mid \pi \in S\}.$$

Definition 3.3. Given a graph G , an *automorphism* of G is a permutation π on the set of vertices that preserves adjacency relation, i.e., for each pair of vertices $i, j \in V(G)$, $\pi(i) \sim \pi(j)$ if and only if $i \sim j$.

Notation 3.4. We note that the set of automorphisms of a graph G is a group under composition. We denote this group by $\text{Aut}(G)$. For a vertex $i \in V(G)$, we denote the orbit of i under $\text{Aut}(G)$ by $\mathcal{O}(i)$.

Observation 3.5. If π is an automorphism of G , then $A_G = M^\pi A_G (M^\pi)^T$.

Proposition 3.6. Given a graph G and $\pi \in \text{Aut}(G)$, if \mathbf{y} is an eigenvector of A_G to eigenvalue ρ , then $\pi(\mathbf{y})$ is also an eigenvector of A_G to ρ .

Proof. Since any permutation matrix is an orthonormal matrix, $(M^\pi)^T M^\pi = I$. Then by Observation 3.5, $A_G M^\pi = M^\pi A_G$. Then $A_G M^\pi \mathbf{y} = M^\pi A_G \mathbf{y} = \rho M^\pi \mathbf{y}$. Therefore $\pi(\mathbf{y})$ is an eigenvector of A_G to ρ . \square

Fact 3.7. The principal eigenvector \mathbf{q} is constant on orbits of $\text{Aut}(G)$, i.e., if $j \in \mathcal{O}(i)$, then $q_i = q_j$.

Proof. Let $j \in \mathcal{O}(i)$. Then there is $\pi \in \text{Aut}(G)$ with $\pi(i) = j$. By Proposition 3.6, $\pi(\mathbf{q})$ is an eigenvector to λ . Since \mathbf{q} is all-positive, $\pi(\mathbf{q})$ is also all-positive. Therefore by Theorem 2.26, $\pi(\mathbf{q}) = \mathbf{q}$ when scaled to the same norm. Therefore $q_i = q_j$. \square

Next we note some basic bounds on the ratio between the coordinates of \mathbf{q} .

Observation 3.8. For two vertices i, j in G , let $\text{dist}(i, j) = k$. Then

$$\frac{q_j}{q_i} \leq \lambda^k.$$

Proof. If $\text{dist}(i, j) = 0$, then $q_j/q_i = 1 = \lambda^0$. If $\text{dist}(i, j) = 1$, then by Corollary 2.21, $\lambda q_i = \sum_{w:w \sim i} q_w \geq q_j$ since all q_w are positive. Now, suppose $k \geq 2$ and $q_w/q_i \leq \lambda^{k-1}$ for all vertices w at distance $k-1$ from i . We know j is adjacent to at least one vertex w at distance $k-1$ from i . Then

$$\frac{q_j}{q_i} = \frac{q_j}{q_w} \cdot \frac{q_w}{q_i} \leq \lambda \cdot \lambda^{k-1} = \lambda^k. \quad \square$$

Recall that D denotes the diameter of the graph.

Corollary 3.9. *For every connected graph G with diameter D ,*

$$\gamma \leq \lambda^D \leq \Delta^D \leq (n-1)^D.$$

Corollary 3.10. *If D is bounded for some family \mathcal{G} of graphs, then $\gamma(\mathcal{G})$ is polynomially bounded in n .*

Since D is relevant in bounding the ratio, we introduce a bound on D for regular graphs.

Fact 3.11. Let G be a connected d -regular graph. Then $D \leq \frac{3n}{d}$.

Proof. Pick v_0, v_D in G so that $\text{dist}(v_0, v_D) = D$. Let v_0, v_1, \dots, v_D be a shortest path from v_0 to v_D . Any v_i, v_j with $|i-j| \geq 3$ cannot have any common neighbors, since otherwise the path will not be a shortest path. Thus

$$d \left\lceil \frac{D}{3} \right\rceil \leq n.$$

Therefore $D \leq 3n/d$. \square

We know $D(G+e) \leq D(G)$ for any $e \in \overline{G}$. The following result shows that $D(G+e)$, and consequently, also $D(G-e)$ (if still connected), cannot differ from D_G by more than a factor of 2.

Fact 3.12. For any connected graph G with $D(G) = D$ and $e \in E(\overline{G})$,

$$D(G+e) \geq \frac{1}{2}D(G).$$

Proof.

(Notation: By $\text{dist}_l(x, y)$, where l is a path, we mean the distance between x and y along the path.)

