HARMONIC FUNCTIONS WITH THE DIRICHLET CONDITION

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Abstract. Harmonic functions under the Dirichlet condition can be shown to have unique solutions. In this paper we will do so using calculus of variations and discussing properties of harmonic functions such as the mean value property and the maximum principle.

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1. Introduction

The Dirichlet problem, which we will define later in this paper, is a problem that has been studied for years. Though there are many methods used in solving for unique solutions of partial differential equations with the Dirichlet condition, in this paper we will take the approach of calculus of variations to solve harmonic functions with the specific boundary conditions outlined by Dirichlet. In doing this, we will use special properties of harmonic functions such as the mean value property and maximum principle, two principles that are exceptionally useful in proving additional properties of harmonic functions like the Harnak inequality and the fact that non-constant harmonic functions defined on all of $\mathbb{R}^n$ are necessarily unbounded, though these are not addressed in the scope of this project.

The first notable study of the Dirichlet problem was undertaken in 'Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism' by George Green in 1828 [1]. Many others worked on this problem through the

Date: DEADLINES: Draft AUGUST 19 and Final version AUGUST 31, 2019.

years, but in 1840 Gauss applied the study of the Dirichlet problem to harmonic functions. Because of the work done in the study of the Dirichlet problem applied to harmonic functions by Gauss there are many important applications in physics [2]. One such application of this problem is found in electricity and magnetism.In physics, harmonic functions can model systems without an external field. If the system has a boundary made from a conductor, then the Dirichlet condition will apply. With the combination of these two conditions the study of the Dirichlet condition with harmonic functions will model the physical system.

2. Dirichlet Condition

Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded domain with $\partial\Omega$ smooth. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.
We wish to show that there exists a unique solution $u : \Omega \to \mathbb{R}$ with $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u(x) = 0 & x \in \Omega \\ u(x) = g & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Here we call the function u a harmonic function, since its laplacian is zero on the interior. This boundary condition is called a Dirichlet condition. This is opposed to other boundary conditions such as the Neumann condition, where the derivative of u on the boundary is prescribed:

$$\tfrac{du}{d\nu} = g \qquad x \in \partial\Omega.$$

The Dirichlet condition is distinct in that u must take the values of a prescribed smooth function though there are no restrictions on the derivative of u at the boundary. Now that we have introduced the Dirichlet condition, we will outline our approach to understanding this problem.

3. Plan

We will show the existence of a unique u by setting up a calculus of variations problem, defining a functional I that provides us with an easier way to find a solution. We will do this by showing there exists a minimizer for the functional and that it is a weak solution to the equation $\Delta u = 0$. The function $u(x)$ is a weak solution to the Dirichlet problem if

$$\textstyle\int_\Omega \nabla u \cdot \nabla\varphi = 0 \text{ for all test functions } \varphi \in H_0^1(\Omega)$$

where

$$H_0^1(\Omega) = \{u \in L^2(\Omega);\ \exists g \in L^p(\Omega) \int_\Omega u\varphi' = -\int_\Omega g\varphi\ \forall \varphi \in C_C^1(\Omega)\}.$$

We then prove that all weak solutions are strong solutions. A function $u \in C^2(\Omega)$ is a strong solution if it satisfies our harmonic function with the Dirichlet condition in the usual sense. Finally all that remains is to show the solution is unique. We do this by using the maximum principle, a property of harmonic functions.

4. Setting Up Calculus of Variations

For this section and the next, we primarily follow the work shown in Evans[3]. We define the Lagrangian L to be a smooth function $L = L(Dw(x), w(x), x)$ where $L : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$. Call $Dw(x)$ the variable p. For the Dirichlet problem, we set

$$L = \frac{1}{2}|\nabla w|^2.$$

Given the partial differential equation $\Delta[\cdot] = 0$ We set up the calculus of variations problem by using a functional I such that any critical point u of I satisfies $\Delta u = 0$ in the weak sense.
Therefore, if I has a minimum that occurs at a function u, u will be a solution to $\Delta[u] = 0$. This provides a much easier method for finding a solution, since it can be easier to find a minimum to I. To make our search for a minimizer of I more explicit, we define

$$I[w] = \int_\Omega L(Dw(x), w(x), x)dx, \tag{4.1}$$

where $w \in g + H_0^1(\Omega)$. This definition of our functional will allow us to show the existence of a minimizer, which we will do next.

5. Existence of a Minimizer

Given our functional $I[w] = \int_\Omega L(Dw(x), w(x), x)dx$, it is not obvious that $I[w]$ will have a minimum. It is possible for I to be such that I does not achieve its infimum, even if we know I is bounded below. Therefore, in order to ensure $I[w]$ has a minimum, it is important for us to assume some properties about $L(Dw(x), w(x), x)$. By developing convexity and coercivity, two conditions which our lagrangian will meet, we will prove the existence of a minimizer for our functional I. First we will express convexity.

