CLASSIFICATIONS SIMPLE COMPLEX LIE ALGEBRAS

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Abstract. This paper is a quick introduction to Lie algebras, especially semisimple and simple complex Lie algebras. Our final goal for this paper is to classify the simple Lie algebras, but, along the way, we will introduce a few definitions and theorems that have wider implications to the general study of Lie algebras and representation theory.

Contents

1. Introduction

In this paper, we will first introduce the subject of Lie algebras and some very basic definitions and theorems of the subject, along with some examples that will motivate the study of Lie algebras, including its connection to the study of Lie groups. In the process, we will introduce the idea of a representation of a Lie algebra on a vector space, which will prove to be a crucial part of understanding Lie algebras themselves, especially via their adjoint representations. Then we will introduce the Killing form, another tool that will help us characterize important classes of Lie algebras (i.e. solvable Lie algebras and semisimple Lie algebras) and provide us with structures for further analyzing Lie algebras. Section 4 will look closely at a special Lie algebra, $\mathfrak{sl}_2(\mathbb{C})$, especially at the element H that acts diagonally on $\mathfrak{sl}_2(\mathbb{C})$. This will provide us with a template on how we should proceed with analyzing semisimple Lie algebras. Provided with the framework obtained by studying $\mathfrak{sl}_2(\mathbb{C})$, we will introduce the Cartan subalgebras, which will play the role of H in general semisimple Lie algebras. In section 6, we will start to analyze structures of semisimple complex Lie algebras: we introduce the notion of roots, give a Euclidean structure to the real vector space spanned by roots, and state a

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few results on how the sets of roots should sit in this Euclidean space. Finally, in the last section, we will use what we have built in the previous sections to classify the simple complex Lie algebras. This paper will follow closely [1] and [3], with some theorems and proofs taken from [2].

2. Preliminaries

In this section, we will first introduce what a Lie algebra is, followed by occasions where we see Lie algebras appear in the study of other subjects in mathematics. Then, we will give some basic definitions related to Lie algebras, followed by the first structures we have in Lie algebras, which we will use throughout this paper, especially in section 6.

Definition 2.1. A Lie algebra $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear map (Lie bracket):

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

At this point, readers may wonder why we should consider this particular structure. The following example will show a few cases where Lie algebras arise naturally in other principles of mathematics.

Example 2.2. (a) Given a smooth manifold M, we define a smooth vector field X on M to be a smooth map $X : M \to TM$, where TM is the tangent bundle of M that maps $p \in M$ into a tangent vector X_p of M at p. Then X can be considered as a function from $C^\infty(M) \to C^\infty(M)$ by defining $X(f) : M \to \mathbb{R}$ by $X(f)(p) = (X(p))(f)$. Note that X is a linear map. By a theorem in smooth manifolds (see chapter 8 in [4], especially Theorem 8.15), a linear map $D : C^\infty(M) \to C^\infty(M)$ satisfies

$$D(fg) = fD(g) + gD(f)$$

for all f, g in $C^\infty(M)$ if and only if $D(f) = X(f)$ for all f in $C^\infty(M)$ for some smooth vector field X. We may ask if XY is a vector field, but, as it turns out,

$$XY(fg) = X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f)$$

, which contradicts the previous theorem. However, [X,Y](fg)=f[X,Y](g)+g[X,Y](f), which, also by the previous theorem, proves that $[X, Y]$ is also a vector field. Then, the space $\mathfrak{X}(M)$ of smooth vector fields is a Lie algebra.
(b) We know for a fact that a connected Lie group is generated by any open neighborhood around the identity (see chapter 7, 8 in [1]). This means that the tangent space at the identity of a Lie group gives a substantial amount of information about the Lie group. This inspires us to study the tangent space at the identity, especially maps in the tangent space. This then gives a natural Lie algebra structure on the tangent space that is defined in chapter 8 of [1].

From this point forward, we will assume the Lie algebra $\mathfrak{g}$ to be a finite-dimensional complex vector space. We have another interesting example of Lie algebras that is useful to us later on:

Example 2.3. $\mathfrak{g} = \mathrm{End}(V)$ is a Lie algebra with the bilinear operator $[\ ,\ ]$ satisfying $[X, Y] = X \cdot Y - Y \cdot X$, where $X, Y \in \mathrm{End}(V)$. When we refers to $\mathrm{End}(V)$ as a Lie algebra, we denote $\mathrm{End}(V) = \mathfrak{gl}(V)$.

The following examples may seem forced at the moment, but they are the key for our analysis in this paper. We call them the classical Lie algebras.

Example 2.4. We denote the Lie bracket of the following examples to be $[A, B] = AB - BA$.
(a) Let $\mathfrak{sl}_n(\mathbb{C})$ be the set of $n \times n$ complex matrices with traces 0 i.e. $\mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \mathrm{tr}(A) = 0\}$. Then, $\mathfrak{sl}_n(\mathbb{C})$ is a Lie algebra since for all A, B in $\mathfrak{sl}_n(\mathbb{C})$, we have $\mathrm{tr}([A, B]) = \mathrm{tr}(AB - BA) = 0$ i.e. $[A, B]$ is in $\mathfrak{sl}_n(\mathbb{C})$.
(b) Let $\mathfrak{so}_{2n+1}(\mathbb{C})$ be the set of all $(2n+1)\times(2n+1)$ matrices $\begin{pmatrix} A & B & E \\ C & -A^T & F \\ -F^T & -E^T & 0 \end{pmatrix}$ where B and C are $n \times n$ skew-symmetric matrices.
(c) Let $\mathfrak{sp}_{2n}(\mathbb{C})$ be the set of all $2n \times 2n$ complex matrices $\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$ where B and C are symmetric matrices.
(d) Let $\mathfrak{so}_{2n}(\mathbb{C})$ be the set of all $2n \times 2n$ matrices $\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$ where B and C are $n \times n$ skew-symmetric matrices.

We will leave the proof of parts (b), (c), and (d) as an exercise for the reader.