We need to prove that there is a pair of vertices at distance at least $\frac{D}{2}$ in $G+e$. Let $u, v \in V(G)$ be such that $\text{dist}_G(u, v) = D$. If $\text{dist}_{G+e}(u, v) = D$, we are done. Otherwise, let p be a shortest path between u and v in G , and q a shortest path between u and v in $G+e$. Then e must be on q . Denote by x the endpoint of e which is closer to u on q , and by y the other endpoint. Pick the middle vertex w of p with $\text{dist}_p(u, w) = \lceil \frac{D}{2} \rceil$. If $\text{dist}_{G+e}(u, w) = \text{dist}_G(u, w) = \lceil \frac{D}{2} \rceil$, we are done. Otherwise, any shortest path r in $G+e$ from u to w must pass through edge e . Suppose we go along r from u to w , by the optimality of q , we can assume that r and q overlap from u to y . Now we look at the vertex w' adjacent to w on p which is closer to u than to v . We have

$$(3.13) \quad \text{dist}_G(w', v) = \text{dist}_p(w', v) = \left\lfloor \frac{D}{2} \right\rfloor + 1.$$

We claim that there is a shortest path in $G+e$ from w' to v that does not pass through e . Let s be a shortest path in $G+e$ from w' to v that passes through e . By the optimality of q , we may assume that s and q overlap from x to v . Then by the optimality of r ,

$$\text{dist}_s(x, w') + \text{dist}_{G+e}(w', w) \geq \text{dist}_r(x, w) = \text{dist}_{G+e}(x, y) + \text{dist}_r(y, w),$$

that is,

$$\text{dist}_s(x, w') \geq \text{dist}_r(y, w).$$

Therefore

$$\text{dist}_s(w', y) = \text{dist}_s(w', x) + \text{dist}_{G+e}(x, y) \geq \text{dist}_{G+e}(w', w) + \text{dist}_r(w, y).$$

Thus s is equivalent to a path that does not pass through e in $G + e$. As a result, a shortest path between w' and v in $G + e$ is also available in G . Then by (3.13),

$$\text{dist}_{G+e}(w', v) = \text{dist}_G(w', v) = \left\lfloor \frac{D}{2} \right\rfloor + 1. \quad \square$$

3.2. Chebyshev polynomials and principal eigenvectors.

3.2.1. Chebyshev polynomials.

The Chebyshev polynomials of the first kind, T_n , can be characterized by the recurrence

$$(3.14) \quad T_{n+1}(t) = 2t \cdot T_n(t) - T_{n-1}(t),$$

with initial values $T_0(t) = 1$ and $T_1(t) = t$.

The Chebyshev polynomials of the second kind, U_n , can be characterized by the same recurrence

$$(3.15) \quad U_{n+1}(t) = 2t \cdot U_n(t) - U_{n-1}(t),$$

with initial values $U_0(t) = 1$ and $U_1(t) = 2t$.

Fact 3.16. When $|t| \geq 1$, the explicit formula for T_n is

$$(3.17) \quad T_n(t) = \frac{1}{2} \left(\left(t - \sqrt{t^2 - 1} \right)^n + \left(t + \sqrt{t^2 - 1} \right)^n \right),$$

and the explicit formula for U_n is

$$(3.18) \quad U_n(t) = \frac{\left(t + \sqrt{t^2 - 1} \right)^{n+1} - \left(t - \sqrt{t^2 - 1} \right)^{n+1}}{2\sqrt{t^2 - 1}}.$$

Fact 3.19. The roots of T_n are $\cos\left(\frac{\pi(k+1/2)}{n}\right)$, $k = 0, \dots, n-1$. The roots of U_n are $\cos\left(\frac{k\pi}{n+1}\right)$, $k = 1, \dots, n$.

3.2.2. Applications to principal eigenvectors.

Fact 3.20. When $x > 1$, both $T_n(x)$ and $U_n(x)$ are strictly increasing.

Definition 3.21. A pendant path of length k in G consists of k vertices such that the induced subgraph on them is a path; moreover, one vertex has degree 1 in G and $k-2$ vertices have degree 2 in G . For example, in the graph $P_r \cdot K_s$, there is a pendant path of length r .

Observation 3.22. Let $1, 2, \dots, k$ be a pendant path in G where consecutive vertices are adjacent and $\deg(1) = 1$. Then for $1 \leq j \leq k$,

$$\frac{q_j}{q_1} = U_{j-1}\left(\frac{\lambda}{2}\right).$$

Proof. By Corollary 2.21, $\lambda q_1 = q_2$ and $q_{j+1} = \lambda q_j - q_{j-1}$ for $1 \leq j \leq n-1$. Therefore q_j/q_1 satisfies the initial values and recurrence relation of $U_{j-1}(\lambda/2)$. \square

Observation 3.22 along with Fact 2.36 can be used to show that most kite graphs have a very large (factorial) principal ratio, since $P_r \cdot K_s$ has a pendant path of length r and λ is larger than $s - 1$. In fact, Tait and Tobin [13] proved that the maximum principal ratio over all graphs of n vertices is attained by a kite graph.

4. MAIN RESULTS

Let G be a d -regular graph. As introduced before, we use $G + e$ to denote the graph obtained by adding an edge $e \in E(\overline{G})$ to G , and $G - e$ to denote the graph obtained by deleting an edge $e \in E(G)$ from G . We always assume $G - e$ is still connected. We are interested in the possible asymptotic behaviors of $\gamma(G + e)$ and $\gamma(G - e)$.

We first make two simple observations.

Observation 4.1. If the diameter $D(G)$ is bounded, then $\gamma(G \pm e)$ is polynomially bounded in n .

Proof. Fact 3.12 shows that $D(G \pm e)$ is also bounded, and the statement follows from Corollary 3.10. \square

Observation 4.2. If d is linear in n , then $\gamma(G \pm e)$ is polynomially bounded in n .

