PRIMER TO CHARACTERISTIC CLASSES WITH APPLICATION TO GAUGE THEORY.

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ABSTRACT. In this paper, we explore characteristic classes of real and complex vector bundles. We then provide an application of Chern classes to the Dirac magnetic monopole.

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1. INTRODUCTION.

The purpose of this paper is to give a brief overview of the theory of characteristic classes. In the literature, there are generally two approaches to characteristic classes. The first approach heavily relies on algebraic topology. See [4] for this approach. The second approach, generally referred to as Chern-Weil theory, uses machineries from differential geometry. The interested reader should consult [3], [5], and [6] for this approach. [1] We will follow the second approach because it allows us to write the characteristic classes explicitly in terms of connections of fiber bundles. We give a brief discussion of Stiefel-Whitney classes in the appendix.

Thus, the prerequisites of this paper are elementary algebraic and differential topology as well as differential geometry. In particular, we will assume results from the theory of fiber bundles and de Rham cohomology. Excellent references for these two topics include [3] and [5].

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[1]It is worth noting that this second approach is restricted to real coefficients for Pontrjagin classes. For our purposes, this will not pose any inconveniences.

1.1. **Notation and Conventions.** Manifolds are taken to be smooth, Hausdorff, and paracompact. Bundles will be denoted by $\xi = (E, \pi, M)$ or the standard $E \xrightarrow{\pi} M$.

2. Pontrjagin Classes.

Let ∇ be a connection on the n-dimensional vector bundle E over M. If f is an invariant polynomial of degree k, then $f(\Omega)$ is a $2k$-form.

Definition 2.1. The **Pontrjagin class** of degree k is the characteristic class

$$p_k(E) = \frac{1}{(2\pi)^{2k}} \sigma_{2k}(\Omega) \in H^{4k}_{dR}(M) = H^{4k}(M; \mathbb{R}) \tag{2.2}$$

where σ_i is the ith symmetric polynomial.

When we say we take the polynomial of the curvature, we mean the following. For a neighborhood U in the base space, we can choose a frame $\mathbf{e}$ on $E|_U$, and write the curvature Ω as a matrix of 2-forms, $R(\mathbf{e})$. We can thus make sense of what we mean by taking a polynomial of these curvatures. For a different choice of frame $\mathbf{e}'$, we would have a different matrix $R(\mathbf{e}')$ related to the first by $R(\mathbf{e}) = gR(\mathbf{e})g^{-1}$ where g is a smooth map from U to the $n \times n$-invertible matrices. We can restrict to a special class of polynomials P for which the resulting expression $P(R(\mathbf{e}))$ does not depend on the choice of frames:

Definition 2.3. Let $A = (A_{ij}) \in M_n(\mathbb{F})$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). The homogeneous polynomial $P \in \mathbb{F}[A_{11}, ..., A_{nn}]$ in n^2 variables is **invariant** if for all $g \in GL_n(\mathbb{F})$,

$$P(gAg^{-1}) = P(A) \tag{2.4}$$

One can show that $P(\Omega)$ is a closed form for any choice of P and ∇, and in fact, the cohomology class $[P(\Omega)]$ is independent of P and ∇. All of the above are general constructions that apply to the other characteristic classes as well. See Chapter 18 of [3] for details.

We only consider even degree polynomials because of the following fact.

Proposition 2.5. *If an invariant polynomial f has odd degree, then $[f(\Omega)] = 0$ in $H^{2k}_{dR}(M)$.*

Proof. Since any invariant polynomial can be written in terms of $s_k(A) := Tr(A^k)$, it suffices to show $[s_k(\Omega)] = 0$[2]. Place a Riemannian metric on E, and take a connection ∇ on E compatible with the metric. Since the connection forms are antisymmetric, the curvature forms are also antisymmetric. Hence, for k odd,

$$s_k(\Omega) = \mathrm{Tr}(\Omega^k) = 0 \tag{2.6}$$

where Ω is the skew symmetric matrix corresponding to Ω. □

[2]See Appendix B of [3] for details.