5.1. **Convexity.**

Definition 5.1. A function $L(p, w, x)$ is convex in p if it meets the convexity condition for all ξ_i, ξ_j:

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p, w, x)\xi_i\xi_j \geq 0 \tag{5.2}$$

with $L_{p_i p_j}$ meaning $\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}L(p, w, x)$.

To show that our lagrangian is convex, we first note that $L = \frac{1}{2}|p|^2$ with $p \in \mathbb{R}^n$. Here $L_{p_i} = p_i$ for all $i \in (1 \dots n)$. Then,

$$L_{p_i p_j} = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix} .$$

It follows that $L_{p_i p_j}$ is the $n \times n$ identity matrix, which is positive semidefinite. Therefore the $\sum_{i,j=1}^{n} L_{p_i p_j}(p, w, x)\xi_i\xi_j$ is non negative as required. Thus L is convex. Now that we have developed that convexity of L, we can move on to understanding the coercivity of L.

5.2. **Coercivity.** Next we show that L is coercive. This will tell us that $I[w] \to \infty$ as $\|w\| \to \infty$ and will ensure that $I[w]$ does not approach its infimum when $\|w\|$ is large.

Definition 5.3. A function $L(x) : \mathbb{R}^n \to \mathbb{R}^n$ is coercive if it satisfies the coercivity condition, which states for some fixed q such that $1 < q < \infty$, there exists an $\alpha > 0$ and $\beta \geq 0$ such that

$$L(p, w, x,) \geq \alpha |p|^q - \beta \tag{5.4}$$

for all $(p, w, x) \in \mathbb{R}^n \times \mathbb{R} \times \Omega$.
Then for $L(p, w, x)$ satisfying the coercivity condition,

$$I[w] = \int_\Omega L(Dw(x), w(x), x)dx \geq \int_\Omega \alpha |p|^q dx - \beta |\Omega|. \tag{5.5}$$

Now let us consider specifically for our problem $q = 2$. Then for our lagrangian $L = \frac{1}{2}|p|^2$ is simple to show the coercivity condition holds. We will show that by choosing $\alpha = \frac{1}{2}$ and $\beta = 0$. Then,

$$L(p, w, x) = \tfrac{1}{2}|p|^2 \geq \tfrac{1}{2}|p|^2$$

and

$$\textstyle\int_\Omega \frac{1}{2}|p|^2 dx \geq \int_\Omega \frac{1}{2}|p|^2 dx$$

as required.
Therefore, L is both coercive and convex. With these conditions, we will now prove the following theorem from Evans[3], that shows the existence of a minimizer for our functional I.

Theorem 5.6. *Assume L is both convex and coercive. Define the set*

$$A := \{w \in H^1(\Omega) | \omega = g \text{ on } \partial\Omega \text{ in the trace sense}\}$$

and suppose A is non empty. Then, there exists at least one $u \in A$ such that

$$I[u] = \inf_{\omega \in A} I[\omega].$$

Proof. Let $m = \inf_{\omega \in A} I[w]$. For our first case, suppose $m = +\infty$. Then suppose for contradiction there does not exists a $u \in A$ such that $I[u] = m$. This would result in A being empty, which is a contradiction. So if $m = +\infty$, there exists a $u \in A$ such that $I[u] = m$.

Now for the finite case, since I is coercive, we know that the

$$m = \inf_{\omega \in A} I[\omega] = \inf_{\omega \in B} I[\omega],$$

where B is a closed ball in A of large enough radius. The key here is that we've restricted our search for a minimizer to a bounded set B.

Now we take a sequence $\{x_n\}_n^\infty$ such that

$$\lim_{n \to \infty} I(x_n) = m$$

and $x_n \in B$ for all n. Since x_n is bounded, we can take a subsequence $\{x_{n_k}\}$ that converges weakly to some $u \in B$ by the Banach-Alaoglu theorem found in Brezis [4] which tells us that B will be compact with respect to the weak* topology. Since I is convex, I is weakly lower semicontinuous, so

$$m = \leq I(u) \leq \lim_{n \to \infty} I(x_n) = m,$$

so we have shown that u is a minimizer.

□

This minimizer, u, will be our solution to the Dirichlet Condition applied to the Harmonic function, which we have now shown exists.