Proof. Fact 3.11 shows that $D(G)$ is bounded, and the statement follows from Observation 4.1. \square

4.1. Adding or removing an edge in bounded distance.

4.1.1. Adding an edge.

We show that if we add an edge e between two vertices at distance 2 to a connected regular graph G of bounded degree, then $\gamma(G + e)$ can be exponential.

Let $e = \{1, 2\}$.

Theorem 4.3. *For any fixed d , there is a family \mathcal{G} of connected d -regular graphs where for each $G_i \in \mathcal{G}$, there is an edge $e_i \in \overline{G_i}$ whose endpoints are at distance two in $G_i - e_i$, such that for the family*

$$\mathcal{G}' = \{G_i + e_i \mid G_i \in \mathcal{G}\},$$

$\gamma(\mathcal{G}')$ grows exponentially in n .

The proof of this theorem will be based on a series of constructions. The graphs produced by Construction 4.6 and Construction 4.7 are the pair of graphs that are used in the proof.

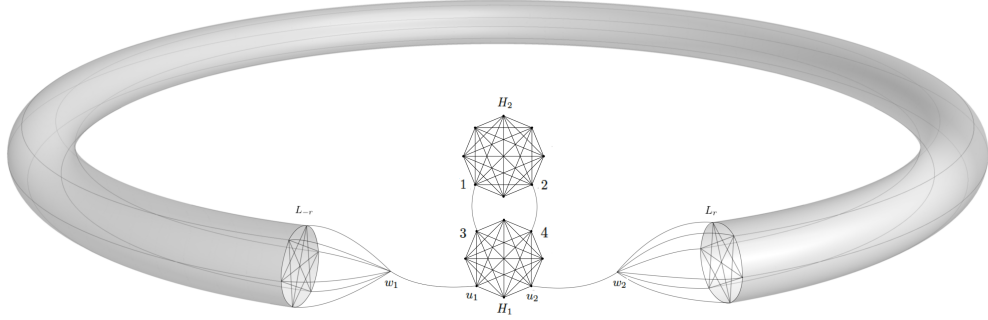
Construction 4.4 (Ring_{r,d,G_2}). Let $r \geq 0$ be a parameter. We label the vertices in P_{2r+1} from one end to the other end as $p_{-r}, p_{-r+1}, \dots, p_{-1}, p_0, p_1, \dots, p_{r-1}, p_r$. Let $G_1 = P_{2r+1} \square K_{d-1}$. We label the vertical layer corresponding to p_i as L_i . Let G_2 , which we will call a “gadget,” be a connected graph with two vertices v_1, v_2 of degree 1 and all other vertices of degree d , and an automorphism that switches v_1 and v_2 . We connect v_1 with every vertex in L_{-r} and connect v_2 with every vertex in L_r . We call the graph obtained Ring_{r,d,G_2} .

Observation 4.5. Let $X \leq \text{Aut}(G_1)$ be the subgroup of automorphisms of G_1 that fixes the vertical layers and permutes the horizontal layers. The orbits of X are the vertical layers. X is isomorphic to the symmetric group S_{d-1} . This group extends to G . The automorphism of G_1 that switches L_j and L_{-j} for $0 \leq j \leq r$ also

extends to G . Therefore the coordinates of the principal eigenvector of Ring_{r,d,G_2} corresponding to all vertices in $L_j \cup L_{-j}$ are the same. We denote this value by a_j , for $0 \leq j \leq r$.

Construction 4.6. $[\text{Ring}_{r,d}]$ Now we specify the gadget G_2 . We take two copies of K_{d+1} and call them H_1, H_2 . We label the vertices in H_1, H_2 from 1 to $d+1$. We remove the edge $\{1, 2\}$ from H_2 , the edge $\{3, 4\}$ from H_1 , and add the edges $\{1, 3\}$ and $\{2, 4\}$. We find two vertices u_1, u_2 in $V(H_1)$ such that $u_2 \in \mathcal{O}(u_1)$ in $H_1 - \{3, 4\}$. We remove the edge $\{u_1, u_2\}$. Finally, we attach a dangling vertex w_1 to u_1 , and a dangling vertex w_2 to u_2 . w_1 and w_2 are the vertices that connect with L_{-r} and L_r , respectively. For this specific G_2 , $\text{Ring}(r, d, G_2)$ is a d -regular graph on $n = (2r + 1)(d - 1) + 2 + 2(d + 1) = 2rd - 2r + 3d + 3$ vertices. We write $\text{Ring}(r, d)$ for this graph.

Construction 4.7. $[\text{Ring}_{r,d} + e]$ We add the edge $e = \{3, 4\} \in E(\overline{\text{Ring}_{r,d}})$ to $\text{Ring}_{r,d}$.



Proposition 4.8. For $0 \leq j \leq r$,

$$\frac{a_j}{a_0} = T_j \left(\frac{\lambda(\text{Ring}_{r,d} + e) - d + 2}{2} \right)$$

where a_j is as defined in Observation 4.5.