3. Chern Classes.

In analogy to the construction of Pontrjagin classes, we consider the invariant polynomial $\sigma_k(A)$ given by

$$\det(\mathbf{Id} + tA) = \sum_{k=0}^{n} \sigma_k(A) t^k \tag{3.1}$$

which is the most elementary example of an invariant polynomial.

Definition 3.2. The **kth Chern class** of the complex vector bundle ξ is the cohomology class

$$c_k(\xi) = \left[\left(\frac{1}{2\pi i} \right)^k \sigma_k(\Omega) \right] \in H^{2k}(M; \mathbb{C}) \tag{3.3}$$

and the **total Chern class** of ξ is the graded class

$$c(\xi) = 1 + c_1(\xi) + c_2(\xi) + ... \in H^*(M; \mathbb{C}) \tag{3.4}$$

Let's motivate the coefficient given in the definition of $c_k(\xi)$. Take the cohomology class $C \cdot \sigma_k(\Omega)$ for some $C \in \mathbb{C}$. The idea is that the Chern classes must lie in cohomology groups with integer coefficients.

Since $\mathbb{C}P^1$ is a connected, compact, smooth manifold, the integration map

$$I : H^2(\mathbb{C}P^1; \mathbb{C}) \to \mathbb{C} \tag{3.5}$$

is an isomorphism. The canonical inclusion $j : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$ also induces an isomorphism

$$j^* : H^2(\mathbb{C}P^n; \mathbb{C}) \to H^2(\mathbb{C}P^1; \mathbb{C}) \tag{3.6}$$

Our goal is to choose $C \in \mathbb{C}$ so that $C \cdot \sigma_k(\Omega)$ maps to an integer which we choose wlog to be -1 under the isomophism $I \circ j^*$.

Take the canonical line bundle H^1 over $\mathbb{C}P^1$, and consider the stereographic projection maps

$$\psi_-(z,t) = \frac{z}{1+t}, \ \psi_+ = \frac{z}{1-t} \qquad (z,t) \in \mathbb{S}^2 \setminus \{N, S\} \tag{3.7}$$

for the north and south poles N, S. Using this, we have

$$(\psi_-^{-1})^*(\Omega) = \frac{2i}{(1+|z|^2)} dx \wedge dy \tag{3.8}$$

Passing to polar coordinates and computing, we have

$$\int_{\mathbb{R}^2} (\psi_-^{-1})^*(\Omega) = 2\pi i \tag{3.9}$$

Therefore,

$$\int_{\mathbb{C}P^1} \Omega = 2\pi i \tag{3.10}$$

which implies that we must indeed take $C = \frac{1}{2\pi i}$.

4. Euler Class.

We recall from our discussion of Pontrjagin classes that $f(\Omega)$ for an invariant polynomial of odd degree gives a trivial cohomology class. As a result, the highest cohomology class has degree divisible by 4. The idea for Euler class is to take the "square root" of this highest Pontrjagin class.

Let E be a real oriented vector bundle over M with Riemannian metric and connection compatible with the metric. Let PfX be the Pfaffian. From the identity $\det X = Pf(X)^2$, we have

$$e(E)^2 = p_n(E) \tag{4.1}$$

where we define:

Definition 4.2. The **Euler class** of a real oriented vector bundle E is the cohomology class

$$e(E) = \left[\frac{1}{(2\pi)^n} Pf(\Omega)\right] \in H^{2n}(M; \mathbb{R}) \tag{4.3}$$

For an alternative definition, we first define the Thom isomorphism.

Proposition 4.4. *Let ξ be an oriented n-plane bundle with total space E. Then the cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < n$, where $E_0 = E \setminus B$. Furthermore, $H^n(E, E^0; \mathbb{Z})$ contains one and only one cohomology class u whose restriction*

$$u|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z}) \tag{4.5}$$

is equal to the preferred generator u_F for every fiber F of ξ, where F_0 is F minus the base point. Furthermore, the correspondence $y \mapsto y \cup u$ maps $H^k(E; \mathbb{Z})$ isomorphically onto $H^{k+n}(E, E_0; \mathbb{Z})$ for every integer k.