Remark 2.5. These are the Lie algebras that are derived from Lie groups as in part (b) of Example 2.2.
(a) The tangent space at the identity of the Lie group $\mathrm{SL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) | \mathrm{tr}(A) = 1\}$ gives us $\mathfrak{sl}_n(\mathbb{C})$.
(b) Given a vector space V of dimension n and a non-degenerate symmetric bilinear form $Q : V \times V \to \mathbb{C}$, the tangent space at the identity of the Lie group $\mathrm{SO}_n(\mathbb{C}) = \{A \in \mathrm{Aut}(V) | Q(Av, Aw) = Q(v, w)$ for all v, w in $V\}$ gives us $\mathfrak{so}_n(\mathbb{C})$.
(c) Given a vector space V of dimension $2n$ and a non-degenerate skew-symmetric bilinear form $Q : V \times V \to \mathbb{C}$, the tangent space at the identity of the Lie group $\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{Aut}(V) | Q(Av, Aw) = Q(v, w)$ for all v, w in $V\}$ gives us $\mathfrak{sp}_{2n}(\mathbb{C})$.

Now we will give some basic definitions that will help us understand the Lie algebras.

Definition 2.6. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ closed under Lie bracket.

Example 2.7. The Lie algebra $[\mathfrak{g}, \mathfrak{g}] = span\{[X, Y] | X, Y \in \mathfrak{g}\}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$ for all Lie algebras $\mathfrak{g}$.

Definition 2.8. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal of a Lie algebra $\mathfrak{g}$ if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Note that there are similarities between an ideal of a Lie algebra and an ideal of a ring; one of them is the fact that we can define a quotient Lie algebra $\mathfrak{g}/\mathfrak{h} = \{X + \mathfrak{h} | X \in \mathfrak{g}\}$ with an induced Lie algebra structure, namely $[X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h}$.

Example 2.9. The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$.

Definition 2.10. A map of Lie algebras $\mathfrak{g}, \mathfrak{h}$ is a linear function $\phi : \mathfrak{g} \to \mathfrak{h}$ preserving the Lie bracket i.e.

$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

We say a map ϕ between the two Lie algebras is an isomorphism if it is a vector space isomorphism. We say two Lie algebras are isomorphic if there exists an isomorphism between the two. If two Lie algebras are isomorphic, we will write $\mathfrak{g} \cong \mathfrak{h}$.

Definition 2.11. A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a map of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

We can view the representation of a Lie algebra as an instruction on how the Lie algebra acts on a vector space V. Hence, the importance of the representation to the study of Lie algebras is similar to the importance of group action to the study of groups.

For the rest of this paper, we will assume V to be a finite-dimensional vector space i.e. we only consider finite-dimensional representations.

Remark 2.12. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\gamma : \mathfrak{g} \to \mathfrak{gl}(W)$ be representations. We define the representations $\rho \oplus \gamma : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ by $(\rho \oplus \gamma)(X) = \rho(X) \oplus \gamma(X)$ and $\rho \otimes \gamma : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$ by $(\rho \otimes \gamma)(X) = \rho(X) \otimes \gamma(X)$.

The following representation is one of the most important representations of a Lie algebra as it gives us an action of the Lie algebra on itself. We will use this representation repeatedly throughout the paper.

Example 2.13. Let $\mathfrak{g}$ be a Lie algebra, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be a representation defined by $\mathrm{ad}(X)(Y) = [X, Y]$. We note that $\mathfrak{g}$ is a vector space and the map ad is a linear map since $\mathrm{ad}(cX+Y)(Z) = [cX+Y, Z] = c[X, Z]+[Y, Z] = c\cdot\mathrm{ad}(X)(Z)+\mathrm{ad}(Y)(Z)$. Moreover, the map ad preserves the Lie bracket as

$$\begin{aligned}\mathrm{ad}([X, Y])(Z) &= [[X, Y], Z] = -[[Y, Z], X] - [[Z, X], Y] = [X, [Y, Z]] - [Y, [X, Z]] \\ &= \mathrm{ad}(X)(\mathrm{ad}(Y)(Z)) - \mathrm{ad}(Y)(\mathrm{ad}(X)(Z)) = [\mathrm{ad}(X), \mathrm{ad}(Y)](Z)\end{aligned}$$

Remark 2.14. The image of ad is a Lie algebra with the Lie bracket satisfying $[\mathrm{ad}(X), \mathrm{ad}(Y)] = \mathrm{ad}([X, Y])$.

Definition 2.15. A Lie algebra $\mathfrak{g}$ is simple if $\dim(\mathfrak{g}) > 1$ and it contains no nontrivial ideals.

Remark 2.16. $\mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{so}_{n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$ are simple complex Lie algebras for all $n \geq 1$. We will delay the proof of this remark until Lemma 7.6.

In an attempt to classify the Lie algebras, we introduce the following chains of subalgebras.

Definition 2.17. The lower central series of subalgebras $\mathcal{D}_k\mathfrak{g}$ is defined by $\mathcal{D}\mathfrak{g} = \mathcal{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}_k\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{k-1}\mathfrak{g}]$. The derived series of subalgebras $\mathcal{D}^k\mathfrak{g}$ is defined by $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}^k\mathfrak{g} = [\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}]$

Definition 2.18. (a) $\mathfrak{g}$ is nilpotent if $\mathcal{D}_k\mathfrak{g} = 0$ for some k.
(b) $\mathfrak{g}$ is solvable if $\mathcal{D}^k\mathfrak{g} = 0$ for some k.
(c) $\mathfrak{g}$ is semisimple if $\mathfrak{g}$ has no nonzero solvable ideals.

Now we will introduce the first two main results of this section: Engel's theorems and Lie's theorem. The first part of Engel's theorem gives us some structure for a nilpotent Lie algebra. Lie's theorem provides us with a generalized result of Engel's theorems for solvable Lie algebra.