Proof. By Corollary 2.21,

$$\lambda a_0 = (d - 2)a_0 + 2a_1$$

and

$$\lambda a_j = (d - 2)a_j + a_{j-1} + a_{j+1}$$

for $1 \leq j \leq n - 1$. Therefore a_j/a_0 satisfies the initial values and recurrence relation of $T_j((\lambda - d + 2)/2)$ according to (3.14). \square

Lemma 4.9. $\lambda(\text{Ring}_{r,d} + e) > d + c(d)$, where $c(d)$ is a constant depending only on d .

Proof. Let H be the induced subgraph of $\text{Ring}_{r,d}$ on $V(H_1) \cup V(H_2)$. Then

$$\deg_{H+e}(1) = \deg_{H+e}(2) = d + 1, \quad \deg_{H+e}(u_1) = \deg_{H+e}(u_2) = d - 1,$$

while the rest of the vertices in $H + e$ are of degree d . Then by Fact 2.36 and Fact 2.35,

$$\lambda(\text{Ring}_{r,d} + e) > \lambda(H + e) \geq d_{\text{qavg}}(H + e) = \sqrt{\frac{(2d+2)d^2+4}{2d+2}} = d\sqrt{1 + \frac{2}{d^2(d+1)}}.$$

Let

$$c(d) := \frac{2}{3d(d+1)}.$$

Since $\sqrt{1+x} > 1 + \frac{1}{3}x$ when $0 < x < 3$,

$$\lambda(G + e) > d + c(d). \quad \square$$

Proof of Theorem 4.3. By Lemma 4.9,

$$\frac{\lambda(\text{Ring}_{r,d} + e) - d + 2}{2} > 1 + \frac{1}{3d(d+1)}.$$

By Observation 4.8, Fact 3.20, and Fact 3.16,

$$\gamma(\text{Ring}_{r,d}) \geq \frac{a_r}{a_0} = T_r \left(\frac{\lambda(\text{Ring}_{r,d} + e) - d + 2}{2} \right) > T_j \left(1 + \frac{1}{3d(d+1)} \right) > \frac{1}{2} \left(1 + \frac{1}{3d(d+1)} \right)^r.$$

Since $r = \frac{n-3d-3}{2d-2}$ and d is bounded,

$$\gamma(\text{Ring}_{r,d}) > (a(d) - \epsilon)^n$$

where

$$a(d) := \left(1 + \frac{1}{3d(d+1)} \right)^{1/(2d-2)}$$

and $0 < \epsilon < a - 1$ is any fixed constant. \square

4.1.2. Removing an edge.

We show that in the case of removing an edge $e = \{1, 2\}$ when $\text{dist}_{G-e}(1, 2)$ is bounded, $\gamma(G - e)$ is polynomially bounded for all D . We make use of the following theorem.

Let $e = \{1, 2\}$.

Theorem 4.10 (Cioabă, Gregory, Nikiforov[12]). *If G is a connected nonregular graph with n vertices, diameter D , and maximum degree Δ , then*

$$\Delta - \lambda_G \geq \frac{1}{n(D+1)}.$$

Theorem 4.11. *For a connected d -regular graph G and an edge $e = \{1, 2\} \in E(G)$, if $\text{dist}_{G-e}(1, 2) < c$ where c is some constant, then $\gamma(G - e)$ is polynomially bounded in n .*

Lemma 4.12. *$q_{\min}(G - e)$ is either q_1 or q_2 .*

Proof. If q_{\min} corresponds to some vertex j with degree d , then the average of the coordinates corresponding to the neighbors of j would be

$$\frac{\lambda(G - e)q_j}{d} < \frac{\lambda(G - e)q_j}{\lambda(G - e)} = q_j,$$

which is a contradiction. \square

Proof of Theorem 4.11. Without loss of generality suppose $q_1 = q_{\min}$. Then summing n equations of the form $\lambda(G - e)q_i = \sum_{j:j \sim i} q_j$, we have

$$\lambda(G - e) \sum_{i=1}^n q_i = (d-1)q_1 + (d-1)q_2 + dq_3 + \cdots + dq_n = d \left(\sum_{i=1}^n q_i \right) - q_1 - q_2.$$

Therefore by Theorem 4.10,

$$\frac{q_1 + q_2}{\sum_{i=1}^n q_i} = d - \lambda(G - e) \geq \frac{1}{n(D(G - e) + 1)} \geq \frac{1}{n^2}.$$

By Observation 3.8, $q_2 \leq \lambda(G - e)^c q_1$. Therefore

$$\gamma(G - e) < \frac{\sum_{i=1}^n q_i}{q_1} \leq n^2 \left(1 + \frac{q_2}{q_1} \right) \leq n^2(1 + \lambda(G - e)^c) < n^2(1 + d^c). \quad \square$$

4.2. Multiplicative spectral gap and stability of the ratio.

We use a known bound on the diameter of a graph in terms of the spectral gap to show that $\gamma(G \pm e)$ is polynomially bounded for bounded-degree expanders.

Definition 4.13. For $0 < \epsilon < 1$, a regular graph of degree d is an ϵ -expander if $\lambda_2 \leq (1 - \epsilon)d$, i.e., $\delta \geq \epsilon d$.

We note that this definition implies that an expander graph is connected.