Definition 4.6. Notation as in the previous proposition, the map

$$\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z}) \tag{4.7}$$

$$x \mapsto (\pi^* x) \cup u \tag{4.8}$$

is the **Thom isomorphism**.

Using the above, we can alternatively define the Euler class as the cohomology class $e(\xi) \in H^n(B; \mathbb{Z})$ which corresponds to $u|_E$ under the isomorphism $\pi^* : H^n(B; \mathbb{Z}) \to H^n(E; \mathbb{Z})$. See section I.6 of [1] for an explicit construction of the Euler class for a rank 2 vector bundle.

We recall that the Gauss-Bonnet theorem relates the local geometry (given by Gaussian curvature) to the global topology (given by the Euler characteristic). We give a generalization of this fact here.

Proposition 4.9 (Generalized Gauss-Bonnet.)**.** *Let M be an oriented $2n$-dimensional closed C^∞ manifold. Then*

$$\langle e(TM), [M]\rangle = \chi(M) \tag{4.10}$$

Hence, for the curvature form Ω of a connection compatible with the Riemannian metric on M,

$$\int_M e(TM) = \chi(M) \tag{4.11}$$

Remark 4.12. The above is the origin of the name "Euler class."

See [3] or [5] for a proof of this.

5. Characteristic Numbers.

Characteristic numbers describe the global curving properties of vector bundles. Here, we will look at some examples of characteristic numbers and state some results in cobordism theory. See section 5.7 of [5], and section 4, 16, and 17 of [4].

Definition 5.1. Let M be a $4n$-dimensional, oriented closed manifold. For a homogeneous polynomial f of degree n, a **Pontrjagin number** is the number

$$\langle f(p_1(M), p_2(M), ...), [M]\rangle \tag{5.2}$$

where $[M]$ is the fundamental class of M and p_k are the Pontrjagin classes.

Chern numbers and Stiefel-Whitney numbers are defined analogously. See the appendix for a treatment of Stiefel-Whitney classes.

Theorem 5.3. *Let B be a smooth, compact n-dimensional manifold. B can be realized as a boundary of some smooth compact manifold M iff all of its Stiefel-Whitney numbers are zero.*

The forward direction is due to Pontrjagin. See [4] for a proof. The converse is much harder to prove, and it is due to Thom.

For instance, one can deduce from this that odd dimensional projective space is a boundary of some smooth, compact manifold.

Definition 5.4. Two smooth closed n-manifolds M_1, M_2 belong to the same **(unoriented) cobordism class** (or that they are **cobordant**) if their exists a compact $(n+1)$-dimensional manifold W such that $\partial W = M \amalg (-N)$. Here, $\amalg$ is a topological sum, and $-N$ is N endowed with the opposite orientation.

Theorem 5.5. *Two smooth closed n-manifolds belong to the same cobordism class iff all of their corresponding Stiefel-Whitney numbers are equal.*

Theorem 5.6 (Pontrjagin)**.** *Two cobordant closed manifolds have the same Pontrjagin numbers.*

Remark 5.7. The converse to Theorem 5.6 is false, but the necessary and sufficient conditions for cobordism is known.

6. Generalizations.

So far, all of our characteristic classes were defined for vector bundles. Here, we will make a few remarks that generalizes these ideas to principal bundles.

Proposition 6.1 (Weil Homomorphism)**.** *Let $I(G)$ be the collection of invariant polynomials with respect to the group G, and let E be a fiber bundle over M with structure group G. Then the map*

$$w : I(G) \to H^*(M) \tag{6.2}$$

is a homomorphism.