Theorem 2.19. *(Engel's theorems)*

(a) Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any complex Lie subalgebra such that for all X in $\mathfrak{g}$, X is a nilpotent endomorphism of V. Then there exists a nonzero vector v in V such that $X(v) = 0$ for all $X \in \mathfrak{g}$.
(b) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, then $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(X)$ *is nilpotent for all X in $\mathfrak{g}$.*

A detailed proof can be found in Theorem 9.9 and Exercise 9.10 in [1].

Corollary 2.20. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite-dimensional vector space V such that $\rho(X)$ is a nilpotent operator for all X in $\mathfrak{g}$. Then, there exists a basis for V such that $\rho(X)$ is strictly upper triangular matrices for all X in $\mathfrak{g}$.*

Theorem 2.21. *(Lie's theorem) Let $\mathfrak{g}$ be a solvable Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Then there exists $v \in V$ that is an eigenvector for $\rho(X)$ for all $X \in \mathfrak{g}$.*

A detailed proof can be found in Theorem 9.11 in [1].

Now, we will provide some more definitions for our last main result of this section: the decomposition theorem.

Definition 2.22. Let $\mathfrak{g}$ be a Lie algebra, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. For a fixed $X \in \mathfrak{g}$, we define $V_\lambda^X = \{v \in V | \exists N \in \mathbb{N} : (\rho(X) - \lambda I)^N v = 0\}$. We can apply this definition to the adjoint representation and obtain $\mathfrak{g}_\lambda^X = \{Y \in \mathfrak{g} | \exists N \in \mathbb{N} : (\mathrm{ad}(X) - \lambda I)^N Y = 0\}$.

Remark 2.23. With the notation above, we have the following decomposition:

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^X$$

We notice that this is just the generalized eigenspace of $\rho(X)$ in the context of linear algebra, and the decomposition above is already encountered in linear algebra. The following lemma will makes use of the fact that $\mathfrak{g}$ is more than just an action on V.

Lemma 2.24. $\rho(\mathfrak{g}_\alpha^X) V_\lambda^X \subset V_{\alpha+\lambda}^X$.

Proof. Fix $v \in V_\lambda^X$ and $Y \in \mathfrak{g}_\alpha^X$. Choose $M \in \mathbb{N}$ such that $(\rho(X) - \lambda I)^M v = 0$ and $(ad(X) - \alpha I)^M Y = 0$. Choose $M = 2N$. For all $Z \in \mathfrak{g}$, we have:

$$(\rho(X) - (\alpha + \lambda)I)\rho(Z) = (ad(\rho(X)) - \alpha I)\rho(Z) + \rho(Z)(\rho(X) - \lambda I)$$

Hence, using the binomial theorem, we obtain:

$$(\rho(X) - (\alpha + \lambda)I)^N \rho(Y) = \sum_{i=0}^{N} (ad(\rho(X)) - \alpha I)^i \rho(Y)(\rho(X) - \lambda I)^{N-i}$$

, which makes

$$(\rho(X) - (\alpha + \lambda)I)^N \rho(Y) v = \sum_{i=0}^{N} (ad(\rho(X)) - \alpha I)^i \rho(Y)(\rho(X) - \lambda I)^{N-i} v = 0$$

because if $i \leq M$, then $\rho(Y)(\rho(X) - \lambda I)^{N-i} v = 0$ and if $i > M$, then $(ad(\rho(X)) - \alpha I)^i \rho(Y) = 0$. This makes $\rho(Y)v \in V_{\alpha+\lambda}^X$ for all $v \in V_\lambda^X$ and $Y \in \mathfrak{g}_\alpha^X$ i.e. $\rho(\mathfrak{g}_\alpha^X) V_\lambda^X \subset V_{\lambda+\alpha}^X$. □

Definition 2.25. Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$ be a linear functional on $\mathfrak{h}$. The generalized weight space of $\mathfrak{g}$ in V attached to λ is defined to be:

$$V_\lambda^{\mathfrak{h}} = \{v \in V | \exists N : \mathfrak{h} \to \mathbb{N} \ st \ (\rho(X) - \lambda(X)I)^{N(X)} v = 0 \ \forall \ X \in \mathfrak{h}\}$$

Note that, $V_\lambda^{\mathfrak{h}}$ is just $\bigcap_{X \in \mathfrak{h}} V_{\lambda(X)}^X$ by definition.

Theorem 2.26. *$\rho(\mathfrak{g}_\alpha^{\mathfrak{h}}) V_\beta^{\mathfrak{h}} \subset V_{\alpha+\beta}^{\mathfrak{h}}$ for all nilpotent subagebra $\mathfrak{h}$ and $\forall \alpha, \beta \in \mathfrak{h}^*$.*

Proof. Let $X \in \mathfrak{g}_\alpha^{\mathfrak{h}}$, then $X \in \mathfrak{g}_{\alpha(Y)}^Y \ \forall Y \in \mathfrak{h}$. By Lemma 2.24, $\rho(X) V_{\beta(Y)}^Y \subset V_{\alpha(Y)+\beta(Y)}^Y$. Hence, $\bigcap_{Y \in \mathfrak{h}} \rho(X) V_{\beta(Y)}^Y \subset \bigcap_{Y \in \mathfrak{h}} V_{\alpha(Y)+\beta(Y)}^Y$. By definition, this is $\rho(X) V_\beta^{\mathfrak{h}} \subset V_{\alpha+\beta}^{\mathfrak{h}}$ for all $X \in \mathfrak{g}_\alpha^{\mathfrak{h}}$. Thus, $\rho(\mathfrak{g}_\alpha^{\mathfrak{h}}) V_\beta^{\mathfrak{h}} \subset \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}}$. □

Corollary 2.27. *If ρ is the adjoint representation i.e. $\rho = \mathrm{ad}$, then we obtain the result $[\mathfrak{g}_\alpha^{\mathfrak{h}}, \mathfrak{g}_\beta^{\mathfrak{h}}] \subset \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}}$.*

Now, we will prove the last main result of this section.