Theorem 4.14 (N. Alon, V. D. Milman[3]). *Let G be a connected graph on n vertices with maximum degree Δ and let δ denote the smallest positive eigenvalue of the Laplacian matrix δ . Then*

$$D(G) \leq 2 \left\lceil \sqrt{\frac{2\Delta}{\delta}} \log_2 n \right\rceil.$$

Corollary 4.15. *For expander graphs G with bounded degree, $\gamma(G \pm e)$ is polynomially bounded in n . More specifically, for an ϵ -expander graph G with bounded degree d ,*

$$(4.16) \quad \gamma(G + e) \leq n^{4\sqrt{2/\epsilon} \log_2(d+1)}.$$

Proof. By definition,

$$\frac{\Delta(G)}{\delta(G)} = \frac{d}{\delta(G)} \leq \frac{1}{\epsilon}.$$

Then

$$D(G) \leq 2 \left\lceil \sqrt{\frac{2}{\epsilon}} \log_2 n \right\rceil.$$

By Fact 3.12,

$$D(G \pm e) \leq 4 \left\lceil \sqrt{\frac{2}{\epsilon}} \log_2 n \right\rceil.$$

Since $\gamma \leq \Delta^D$ (Corollary 3.9),

$$\begin{aligned}
 \gamma(G \pm e) &\leq (d+1)^4 \left\lfloor \sqrt{2/\epsilon} \log_2 n \right\rfloor \\
 &\leq (d+1)^4 \sqrt{2/\epsilon} \log_2 n \\
 &= 2^4 \sqrt{2/\epsilon} \log_2 n \log_2(d+1) \\
 &= n^4 \sqrt{2/\epsilon} \log_2(d+1). \quad \square
 \end{aligned}$$

4.3. Additive spectral gap and stability of the ratio.

We show that a large $((2+\epsilon)\sqrt{n})$ additive spectral gap implies that $\gamma(G \pm e)$ is bounded. Specifically, we prove the following.

Theorem 4.17. *Let G be a connected d -regular graph. If the spectral gap $\delta = d - \lambda_2$ of G satisfies $\delta > \frac{2}{c}\sqrt{n} + 2$ for some value $0 < c < 1$, then*

$$\gamma(G \pm e) < \frac{1+c}{1-c}.$$

To motivate this result, we first point out that graphs with such an additive spectral gap are not necessarily expanders. Indeed, when d is larger than $\Theta(\sqrt{n})$, the multiplicative spectral gap of graphs with a $\Theta(\sqrt{n})$ additive spectral gap will go to zero. Moreover, the diameter of graphs with such an additive spectral gap can still grow quite fast (polynomially in n), approaching the upper bound derived from Theorem 4.14.

4.3.1. Existence of graphs with large additive spectral gap and large diameter.

Proposition 4.18. *For a regular graph with additive spectral gap δ , the diameter D is $O((n/\delta)^{1/3}(\log n)^{2/3})$.*

Proof. Let the regular graph have degree d . From Theorem 4.14, we know that

$$(4.19) \quad D^2 \leq c \frac{d}{\delta} (\log n)^2$$

where c is some constant. By Fact 3.11, we also have

$$(4.20) \quad D \leq \frac{3n}{d}.$$

Multiplying (4.19) and (4.20), we have

$$(4.21) \quad D \leq (3c)^{1/3} \left(\frac{n}{\delta} \right)^{1/3} (\log n)^{2/3}. \quad \square$$

Corollary 4.22. *For a regular graph with an $\Omega(\sqrt{n})$ additive spectral gap, the diameter is $O(n^{1/6}(\log n)^{2/3})$.*

We show that this bound is nearly tight.

Proposition 4.23. *There are connected regular graphs with diameter $(1/2)n^{1/6}$ and an additive spectral gap of $cn^{1/2}$ where $c = 2\pi^2(1 + O(n^{-1/3}))$.*

We prove a more general statement.

Proposition 4.24. *For any constant $0 < t < 1$, there are connected regular graphs with diameter $(1/2)n^{(1-t)/3}$ and an additive spectral gap of cn^t where $c = 2\pi^2(1 + O(n^{(2t-2)/3}))$.*

Proof. Recall the lexicographic product and its properties in Definition 2.39, Observation 2.40, and Fact 2.19. Let $G := C_r \circ \overline{K}_s$ where $r \cdot s = n$. Then G is a connected regular graph of degree $2s$ and diameter $\lfloor r/2 \rfloor$. The adjacency matrix of G can be expressed as

$$(4.25) \quad A_G = A_{C_r} \otimes J_s + I_r \otimes \mathbf{0} = A_{C_r} \otimes J_s.$$

Since

$$(4.26) \quad \text{spec}(J_s) = \{\{s, 0^{s-1}\}\},$$

we have

$$(4.27) \quad \text{spec}(G) = \{\{s\lambda_1(C_r), s\lambda_2(C_r), \dots, s\lambda_r(C_r), 0^{(s-1)r}\}\}$$