Theorem 6.3 (Principal Theorem of Chern-Weil Theory.)**.** *Let $\xi = (P, \pi, M, G)$ be a principal G-bundle. If we choose a connection in ξ, then the Weil homomorphism $w : I(G) \to H^*(M; \mathbb{R})$ is determined. If we set $w(f) \in H^{2k}(M; \mathbb{R})$ for $f \in I^k(G)$, then we get a characteristic class of the principal G-bundle.*

See [6] and [5] for details.

7. Application: Dirac Magnetic Monopole.

We provide an application of Chern classes to gauge theory, namely to the Dirac magnetic monopole. Recall that Maxwell's equations state that

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \end{aligned}$$

where $\rho, \mathbf{j}$ are respectively charge and current density, and $\mathbf{E}, \mathbf{B}$ are the electric and magnetic fields. This equation has some noticeable symmetries. For instance,we have charge conjugation, namely the equations are invariant if we swap the signs of $\rho, \mathbf{j}, \mathbf{E}, \mathbf{B}$. We also have the electromagnetic duality, i.e., if we take $\rho = 0,\ \mathbf{j} = 0$, then the equation remains invariant under the transformation $\mathbf{E} \mapsto \mathbf{B},\ \mathbf{B} \mapsto -\mathbf{E}$.

Dirac noticed an asymmetry in the above equations, namely $\nabla \cdot \mathbf{B} = 0$, i.e. that there are no sources for magnetic fields. Dirac then considered what happens if there is a magnetic monopole. He deduced that the existence of a magnetic monopole implies that all charges come in integer multiples of the charge of an electron.

It turns out that the bundle $P(\mathbb{S}^2, U(1))$ describes the Dirac monopole. We show that the strength of the monopole g is intimately related to the Chern number of this bundle.

Take the usual neighborhoods U_N, U_S on $\mathbb{S}^2$ around the north and south poles respectively. Let ω be the connection 1-form on P. For local sections σ_N, σ_S on each neighborhood, define the local gauge potentials

$$A_N := \sigma_N^* \omega, \;\; A_S := \sigma_S^* \omega \tag{7.1}$$

We can choose σ_N, σ_S so that in coordinates it is given by (the so called **Wu-Yang form**)

$$A_N = ig(1 - \cos\theta)d\phi, \;\; A_S = -ig(1 + \cos\theta)d\phi \tag{7.2}$$

where g is the strength of the monopole. (This is given so that the magnetic field is $\mathbf{B} = g\mathbf{r}/r^3$. See 1.9.1 of [6] for a detailed discussion.)

Let

$$t_{NS} : U_N \cap U_S \to U(1) \tag{7.3}$$

$$t_{NS}(\varphi) = \exp(i\varphi(\phi)) \tag{7.4}$$

be the transition function on the equator where $\varphi : \mathbb{S}^1 \to \mathbb{R}$. Then A_N, A_S are related on the equator by

$$A_N = t_{NS}^{-1} A_S t_{NS} + t_{NS}^{-1} dt_{NS} = A_S + i \cdot d\varphi \tag{7.5}$$

And so, using the formula for A_N, A_S from above, we have

$$d\varphi = -i(A_N - A_S) = 2g d\phi \tag{7.6}$$

Integrating this over the equator, we have

$$\Delta\varphi = \int_0^{2\pi} 2g d\varphi = 4\pi g \tag{7.7}$$

For t_{NS} to be defined uniquely, $\Delta\varphi$ must be an integer multiple of 2π. Thus,

$$\Delta\varphi / 2\pi = 2g \in \mathbb{Z} \tag{7.8}$$

We observe here that the above quantity is just the magnetic flux of the monopole through $\mathbb{S}^2$. Indeed, by Stokes theorem, we have

$$\Phi = \int_{\mathbb{S}^2} \mathbf{B} \cdot d\mathbf{S} = \int_{U_N} dA_N + \int_{U_S} dA_S = \int_{\mathbb{S}^1} A_N + \int_{\mathbb{S}^1} A_S = 4\pi g \tag{7.9}$$

This also has a topological meaning. In section 6, we generalized characteristic classes to principal bundles. We see that the above is just the **first Chern number** of the principal bundle since

$$\int_{\mathbb{S}^2} c_1(dA_i) = \frac{i}{2\pi}\int_{\mathbb{S}^2} dA_i \tag{7.10}$$

where the field strength dA_i is the curvature.