6. Show u is a Weak Solution

We will first show that u is a weak solution than there are looser conditions outlined above than there is for a strong solution, such as not requiring continuity in u.
Let $i(h) = I(u + h\varphi)$. Since we know that there exists a u such that $I(u)$ is a minimum, we can tell that $i(h)$ has a minimum at $h = 0$.
Therefore in order to show our solution u is a weak solution, we take our functional $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 dx$ and calculate $i'(0)$ which we set equal to zero since $h = 0$ corresponds to a minimum. This gives us

$$i'(0) = \lim_{h\to 0} \frac{1}{h} \int_\Omega \frac{1}{2}\nabla(u + h\varphi)|^2 dx - \int_\Omega \frac{1}{2}|\nabla u|^2 dx. \tag{6.1}$$

Then, by linearity,

$$\begin{aligned} i'(0) &= \lim_{h\to 0} \frac{1}{h}\Big[\int_\Omega \frac{1}{2}|\nabla u + h\nabla\varphi|^2 dx - \int_\Omega \frac{1}{2}|\nabla u|^2 dx\Big] \\ &= \lim_{h\to 0} \frac{1}{h}\Big[\frac{1}{2}\int_\Omega |\nabla u|^2 + 2\nabla u \cdot h\nabla\varphi + |h\nabla\varphi|^2 dx - \int_\Omega \frac{1}{2}|\nabla u|^2 dx\Big]. \end{aligned}$$

Then by applying cancellations we have:

$$\begin{aligned} i'(0) &= \lim_{h\to 0} \frac{1}{h}\Big[\frac{1}{2}\int_\Omega 2\nabla u \cdot h\nabla\varphi + |h\nabla\varphi|^2 dx\Big] \\ &= \lim_{h\to 0} \frac{1}{2}\int_\Omega 2\nabla u \cdot \nabla\varphi + h|\nabla\varphi|^2 dx. \end{aligned} \tag{6.2}$$

Now by taking the limit we get

$$i'(0) = \int_\Omega \nabla u \cdot \nabla\varphi dx. \tag{6.3}$$

Therefore, u is a weak solution to our functional. We can take this one step farther however by proving that a weak solution of our harmonic problem with the Dirichlet condition is also a strong solution.

7. Show a Weak Solution u is a Strong Solution

In order to show a weak solution u is a strong solution, we state a result proved in Evans[3] whose proof is beyond the scope of this paper. This theorem states a weak solution will be smooth.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with a smooth boundary $\partial\Omega$ such that:*

$$\begin{aligned} \Delta w(x) = f \qquad & x \in \Omega \\ w(x) = 0 \qquad & x \in \partial\Omega. \end{aligned}$$

Suppose $w \in H^1(\Omega)$ is a weak solution. Then if $f \in C^\infty(\Omega)$, w is smooth.

Proof. Found in Evans[3]. See Theorem 3 of Chapter 6, section 6.3.1. □

To apply this theorem to the Dirichlet Problem, we let

$$f := -\Delta g$$
and let $u = g + w$

so that

$$\Delta u = \Delta g + \Delta w = -f + f = 0 \text{ on } \Omega.$$

Then, we can say that for $g : \mathbb{R}^n \to \mathbb{R}$ smooth, $u \in C^\infty(\Omega)$.

Therefore we can integrate (6.3) by parts to get

$$\int_\Omega \Delta u \cdot \varphi dx = 0 \tag{7.2}$$

for all test functions φ. Then $\Delta u = 0$ almost everywhere. Then, by continuity, we know $\Delta u = 0$ everywhere, and so u is a strong solution.

8. Useful Properties of Harmonic Functions

Before we show that u is a unique solution, we need to define some properties of harmonic functions. We will first define the mean value property. We will be following the work done by Han and Lin in this section[5].

Definition 8.1. For $u \in C(\Omega)$ We say u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) d\sigma_y \tag{8.2}$$

for any $B_r(x) \in \Omega$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

This definition can be interpreted to mean that if u satisfies the mean value property, then the value of any point in the interior can be determined by the averaging of values on the boundary. We will next define the second mean value property and show that these properties are equivalent.

Definition 8.3. For $u \in C(\Omega)$ We say u satisfies the second mean value property if

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n} \int_{B_r(x)} u(y) dy \tag{8.4}$$

for any $B_r(x) \in \Omega$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

To show Definition 8.1 and Definition 8.3 are equivalent, we can manipulate (8.2) in the following way:

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int_{\partial B_r(x)} u(y) d\sigma_y.$$

Then by integration we get

$$\int_0^r u(x) r^{n-1} = \int_0^r \frac{1}{\omega_n} \int_{\partial B_r(x)} u(y) d\sigma_y,$$

which results in the second mean value property:

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n} \int_{B_r(x)} u(y) dy. \tag{8.5}$$

Now that we have defined the different ways u can satisfy the mean value property, we wish to show that any solution to function will satisfy this property. To do this, we examine the following theorem from Han and Lin [5].