Theorem 2.28. *(Decomposition theorem) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of $\mathfrak{g}$. Then, we have the following decomposition:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}}$$

Proof.
Case 1 : For all $X \in \mathfrak{h}$, $\rho(X)$ has only one eigenvalue. Since $\mathfrak{h}$ is nilpotent, it is solvable. By Lie's theorem, there exists $v \neq 0 \in V$ such that $\rho(X)v = \lambda(X)v$ where λ is a function from $\mathfrak{h} \to \mathbb{C}$. Note that λ is linear since $\lambda(cX+Y)v = \rho(cX+Y)v = c\rho(X)v + \rho(Y)v = [c\lambda(X) + \lambda(Y)]v$. Hence $\lambda \in \mathfrak{h}^*$ and $\lambda(X)$ is the only eigenvalue of $\rho(X)$. Hence $V = V_{\lambda(X)}^X$ for all $X \in \mathfrak{h}$, which means that $V = V_\lambda^h$.
Case 2 : There exists $X_0 \in \mathfrak{h}$ such that $\rho(X_0)$ has at least 2 distinct eigenvalues. Since $\mathfrak{h}$ is nilpotent, $ad(X)$ is nilpotent on $\mathfrak{h}$ for all $X \in \mathfrak{h}$ (by Engel's theorem). Hence $\mathfrak{h} \subset \mathfrak{g}_0^X$ for all $X \in \mathfrak{h}$. Hence $\rho(\mathfrak{h}) V_\lambda^X \subset \rho(\mathfrak{g}_0^X) V_\lambda^X \subset V_\lambda^X$ for all $X \in \mathfrak{h}, \lambda \in \mathbb{C}$. By Remark 2.23, we can decompose V as $\bigoplus_{\lambda \in \mathbb{C}} V_\lambda^X$. Since $V_\lambda^{X_0}$ is invariant under the action of $\mathfrak{h}$, $V_\lambda^{X_0}$ is also a representation of $\mathfrak{h}$. Using the assumption that $\rho(X_0)$ has at least 2 distinct eigenvalues, let α be one of them. Then we prove by induction on $\dim(V)$, using the fact that $V_\alpha^{X_0}$ is also a representation of smaller dimension. □

Corollary 2.29. *In the case of the adjoint representation, we obtain:*

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda^{\mathfrak{h}}$$

where $\mathfrak{g}_\lambda^{\mathfrak{h}} = \{Y \in \mathfrak{g} | \exists N : \mathfrak{h} \to \mathbb{N} \ st \ (\mathrm{ad}(X) - \lambda(X)I)^{N(X)}(Y) = 0 \ \forall X \in \mathfrak{h}\}$.

This decomposition gives us some structures of the Lie algebra $\mathfrak{g}$ in terms of a nilpotent algebra $\mathfrak{h}$. In section 5 of this paper, we will see the decomposition of a semisimple Lie algebra in terms of a special nilpotent subalgebra: the Cartan subalgebra.

3. Killing form and Complete Irreducibility

This section will start off by introducing some more structures on a finite-dimensional Lie algebra. We will introduce an inner product on $\mathfrak{g} \subset \mathfrak{gl}(V)$ and, as a result, introduce the Killing form derived from the subalgebra $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{gl}(\mathfrak{g})$. Then we will conclude this section by stating a few important results that characterize a Lie algebra by its Killing form and other results in the general theory of semisimple Lie algebras.

Definition 3.1. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra, where V is a finite-dimensional vector space (with a fixed basis). We define a bilinear form $B_V : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by $B_V(X, Y) = \mathrm{tr}(X \circ Y)$.

Remark 3.2. $B_V(X, Y) = B_V(Y, X)$ and $B_V([X, Y], Z) = B_V(X, [Y, Z])$.

Definition 3.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, we define the Killing form on $\mathfrak{g}$ by $K(X, Y) = B_{\mathfrak{g}}(\mathrm{ad}(X), \mathrm{ad}(Y))$ (note that the Lie algebra $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$).

In this case, the Killing form gives us a natural inner product that is defined with respect to a given basis of $\mathfrak{g}$. We will state without proof the following lemmas and theorems. The proofs can be found in Appendix C of [1].

Lemma 3.4. *(Cartan's criterion) Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra. Then the followings are equivalent:*
(a) $B_V(X, Y) = 0$ for all $X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$.
(b) $B_V(X, X) = 0$ for all $X \in [\mathfrak{g}, \mathfrak{g}]$.
(c) $\mathfrak{g}$ is solvable.

Lemma 3.5. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

Theorem 3.6. *A semisimple complex Lie algebra is the direct sum of simple complex Lie algebras.*

Theorem 3.7. *(Complete Reducibility) Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ be a subspace invariant under the action of $\mathfrak{g}$. Then there exists a subspace $W' \subset V$ invariant under $\mathfrak{g}$ and $V = W \oplus W'$.*

Remark 3.8. The Cartan's Criterion gives us a characterization of solvable Lie algebras in terms of the bilinear form B_V. Lemma 3.5 provides us with the characterization of semisimple Lie algebras in terms of their Killing forms. Theorem 3.7 explains why, for the rest of the paper, we will focus mainly on semisimple Lie algebras. Theorem 3.6 tells us that in order to understand and classify semisimple Lie algebras, we first need to understand and classify simple Lie algebras, which is the center of this paper.

4. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its representation

In this section, we will pay attention to $\mathfrak{sl}_2(\mathbb{C})$ and its representation. The study of $\mathfrak{sl}_2(\mathbb{C})$ will be important in deriving certain properties of a general semisimple Lie algebra and give us the motivation to study its Cartan subalgebra.