8. Appendix: Stiefel-Whitney Classes.

Like Chern and Euler classes, there are a few ways of defining Stiefel-Whitney classes. Here, we will define it using Steenrod squares. See chapter 8 of [4] for details. See [2] section 4.L for a discussion on Steenrod squares.

Definition 8.1. A transformation $\Theta = \Theta_X : H^m(X;G) \to H^n(X;H)$ defined for all spaces X, groups G, H, and fixed choices of m, n is a **cohomology operation** if for all maps $f : X \to Y$, the map satisfies the naturality condition $\Theta_Y \circ f^* = f^* \circ \Theta_X$, i.e. the following diagram commutes

$$\begin{array}{ccc} H^m(Y;G) & \xrightarrow{\Theta_Y} & H^n(Y;H) \\ \downarrow f^* & & \downarrow f^* \\ H^m(X;G) & \xrightarrow{\Theta_X} & H^n(X;H) \end{array}$$

The cohomology operation we are interested in is the following.

Definition 8.2. The homomorphism $Sq^i : H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2),\ i \geq 0$ satisfying the following axioms is called the **Steenrod square**

- Naturality, i.e. $Sq^i \circ f^* = f^* \circ Sq^i$ for all maps $f : X \to Y$
- $Sq^0 = \mathbf{Id}$
- $Sq^n(x) = x^2$ for $x \in H^n(X;\mathbb{F}_2)$
- If $n \geq \deg(x)$, then $Sq^n(x) = 0$
- Cartan formula: $Sq^n(x \cup y) = \sum_{i+j=n} (Sq^i x) \cup (Sq^j y)$

Definition 8.3. Let (E, B, F) be a vector bundle, and $\phi : H^k(B) \to H^{k+n}(E;E_0)$ be the Thom isomorphism. The **Stiefel-Whitney class** is the cohomology class

$$w_i(\xi) = \phi^{-1} \circ Sq^i \circ \phi(1) \in H^i(B) \tag{8.4}$$

One can verify that under this definition, the Stiefel-Whitney class satisfies the following properties. See [4] for a proof:

- $w_0(\xi) = 1 \in H^0(B;\mathbb{F}_2)$ and if $i \geq \dim \xi + 1$, then $w_i(\xi) = 0$.
- Naturality, i.e. if $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then $w_i(\xi) = f^* w_i(\eta)$.
- Whitney product formula. If ξ, η are vector bundles over the base space B, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta) \tag{8.5}$$

- $w_1(\gamma_1^1) \neq 0$ for the canonical line bundle γ_1^1 over $\mathbb{P}^1$.

We can take the above as axioms and define the Stiefel-Whitney classes as cohomology classes satisfying these axioms. See [4] for this approach.

We can give an alternative characterization of Stiefel-Whitney classes using the Čech cohomology.

Let M be an orientable Riemannian manifold. In particular, the structure group of the tangent bundle of M can be reduced to $SO(m)$. Let LM be the frame bundle over M, and let t_{ij} be the transition function of LM which satisfies the consistency condition

$$t_{ij}t_{jk}t_{ki} = \mathbf{Id} \qquad t_{ii} = \mathbf{Id} \tag{8.6}$$

We define the **spin structure** on M as follows. Let $SPIN(m)$ be the mth spin group, i.e. the topological group which is the universal cover of $SO(m)$. (So for instance, $SPIN(3) = SU(2)$.) Let $\varphi : SPIN(m) \to SO(m)$ be the covering map. Then take $\tilde{t}_{ij} \in SPIN(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij} \qquad \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \mathbf{Id} \qquad \tilde{t}_{ii} = \mathbf{Id} \tag{8.7}$$

We will define the Stiefel-Whitney class so that the first Stiefel-Whitney class measures the obstruction to the orientability of M, and the second Stiefel-Whitney class as the admittance of a spin structure of M.