Thus

$$\delta(G) = s(\lambda_1(C_r) - \lambda_2(C_r)).$$

It is well-known that the eigenvalues of a cycle C_r are $\{2 \cos\left(\frac{2\pi j}{r}\right)\}$,

where $j = 0, 1, \dots, r-1$. Thus

$$\lambda_1(C_r) - \lambda_2(C_r) = 2 \left(1 - \cos\left(\frac{2\pi}{r}\right)\right) = \frac{4\pi^2}{r^2} + \epsilon$$

where $|\epsilon| < \frac{2^5 \pi^4}{4! r^4}$. We take $r = n^{(1-t)/3}$ and $s = n^{(2+t)/3}$. Then

$$\delta(G) = 4\pi^2 n^t \left(1 + O\left(\frac{1}{r^2}\right)\right) = 4\pi^2 n^t \left(1 + O(n^{(2t-2)/3})\right). \quad \square$$

4.3.2. Large additive spectral gap implies bounded ratio.

Now we prove Theorem 4.17 which shows that $\gamma(G \pm e)$ is bounded when G has an additive spectral gap of $(2 + \epsilon)\sqrt{n}$ for some $\epsilon > 0$. This is an application of the theory developed in Chapter V. 2 in Stewart and Sun's book [5], dealing with perturbation of invariant subspaces. The outline of the proof is from this book. We adapt the proofs to our special case and fill in some details.

Notation 4.28. Let $U \in M_n(\mathbb{R})$ be the orthogonal matrix whose columns are eigenvectors of A . We can write it as

$$(4.29) \quad U = (\mathbf{x} \ Y) = (\mathbf{x} \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n),$$

where the columns $\mathbf{x}, \mathbf{y}_2, \mathbf{y}_3, \dots, \mathbf{y}_n$ are eigenvectors of A corresponding to eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$. Then $\lambda_1 = d$, and we can assume

$$(4.30) \quad \mathbf{x} = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)^T.$$

In the context of adding an edge e to G , we label the two endpoints of e to be 1 and 2, and let $E \in M_n(\mathbb{R})$ have $E_{21} = E_{12} = 1$ while all other coordinates are zero. In the context of deleting an edge e from G , we also label the two endpoints of e to be 1 and 2, and let $E \in M_n(\mathbb{R})$ have $E_{21} = E_{12} = -1$ while all other coordinates are zero. In both cases, $A + E$ is the adjacency matrix of the graph obtained.

We know \mathbf{x} is not an eigenvector of $A + E$. We want to know how close \mathbf{x} is to the principal eigenvector of $G \pm e$, in the sense that we want to find a vector \mathbf{v} with small norm such that $\tilde{\mathbf{x}} = \frac{\mathbf{x} + \mathbf{v}}{\|\mathbf{x} + \mathbf{v}\|}$ is the unit principal eigenvector of $G \pm e$.

Proposition 4.31. *Let \mathbf{p} be a vector in \mathbb{R}^{n-1} . Let $U = (\mathbf{x} \ Y) \in M_n(\mathbb{R})$ be orthogonal, where \mathbf{x} is a vector in \mathbb{R}^{n-1} and Y is an $n \times (n-1)$ matrix. Define $\tilde{U} \in M_n(\mathbb{R})$ as $\tilde{U} = (\tilde{\mathbf{x}} \ \tilde{Y})$, where*

$$(4.32) \quad \tilde{\mathbf{x}} = \frac{\mathbf{x} + Y\mathbf{p}}{\sqrt{1 + \|\mathbf{p}\|^2}} \quad \text{and} \quad \tilde{Y} = (Y - \mathbf{x}\mathbf{p}^T)(I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2}.$$

Then \tilde{U} is an orthogonal matrix.

Proof.

$$\begin{aligned} \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} &= \frac{\mathbf{x}^T \mathbf{x} + \mathbf{p}^T Y^T Y \mathbf{p}}{1 + \|\mathbf{p}\|^2} = \frac{1 + \|\mathbf{p}\|^2}{1 + \|\mathbf{p}\|^2} = 1. \\ \tilde{Y}^T \tilde{Y} &= (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2} (Y^T - \mathbf{p}\mathbf{x}^T) (Y - \mathbf{x}\mathbf{p}^T) (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2} \\ &= (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2} (I_{n-1} + \mathbf{p}\mathbf{p}^T) (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2} \\ &= I_{n-1}. \\ \tilde{\mathbf{x}}^T \tilde{Y} &= \frac{(\mathbf{x}^T + \mathbf{p}^T Y^T) (Y - \mathbf{x}\mathbf{p}^T) (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2}}{\sqrt{1 + \|\mathbf{p}\|^2}} \\ &= \frac{(-\mathbf{p}^T + \mathbf{p}^T) (I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1/2}}{\sqrt{1 + \|\mathbf{p}\|^2}} \\ &= 0. \end{aligned}$$

Therefore

$$\tilde{U}^T \tilde{U} = \begin{pmatrix} \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} & \tilde{\mathbf{x}}^T \tilde{Y} \\ \tilde{Y}^T \tilde{\mathbf{x}} & \tilde{Y}^T \tilde{Y} \end{pmatrix} = I_n. \quad \square$$

We want to find \mathbf{p} such that $\tilde{\mathbf{x}}$ is the principal eigenvector of $A + E$ and the columns of $(\tilde{\mathbf{x}} \ \tilde{Y})$ form an orthogonal eigenbasis for $A + E$.