In Example 2.4, we defined $\mathfrak{sl}_2(\mathbb{C}) = \{M \in M_2(\mathbb{C}) | tr(M) = 0\}$. We will take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as a basis for $\mathfrak{sl}_2(\mathbb{C})$. We notice the following relations characterize $\mathfrak{sl}_2(\mathbb{C})$:

$$[H,E]=2E,\ [H,F]=-2F,\ [E,F]=H$$

Indeed, let $\mathfrak{g}$ be any Lie algebra. Assume that there exists $K,X,Y\in\mathfrak{g}$ such that:

$$[K,X]=2X,\ [K,Y]=-2Y,\ [X,Y]=K$$

Then, we have a natural mapping ϕ from $\mathfrak{sl}_2(\mathbb{C})\to\mathfrak{g}$ that maps $H\mapsto K$, $E\mapsto X$, and $F\mapsto Y$. Note that this is a Lie Algebra mapping and also a Lie algebra isomorphism between $\mathfrak{sl}_2(\mathbb{C})$ and the subalgebra $\mathfrak{k}=\mathbb{C}K+\mathbb{C}X+\mathbb{C}Y$ of $\mathfrak{g}$. Hence, when we deal with semisimple Lie algebra $\mathfrak{g}$, we normally starts off with finding a structure within $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Another reason we may want to consider the substructure isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ is because of the following theorem about its representation.

Theorem 4.1. *Let $\rho:\mathfrak{sl}_2(\mathbb{C})\to\mathfrak{gl}(V)$ be a representation. Let $E=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $F=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, and $H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Let $v\in V$ be a nonzero vector such that $\rho(E)v=0$ and $\rho(H)v=\lambda v$ for some $\lambda\in\mathbb{C}$. Then:*
(a) $\rho(H)\rho(F)^n v=(\lambda-2n)\rho(F)^n v\ \forall n\in\mathbb{Z}_{\geq 0}$.
(b) $\rho(E)\rho(F)^n v=n(\lambda-n+1)\rho(F)^{n-1}v\ \forall n\in\mathbb{N}$.
(c) $\lambda\in\mathbb{Z}_{\geq 0}$ and $\rho(F)^j v$ are linear independent for $0\leq j\leq\lambda$ and $\rho(F)^{\lambda+1}(v)=0$

Proof. (a) We will prove this by induction on n. It is true for $n=0$ by assumption. Assume it holds for $n=k-1$, then:

$$\begin{aligned}\rho(H)\rho(F)^k v&=\rho(F)\rho(H)\rho(F)^{k-1}v+[\rho(H),\rho(F)]\rho(F)^{k-1}v\\&=\rho(F)(\lambda-2k+2)\rho(F)^{k-1}v+\rho([H,F])\rho(F)^{k-1}v\\&=(\lambda-2k+2)\rho(F)^k v+\rho(-2F)\rho(F)^{k-1}v\\&=(\lambda-2k)\rho(F)^k v\end{aligned}$$

This proves (a).
(b) The proof of this will be skipped since it follows the same idea of induction as part (a).
(c) If $\lambda\notin\mathbb{Z}_{\geq 0}$, then $n(\lambda-n+1)\neq 0\ \forall n\in\mathbb{Z}$. Therefore, by induction and part (b), $\rho(F)^n v\neq 0$ for all $n\in\mathbb{Z}_{\geq 0}$. However, by part (a), $\rho(F)^n v$ are all eigenvalues of $\rho(H)$ with distinct eigenvalues; hence, V is infinite-dimensional (contradiction). The rest of part (c) follows trivially. □

The heart of this theorem is in part (c), where it shows us that for a finite-dimensional representation, especially the adjoint representation, the eigenvalue of an element H is a non-negative integer. This also motivates us to find the elements in a general semisimple Lie algebra $\mathfrak{g}$ that plays the same role as H in $\mathfrak{g}$. This is our motivation for the main subject of the next chapter: The Cartan subalgebras of a semisimple complex Lie algebra.

Remark 4.2. The existence of the element $v\in V$ satisfying conditions in Theorem 4.1 is always guaranteed. Indeed, by Theorem 2.21 (Lie's theorem) and the fact that Lie subalgebra $\mathfrak{h}=\mathbb{C}H\oplus\mathbb{C}E$ of $\mathfrak{sl}_2(\mathbb{C})$ is solvable, there exists $v\in V$ such that v is a eigenvector of $\rho(H)$ and $\rho(E)$, with the corresponding eigenvalues λ, β. However, we have:

$$\begin{aligned}2\beta v = \rho(2E)v = \rho([H,E])v = [\rho(H),\rho(E)]v = \rho(H)\rho(E)v - \rho(E)\rho(H)v \\ = \beta\rho(H)v - \lambda\rho(E)v = \beta\lambda v - \lambda\beta v = 0\end{aligned}$$

which makes $\beta = 0$.

Remark 4.3. The subspace W of V spanned by $\{\rho(F)^j | 0 \leq j \leq \lambda\}$ is an invariant subspace under $\mathfrak{sl}_2(\mathbb{C})$. Therefore, $\rho' : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(W)$ is an induced representation from ρ. For each λ, we can construct W as followed:
Let $\mathbb{C}[x,y]$ be the polynomial ring over $\mathbb{C}$ in two variables x and y. Consider the subspace $V^{(\lambda)} = \text{Span}\{\frac{x^i y^{\lambda-i}}{(\lambda-i)!} | 0 \leq i \leq \lambda\}$, where $\lambda \in \mathbb{N}$. Let $X = y \cdot \frac{\partial}{\partial x}$, $Y = x \cdot \frac{\partial}{\partial y}$, and $K = [X,Y]$. Note that, X and Y preserve $V^{(d)}$, which means K also preserves $V^{(d)}$. Moreover, we have:

$$\begin{aligned}&X\frac{x^i y^{\lambda-i}}{(\lambda-i)!} = i(\lambda-i+1)\frac{x^{i-1}y^{\lambda-i+1}}{(\lambda-i+1)!}\\ &Y\frac{x^i y^{\lambda-i}}{(\lambda-i)!} = \frac{x^{i+1}y^{\lambda-i-1}}{(\lambda-i-1)!}\\ &K\frac{x^i y^{\lambda-i}}{(\lambda-i)!} = (i+1)(\lambda-i)\frac{x^i y^{\lambda-i}}{(\lambda-i)!} - i(\lambda-i+1)\frac{x^i y^{\lambda-i}}{(\lambda-i)!} = (\lambda-2i)\frac{x^i y^{\lambda-i}}{(\lambda-i)!}\end{aligned}$$

By this, we obtain:

$$[K,X] = 2X, [K,Y] = -2Y, [X,Y] = K$$

and $K, X,$ and Y act on the basis $\{\frac{x^i y^{\lambda-i}}{(\lambda-i)!}\}$ of $V^{(\lambda)}$ the same way $H, E,$ and F act on the basis $\{\rho(F)^i v\}$ of W.