Let $\mathcal{U} = \{U_i\}$ be a contractible open covering of M, and let $\{e_{i\alpha}\}_{1\leq\alpha\leq m}$ be a local orthonormal frame of TM over U_i. By definition of a transition function, we have $e_{i\alpha} = t_{ij}e_{j\alpha}$ where $t_{ij} : U_i \cap U_j \to O(m)$. Define

$$c(i,j) = \det(t_{ij}) = \pm 1 \tag{8.8}$$

We claim that this is an element of $H^1(M;\mathbb{F}_2)$ which does not depend on the choice of local frames chosen, and define the class $[c]$ to be the **first Stiefel-Whitney class** $w_1(M) \in \check{\mathrm{H}}(M;\mathbb{F}_2)$.

Firstly, c must be a Čech 1-chain because $c(i,j) = c(j,i)$ from its definition. Next, for the cocycle condition,

$$\delta c(i,j,k) = \det(t_{ij})\det(t_{jk})\det(t_{ki}) = 1 \tag{8.9}$$

Thus c is a Čech cocyle, so $[c] \in \check{\mathrm{H}}(M;\mathbb{F}_2)$. Choose a different frame $\{\bar{e}_{i\alpha}\}$ over U_i for which $\bar{e}_{i\alpha} = h_i e_{i\alpha}$, $h_i \in O(m)$. Then $\bar{t}_{ij} = h_i t_{ij} h_j^{-1}$, so taking a 0-cochain c_0 such that $c_0(i) = \det h_i$, we have

$$\tilde{c}(i,j) = \det(h_i t_{ij} h_j^{-1}) = \delta c_0(i,j) f(i,j) \tag{8.10}$$

so, c changed by an exact amount, so we still have the same cohomology class.

Proposition 8.11. *Let M be an Riemannian manifold. M is orientable iff $w_1(M)$ is trivial.*

Proof. For the forward direction, if M is orientable, then the structure group can be reduced to $SO(m)$, and so, $c(i, j) = \det(t_{i,j}) = 1$, hence $w_1(M) = 1$.

For the backward direction, if $w_1(M)$ is trivial, then by definition of cohomology, c is a coboundary, i.e. there exists some c_0 such that $c = \delta c_0$. Now $c_0(i) = \pm 1$, so we can choose $h_i \in O(m)$ such that $\det(h_i) = c_0(i)$ for each i. If we define the new frame $\overline{e}_{i\alpha} = h_i e_{i\alpha}$, we have the transition functions $\tilde{t}_{ij}$ such that $\det(\tilde{t}_{ij}) = 1$ for any overlapping pair (i, j). Then M is orientable. □

Next we look at the second Stiefel-Whitney class. Let M^m be orientable. In particular, we can reduce the structure group to $SO(m)$. Again, for the transition functions $t_{ij} \in SO(m)$, we can consider the lift $\tilde{t}_{ij} \in SPIN(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij} \qquad \tilde{t}_{ij}^{-1} = \tilde{t}_{ji} \tag{8.12}$$

The necessary and sufficient condition for $\tilde{t}_{ij}$ to define a spin bundle over M is for the cocycle condition

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \mathbf{Id} \tag{8.13}$$

to be satisfied. The second Stiefel-Whitney class will be defined so to measure how much this fails. Let $c : U_i \cap U_j \cap U_k \to \mathbb{F}_2$ be given by

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = c(i, j, k)\mathbf{Id} \tag{8.14}$$

c is closed and symmetric, so it defines a cohomology class. The **second Stiefel-Whitney class** is the cohomology class $[c] \in \check{\mathrm{H}}^2(M; \mathbb{F}_2)$.

Proposition 8.15. *Let M be an orientable manifold. M admits a spin bundle iff $w_2(M)$ is trivial.*

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