Notation 4.33. We define

$$\begin{aligned} e_{11} &:= \mathbf{x}^T E \mathbf{x} \in \mathbb{R}, \quad e_{21} := Y^T E \mathbf{x} \in \mathbb{R}^{n-1}, \quad E_{22} := Y^T E Y \in M_{n-1}(\mathbb{R}), \\ \text{and } L &:= Y^T A Y = \text{diag}(\lambda_2, \dots, \lambda_n) \in M_{n-1}(\mathbb{R}). \end{aligned}$$

Observation 4.34. Since $\|E\| = 1$ and $(\mathbf{x} \ Y)$ is orthogonal, $|e_{11}|, \|e_{21}\|, \|E_{22}\| \leq 1$.

Proposition 4.35. *If*

$$(4.36) \quad ((d + e_{11})I_{n-1} - (L + E_{22}))\mathbf{p} = e_{21} - \mathbf{p}e_{21}^T \mathbf{p},$$

then $\tilde{\mathbf{x}}$ is an eigenvector of $A + E$.

Proof. (4.36) is equivalent to

$$(Y^T - \mathbf{p}\mathbf{x}^T)(A + E)(\mathbf{x} + Y\mathbf{p}) = 0,$$

which gives

$$\tilde{Y}^T (A + E) \tilde{\mathbf{x}} = 0.$$

Since $(\tilde{\mathbf{x}} \ \tilde{Y})$ is an orthogonal matrix, $\tilde{\mathbf{x}}$ is an eigenvector of $A + E$. \square

Notation 4.37. We define $M \in M_{n-1}(\mathbb{R})$ as $M = (d + e_{11})I_{n-1} - L - E_{22}$.

Proposition 4.38. *If $\delta > \frac{2}{c}\sqrt{n} + 2$ for some value $0 < c < 1$, then M is non-singular and*

$$(4.39) \quad \|M^{-1}\| \leq \frac{1}{\delta - 2} < \frac{c}{2\sqrt{n}}.$$

Proof. Since $(d + e_{11})I_{n-1} - L$ is a diagonal matrix and E_{22} is a symmetric matrix, M is symmetric. Recall that $\|E_{22}\| \leq 1$. By Theorem 2.16 (Rayleigh's principle) and Observation 4.34,

$$\begin{aligned} \min \text{spec}(M) &= \min_{\|\mathbf{x}\|=1} \mathbf{x}^T((d + e_{11})I_{n-1} - L - E_{22})\mathbf{x} \\ &\geq \min_{\|\mathbf{x}\|=1} \mathbf{x}^T((d + e_{11})I_{n-1} - L)\mathbf{x} - 1 \\ &= \min_{\|\mathbf{x}\|=1} \text{spec}((d + e_{11})I - L) - 1 \\ &= d + e_{11} - \lambda_2 - 1 \\ &\geq \delta - 2. \end{aligned}$$

Therefore all eigenvalues of M are positive, and

$$\|M^{-1}\| = \max \text{spec}(M^{-1}) = \frac{1}{\min \text{spec}(M)} \leq \frac{1}{\delta - 2} < \frac{c}{2\sqrt{n}}. \quad \square$$

Proposition 4.40. *We write (4.36) in terms of M :*

$$(4.41) \quad M\mathbf{p} = e_{21} - \mathbf{p}e_{21}^T\mathbf{p}.$$

Let $\theta = \|M^{-1}\|^{-1}$ and $\eta = \|e_{21}\|$. We claim that if $\delta > \frac{2}{c}\sqrt{n} + 2$ for some value $0 < c < 1$, then there exists \mathbf{p} with

$$(4.42) \quad \|\mathbf{p}\| < \frac{2\eta}{\theta + \sqrt{\theta^2 - 4\eta^2}}$$

such that (4.41) holds. The $\tilde{\mathbf{x}}$ defined by (4.32) is an eigenvector of $A + E$.

Proof. We want to find a solution with small norm to the non-linear equation (4.41). We do this by an iterative construction.

We define a sequence of vectors $\mathbf{p}_0, \mathbf{p}_1, \dots$ such that

$$\mathbf{p}_0 = 0 \quad \text{and} \quad \mathbf{p}_i = M^{-1}(e_{21} - \mathbf{p}_{i-1}e_{21}^T\mathbf{p}_{i-1}), \quad \text{for } i \geq 1.$$

Then

$$\|\mathbf{p}_i\| \leq \|M^{-1}\|(|e_{21}| + \|\mathbf{p}_{i-1}\|^2|e_{21}|) \leq \frac{\eta(1 + \|\mathbf{p}_{i-1}\|^2)}{\theta}.$$

We claim that the sequence $\{\mathbf{p}_i\}$ converges. We define

$$\xi_0 = 0, \quad \xi_i = \frac{\eta(1 + \xi_{i-1}^2)}{\theta}, \quad \text{for } i \geq 1.$$

Then $\|\mathbf{p}_i\| \leq \xi_i$. Since $\xi_1 = \frac{\eta}{\theta} > \xi_0$, we can prove by induction that $\xi_0, \xi_1, \xi_2, \dots$ is monotone increasing. Let

$$\phi(\xi) = \frac{\eta(1 + \xi^2)}{\theta}.$$

This function is monotone increasing in ξ , and has a fixed point at $\xi = \frac{2\eta}{\theta + \sqrt{\theta^2 - 4\eta^2}}$.