5. Cartan Subalgebra

With the motivation given to us from the previous chapter, we will begin this chapter by introducing a Cartan subalgebra of a general (not necessary semisimple) complex Lie algebra. We will, of course, give the examples of Cartan subalgebras of classical Lie algebras. Then we will give the theorem about the existence of Cartan subalgebras of a complex Lie algebra and conclude with the isomorphism theorem of Cartan subalgebras of a semisimple complex Lie algebras. The proofs of these theorems are omitted as they are technical and not truly relevant to the rest of the paper. Interested readers can see Chapter III in [2] for complete proofs.

Definition 5.1. A Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra that is self-normalising i.e. if $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$

Example 5.2. We can easily check the following:
(a) The Lie algebra $\mathfrak{h}$ of all $n \times n$ diagonal matrices with trace 0 is a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. We can choose a basis for $\mathfrak{h}$ as H_i is a diagonal matrix with 1 in the i-th place, -1 in the $(i+1)$-th place, and 0 in every place else for all $i = 1, 2, \ldots n-1$.
(b) The Lie algebra $\mathfrak{h}$ of all $(2n+1) \times (2n+1)$ diagonal matrices with the diagonal $(a_1, a_2, \ldots, a_n, -a_1, \ldots, -a_n, 0)$ is a Cartan subalgebra of $\mathfrak{so}_{2n+1}(\mathbb{C})$. We can choose a basis for $\mathfrak{h}$ as H_i is a diagonal matrix with 1 in the i-th place, -1 in the $(i+n)$-th place, and 0 in every place else for all $i = 1, 2, \ldots n$.

(c) The Lie algebra $\mathfrak{h}$ of all $2n \times 2n$ diagonal matrices with the diagonal $(a_1, a_2, \ldots, a_n, -a_1, \ldots, -a_n)$ is a Cartan subalgebra of both $\mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathfrak{so}_{2n}(\mathbb{C})$.We can choose a basis for $\mathfrak{h}$ as H_i is a diagonal matrix with 1 in the i-th place, -1 in the $(i+n)$-th place, and 0 in every place else for all $i = 1, 2, \ldots n$.

Theorem 5.3. *(The existence of a Cartan subalgebra of a semi-simple Lie algebra) Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Then, it has a Cartan subalgebra $\mathfrak{h}$.*

The detailed proof can be found in Chapter III in [2].

The main reason we choose to study the complex Lie algebra instead of real Lie algebras is because of this theorem as a finite-dimensional complex Lie algebra always has a Cartan subalgebra, but this is not necessary for a real Lie algebra.

Theorem 5.4. *(Chevalley's theorem) Every two Cartan subalgebras of $\mathfrak{g}$ are isomorphic.*

The detailed proof can be found in Chapter III in [2].

The next section will introduce more properties of the Cartan subalgebras of a semisimple complex Lie algebra.

6. Structures of semisimple complex Lie algebras

In this chapter, we will begin by proving a few facts about the Killing form and the decomposition of $\mathfrak{g}$ in term of its Cartan subalgebra $\mathfrak{h}$ (note that $\mathfrak{h}$ is nilpotent by definition). This will lead us to a few desirable properties of the Cartan subalgebras of semisimple complex Lie algebras, namely the abelianness of $\mathfrak{h}$ and the diagonality of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ (which is what we want as in section 4 for $\mathfrak{sl}_2(\mathbb{C})$). This will give us a stronger decomposition theorem than the one we obtain in section 2. Then we will introduce the definition of roots and define a symmetric bilinear form $\bar{K}$ on the $\mathfrak{h}^*$. Next, we will prove a few results about the root space, the set of roots, and the symmetric form we just define. Finally, we will study the behavior of roots. Throughout the chapter, we will do some calculations on roots and the form $\bar{K}$ for classical Lie algebras.

In this section, we will assume $\mathfrak{g}$ to be a semisimple Lie algebra and $\mathfrak{h}$ to be a Cartan subalgebra. We simply denote $\mathfrak{g}_\lambda^{\mathfrak{h}} = \mathfrak{g}_\lambda$. We will first introduce the Jordan decomposition and a theorem about the representation of Jordan decomposition, as it will help us prove the diagonality of $\mathfrak{h}$.

Definition 6.1. Let $\mathfrak{g}$ be a Lie algebra. A Jordan decomposition of X in $\mathfrak{g}$ is a decomposition of the form $X = X_s + X_n$ where

(a) $X_s, X_n \in \mathfrak{g}$.
(b) $\mathrm{ad}(X_s)$ is diagonalizable.
(c) $\mathrm{ad}(X_n)$ is nilpotent.
(d) $[X_s, X_n] = 0$.

Theorem 6.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then, for all X in $\mathfrak{g}$, there exists a Jordan decomposition $X = X_s + X_n$ such that for every representation ρ, $\rho(X)_s = \rho(X_s)$ and $\rho(X)_n = \rho(X_n)$.*

The proof of this theorem can be found in Appendix C of [1].

The following theorem is the first main result of this section.

Theorem 6.3. *Let K be the Killing form on $\mathfrak{g}$. Then:*
(a) $K_{|\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} = 0$ if $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta \not\equiv 0$
(b) $K_{|\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-degenerate. Moreover, $K_{|\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate.
(c) $\mathfrak{h}$ is abelian.
(d) The action of $\mathfrak{h}$ on $\mathfrak{g}$ via adjoint representation is diagonalizable. Therefore, $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | \operatorname{ad}(H)(X) = \lambda(H) \cdot X \ \forall\ H \in \mathfrak{h}\}$.