Theorem 8.6. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean value property in Ω.*

Proof. Take any ball $B_r(x) \in \Omega$. Then for $\rho \in (0,4)$

$$\int_{B_\rho(x)} \Delta u(y)dy = \int_{B\rho(x)} \nabla \cdot \nabla u dy = \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu} d\sigma \tag{8.7}$$

by the divergence theorem, where ν is the normal to the surface.
Then, using a change of variables and scaling,

$$\begin{aligned} \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu} d\sigma &= \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial \nu}(x+\rho\omega) d\sigma_\omega \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega) d\sigma_\omega . \end{aligned} \tag{8.8}$$

Then, since $\Delta u = 0$ as u is harmonic,

$$0 = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega) d\sigma_\omega . \tag{8.9}$$

This tells us that $\int_{|\omega|=1} u(x+\rho\omega)d\sigma_\omega$ is constant with respect to ρ. Therefore, since $\rho \in (0,r)$, we can evaluate this integral at the endpoints. This gives us:

$$\begin{aligned} 0 &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(x+\rho\omega) d\sigma_\omega \\ &= \int_{|\omega|=1} u(x+r\omega) d\sigma_\omega - \int_{|\omega|=1} u(x) d\sigma_\omega . \end{aligned} \tag{8.10}$$

Then by simple manipulation,

$$\int_{|\omega|=1} u(x+r\omega) d\sigma_\omega = \int_{|\omega|=1} u(x) d\sigma_\omega = u(x)\omega_n . \tag{8.11}$$

Then,

$$u(x) = \frac{1}{\omega_n} \int_{|\omega|=1} u(x+r\omega) d\sigma_\omega = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) d\sigma_y \qquad . \tag{8.12}$$

Therefore, u satisfies the mean value property as required. □

Next we define the maximum principle.

Definition 8.13. The function $u \in \Omega$ has the maximum principle if it either assumes its maximum and minimum values on the boundary of Ω or u is constant in which case it will also attain a maximum or minimum inside Ω.

Next we use the mean value property to show that our solution u will have the maximum principle.

Proposition 8.14. *If $u \in C(\overline{\Omega})$ satisfies the mean value property in Ω then it has the maximum principle.*

Proof. For this proof we only prove the maximum case. The case for the minimum follows similarly.
First create a set A where

$$A = \{x \in \Omega | u(x) = \max_{\Omega} u = M\} \subset \Omega$$

We want to show that if either $A = \emptyset$ or $A = \Omega$ then u satisfies the maximum

principle. If $A = \emptyset$ there does not exist an $x \in \Omega$ such that $u(x)$ is a maximum . Then, the maximum will occur on the boundary of Ω as required. If $A = \Omega$, then u takes on the maximum for all $x \in \Omega$ so u is constant as required.
The proof that A is closed comes from the fact that the preimage of a closed set is closed.
To show A is open, we use the mean value property. Let $x_0 \in A$. Make a ball $B_r(x_0) \in \Omega$ for $r > 0$. Since $x_0 \in A$,

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int_{B_r(x_0)} u(y)dy$$

by the mean value property. Since M is the maximum of u over Ω,

$$\frac{n}{\omega_n r^n} \int_{B_r(x_0)} u(y)dy \leq M \frac{n}{\omega_n r^n} \int_{B_r(x_0)} dy = M$$

This implies that the inequality is strictly equal, thus $u(y) = M$ for any $u \in B_r(x_0)$. Therefore A is open as required.

□

Now that we know a solution to our harmonic function will have the maximum principle, we can move on to showing uniqueness.

9. Uniqueness

To show that our solution u for the Dirichlet problem is unique, we apply the maximum principle. Define a linear transformation $H : u \to g$. Then, the uniqueness of u is equivalent to H being invertible. Since H is invertible if and only if $Ker(H) = \{0\}$, the uniqueness of our solution u holds if $u(x) = 0$ for

$$\begin{aligned} \Delta u_0(x) = 0 \qquad & x \in \Omega \\ u_0(x) = 0 \qquad & x \in \delta\Omega. \end{aligned}$$

To show this we apply the maximum principle. By the maximum principle, the maximum and minimum values of u for $x \in \overline{\Omega}$ takes place on the boundary. Then, for $x \in \Omega$, $0 \leq u(x) \leq 0$. So $u(x) = 0$. Thus, u is uniquely determined by g.

Acknowledgments

It is a pleasure to thank Peter May and his support that makes the University of Chicago REU program possible. I am also happy to thank my mentor, William Cooperman, in providing me with help, guidance, and support throughout the writing of this paper.

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