Then $\xi_i < \xi_{i+1} = \phi(\xi_i) < \phi(\xi) = \xi$. Therefore the sequence $\{\xi_i\}$ converges to ξ .

Thus

$$\|\mathbf{p}_i\| \leq \xi_i \leq \xi = \frac{2\eta}{\theta + \sqrt{\theta^2 - 4\eta^2}} < \frac{2\eta}{\theta}.$$

Next we prove the convergence of $\{\mathbf{p}_0, \mathbf{p}_1, \dots\}$. For any $i \geq 2$,

$$\begin{aligned} \|\mathbf{p}_i - \mathbf{p}_{i-1}\| &= \|M^{-1}(\mathbf{p}_{i-1}e_{21}^T\mathbf{p}_{i-1} - \mathbf{p}_{i-2}e_{21}^T\mathbf{p}_{i-2})\| \\ &= \|M^{-1}(\mathbf{p}_{i-1} + \mathbf{p}_{i-2})e_{21}^T(\mathbf{p}_{i-1} - \mathbf{p}_{i-2})\| \\ &\leq 2\|M^{-1}\|\|\mathbf{p}_{i-1}\|\|\mathbf{p}_{i-1}\| \cdot \eta \cdot \|\mathbf{p}_{i-1} - \mathbf{p}_{i-2}\| \\ &\leq \frac{4\eta^2}{\theta^2}\|\mathbf{p}_{i-1} - \mathbf{p}_{i-2}\|. \end{aligned}$$

Then $\|\mathbf{p}_i - \mathbf{p}_0\| \leq \rho^i\|\mathbf{p}_1 - \mathbf{p}_0\|$, where $\rho = \frac{4\eta^2}{\theta^2} < \frac{c^2}{4n} < 1$. Therefore $\{\mathbf{p}_i\}$ is a Cauchy sequence in \mathbb{R}^{n-1} and has a limit \mathbf{p} . Thus a solution \mathbf{p} exists, with norm satisfying (4.42). \square

Proposition 4.43. *If $\delta > \frac{2}{c}\sqrt{n} + 2$ for some value $0 < c < 1$, then the principal eigenvector of $A + E$, $\tilde{\mathbf{x}}$, can be written in the form*

$$\tilde{\mathbf{x}} = \frac{\mathbf{x} + Y\mathbf{p}}{\sqrt{1 + \|\mathbf{p}\|^2}} \quad \text{where} \quad \|\mathbf{p}\| < \frac{c}{\sqrt{n}}.$$

Proof. We showed that there exists \mathbf{p} with

$$\|\mathbf{p}\| < \frac{2\eta}{\theta + \sqrt{\theta^2 - 4\eta^2}} < \frac{2\eta}{\theta} < \frac{c}{\sqrt{n}}$$

such that $\frac{\mathbf{x} + Y\mathbf{p}}{\sqrt{1 + \|\mathbf{p}\|^2}}$ is an eigenvector of $A + E$. It remains to show that this is the principal eigenvector. Since $\|Y\| = 1$ and $\|\mathbf{p}\| < \frac{c}{\sqrt{n}}$ where $0 < c < 1$, and since $\mathbf{x} = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$, all coordinates of $\tilde{\mathbf{x}}$ are positive. Therefore by Corollary 2.27, $\tilde{\mathbf{x}}$ is the principal eigenvector of $A + E$. \square

Proof of Theorem 4.17. Following the argument above, we know the smallest possible coordinate of $\tilde{\mathbf{x}}$ is $\frac{1-c}{\sqrt{n}}$, and the largest possible coordinate of $\tilde{\mathbf{x}}$ is $\frac{1+c}{\sqrt{n}}$. Therefore the ratio of $G \pm e$ is as claimed. This completes the proof. \square

4.4. Open questions.

In Section 4.1.2, we showed that if we remove a non-bridge edge e from a connected regular graph such that the endpoints of e are of bounded distance in $G - e$, then $\gamma(G - e)$ is polynomially bounded. Is there also a polynomial bound when the endpoints of e are of unbounded distance in $G - e$?

Question 4.44. *If we remove a non-bridge edge e from a connected regular graph G , is $\gamma(G - e)$ always polynomially bounded?*

In Section 4.1.1, we showed that if we add an edge e between two vertices at distance 2 to a connected regular graph G of bounded degree, then $\gamma(G + e)$ can

be exponential. Can $\gamma(G + e)$ be exponential when e is between two vertices of unbounded distance in G ?

Question 4.45. *If we add an edge e to a connected regular graph G with bounded degree d , such that the distance between the endpoints of e in G is unbounded, can $\gamma(G + e)$ be exponential in n ?*

In Section 4.3.2, we showed that for a connected regular graph G , an additive spectral gap larger than $2\sqrt{n}$ implies that $\gamma(G \pm e)$ is bounded (Theorem 4.17). However, this bound ceases to work at all for G with an additive spectral gap $\delta \leq 2\sqrt{n}$. Is this a limitation of the method, or is there really an abrupt change at $\delta = 2\sqrt{n}$?

Question 4.46. *Is there a family of connected regular graphs G with an additive spectral gap slightly less than $2\sqrt{n}$ and with $\gamma(G \pm e)$ not bounded?*

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