Proof. (a) Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ where $\alpha + \beta \not\equiv 0$. Note that: $(\operatorname{ad}(X) \circ \operatorname{ad}(Y))^N \mathfrak{g}_\gamma \subset \mathfrak{g}_{\gamma+N(\alpha+\beta)}$ (by Corollary 2.27). Since $\mathfrak{g}$ is finite-dimensional, there are only finite summands in the decomposition in Theorem 2.28. We can choose N such that $\mathfrak{g}_{\gamma+N(\alpha+\beta)} = 0$ for each $\mathfrak{g}_\gamma \neq 0$. Hence, $(\operatorname{ad}(X) \circ \operatorname{ad}(Y))^N \mathfrak{g} = 0$, or $(\operatorname{ad}(X) \circ \operatorname{ad}(Y))^N = 0$. This means that $\operatorname{ad}(X) \circ \operatorname{ad}(Y)$ is nilpotent. However, this means that $K(X,Y) = Tr(\operatorname{ad}(X) \circ \operatorname{ad}(Y)) = 0$ for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$.
(b) This follows from part (a) and Lemma 3.5.
(c) Since $\mathfrak{h}$ is nilpotent by definition, $\operatorname{ad}(\mathfrak{h})$ is a solvable Lie algebra. By Lemma 3.4, $B_{\mathfrak{g}}(X,Y) = 0$ for all $X \in \operatorname{ad}(\mathfrak{h}), Y \in [\operatorname{ad}(\mathfrak{h}), \operatorname{ad}(\mathfrak{h})]$. Hence $K_{|\mathfrak{h}\times[\mathfrak{h},\mathfrak{h}]} = 0$. However, since $K_{|\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate by part (b), $[\mathfrak{h},\mathfrak{h}] = 0$ i.e. $\mathfrak{h}$ is abelian.
(d) Consider $X \in \mathfrak{h}$. By Theorem 6.2, we can decompose X into X_s, X_n such that $\operatorname{ad}(X_s) = \operatorname{ad}(X)_s$ is diagonalizable, $\operatorname{ad}(X_n) = \operatorname{ad}(X)_n$ is nilpotent. For all $Y \in \mathfrak{h}$, $[\operatorname{ad}(X), \operatorname{ad}(Y)] = \operatorname{ad}([X,Y]) = 0$ since $\mathfrak{h}$ is abelian. Recall a fact in linear algebra that: If $A \in \operatorname{End}(V)$ where V is a finite dimensional vector space, and B commutes with A, then B commutes with A_s, A_n, where $A = A_s + A_n$ is the Jordan decomposition. Apply this fact, we obtain $\operatorname{ad}([X_s,Y]) = [\operatorname{ad}(X)_s, \operatorname{ad}(Y)] = 0$ for all $Y \in \mathfrak{h}$. Because $\mathfrak{g}$ is semisimple, it has no nontrivial ideals, then $[X_s, Y] = 0$ for all $Y \in \mathfrak{h}$. Since $\mathfrak{h}$ is self-normalizing, as it is a Cartan subalgebra, X_s is in $\mathfrak{h}$. We will prove that $X_n = 0$. Note that $X_n = X - X_s \in \mathfrak{h}$. By Lie's theorem, since $\operatorname{ad}(\mathfrak{h})$ is solvable, there exists a basis in which every element of $\operatorname{ad}(\mathfrak{h})$ is in upper triangular form. In particular, since $\operatorname{ad}(X_n) = \operatorname{ad}(X)_n$ is nilpotent, it is in strictly upper triangular form. Hence $K(X_n, Y) = Tr(\operatorname{ad}(X_n) \circ \operatorname{ad}(Y)) = 0$ for all $Y \in \mathfrak{h}$. Since $K_{|\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, we obtain $X_n = 0$. Hence $X = X_S$ is diagonalizable, or $\operatorname{ad}(X) = \operatorname{ad}(X_s) = \operatorname{ad}(X)_s$ is a diagonalizable action on $\mathfrak{g}$.

□

Corollary 6.4. *It follows from (b) that $dim(\mathfrak{g}_\alpha) = dim(\mathfrak{g}_{-\alpha})$.*

Now, we will introduce a bilinear form on $\mathfrak{h}^*$.

Definition 6.5. We define a linear map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that $\nu(X)(Y) = K(X,Y)$ for all $X, Y \in \mathfrak{h}$. We define a symmetric bilinear form $\bar{K}$ on $\mathfrak{h}^*$ satisfying $\bar{K}(\alpha,\beta) = K(\nu^{-1}(\alpha), \nu^{-1}(\beta))$.

Remark 6.6. $\bar{K}$ is well-defined since $\mathfrak{h}$ is finite-dimensional and ν is bijective (since K is non-degenerate).

Recall Corollary 2.29 and Theorem 6.3(d), we obtain $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | H \in \mathfrak{h}, \operatorname{ad}(H)(X) = \lambda(H) \cdot X\ \forall\}$. We call $\lambda \in \mathfrak{h}^*$ a root of the Lie algebra of $\mathfrak{g}$ if $\lambda \not\equiv 0$ and $\mathfrak{g}_\lambda$ is non-trivial. We denote $\mathbf{R}$ the set of such λ, and we will call $\mathbf{R}$ the root system of $\mathfrak{g}$. Then, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \mathbf{R}} \mathfrak{g}_\lambda$

Example 6.7. (a) The roots of $\mathfrak{gl}_n(\mathbb{C})$ are $\{\lambda_i - \lambda_j | i \neq j\}$ where $\lambda_i(H_j) = \delta_{ij}$.
(b) The roots of $\mathfrak{sp}_{2n}(\mathbb{C})$ are $\{\pm\lambda_i \pm \lambda_j | i \neq j\} \cup \{\pm 2\lambda_i\}$ where $\lambda_i(H_j) = \delta_{ij}$.
(c) The roots of $\mathfrak{so}_{2n}(\mathbb{C})$ are $\{\pm\lambda_i \pm \lambda_j | i \neq j\}$ where $\lambda_i(H_j) = \delta_{ij}$.
(d) The roots of $\mathfrak{so}_{2n+1}(\mathbb{C})$ are $\{\pm\lambda_i \pm \lambda_j |\} \cup \{\pm\lambda_i\}$ where $\lambda_i(H_j) = \delta_{ij}$.

Lemma 6.8. *(a) If $\alpha \in \mathbf{R}, X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$, then $[X, Y] = K(X, Y)\nu^{-1}(\alpha)$.*
(b) $\bar{K}(\alpha, \alpha) \neq 0$ if $\alpha \in \mathbf{R}$.

A detailed proof can be found in section 8.3 in [3].

The next theorem is the second main result of this section. We will see from this theorem some specific structures of the semisimple Lie algebras, namely the dimension of a weight space, the string property of the root system, and so on.

Theorem 6.9. *For $\alpha, \beta \in \mathbf{R}$:*

(a) $dim(\mathfrak{g}_\alpha) = 1$.
(b) $\{\beta + n\alpha | n \in \mathbb{Z}\} \cap (\mathbf{R} \cup \{0\})$ is a finite connected string $\{\beta - p\alpha, \beta - (p - 1)\alpha, ..., \beta, ..., \beta + q\alpha\}$ where $q, p \in \mathbb{Z}_{\geq 0}\}$ and $p - q = \frac{2\bar{K}(\alpha,\beta)}{\bar{K}(\alpha,\alpha)}$.
(c) If $\alpha + \beta \in \mathbf{R}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
(d) $n\alpha \in \mathbf{R}$ if and only if $n = \pm 1$.

A detailed proof of this theorem can be found in section 8.4 in [3]

Remark 6.10. For $\mathfrak{sl}_2(\mathbb{C})$, we obtain the Cartan subalgebra $\mathbb{C}H$ that acts diagonally on $\mathfrak{g}$. We knows from the decomposition $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F$ that $\mathbb{C}X = \mathfrak{sl}_2(\mathbb{C})_\lambda$ and $\mathbb{C}Y = \mathfrak{sl}_2(\mathbb{C})_{-\lambda}$ where $\lambda \equiv 2$. We can see that this satisfies the above theorem.

The following theorem will gives us more details on the form $\bar{K}$, especially the fact that $\bar{K}$ provides the real span of $\mathbf{R}$ with an Euclidean structure.

Theorem 6.11. *(a) $\bar{K}(\alpha, \beta) = \sum_{\lambda \in \mathbf{R}} \bar{K}(\alpha, \lambda)\bar{K}(\beta, \lambda)\ \forall \alpha, \beta \in \mathfrak{h}^*$*
(b) $\bar{K}(\alpha, \beta) \in \mathbb{Q}\ \forall \alpha, \beta \in \mathbf{R}$
(c) Let $\mathbb{E} \subset \mathfrak{h}^*$ be the span of the roots in $\mathbf{R}$ over $\mathbb{R}$. Then, $\bar{K}_{|\mathbb{E}\times\mathbb{E}}$ is positive-definite symmetric bilinear form.

Proof. (a) Choose a basis $H_1, H_2, ...H_k$ of $\mathfrak{h}$ and elements $X_\lambda \in \mathfrak{g}_\lambda$ such that $K(H_i, H_i) = K(X_\lambda, X_\lambda) = 1$ for all $i = 1, 2...k$ and $\lambda \in \mathbf{R}$. For $X, Y \in \mathfrak{h}$, $X = \sum_{i=1}^{k} x^i H_i$, $Y = \sum_{i=1}^{j} y^i H_i$. Note that, $\text{ad}(X) \circ \text{ad}(Y)(H_i) = 0$ for all $i = 1, 2...k$. $\text{ad}(X) \circ \text{ad}(Y)(X_\lambda) = \lambda(X)\lambda(Y)$. Therefore, $K(X, Y) = tr(\text{ad}(X) \circ \text{ad}(Y)) = \sum_{\lambda \in \mathbf{R}} \lambda(X)\lambda(Y)$. Hence $\bar{K}(\alpha, \beta) = K(\nu^{-1}(\alpha), \nu^{-1}(\beta)) = \sum_{\lambda \in \mathbf{R}} = \lambda(\nu^{-1}(\alpha))\lambda(\nu^{-1}(\beta)) = \sum_{\lambda \in \mathbf{R}} \bar{K}(\alpha, \lambda)\bar{K}(\beta, \lambda)$, which proves (a).
(b) By part (a), $\bar{K}(\alpha, \alpha) = \sum_{\lambda \in \mathbf{R}} (\bar{K}(\alpha, \lambda))^2$. This means $\frac{4}{\bar{K}(\alpha,\alpha)} = \sum_{\lambda \in \mathbf{R}} (\frac{2\bar{K}(\alpha,\lambda)}{\bar{K}(\alpha,\alpha)})^2 \in \mathbb{N}$. Hence $\bar{K}(\alpha, \beta) = \frac{\bar{K}(\alpha,\alpha)}{2} \frac{2\bar{K}(\alpha,\beta)}{\bar{K}(\alpha,\alpha)} \in \mathbb{Q}$.

(c) Note that $\bar{K}(\lambda, \lambda) \geq 0$ for all $\lambda \in \mathbb{E}$ by part (a). Therefore, $\bar{K}$ is a positive, semi-definite, and non-degenerate form on $\mathbb{E}$, which means that $\bar{K}$ is a positive definite form.

□

Example 6.12. Using Theorem 6.11(a), we obtain the following equalities.

(a) For $\mathfrak{sl}_n(\mathbb{C})$, $\bar{K}(\sum_{i=1}^{n} a^i\lambda_i, \sum_{i=1}^{n} b^i\lambda_i) = \frac{1}{2n}(\sum_{i=1}^{n} a^ib^i - \frac{1}{n}\sum_{i,j} a^ib^j)$.
(b) For $\mathfrak{sp}_{2n}(\mathbb{C})$, $\bar{K}(\sum_{i=1}^{n} a^i\lambda_i, \sum_{i=1}^{n} b^i\lambda_i) = \frac{1}{4n+4}\sum_{i=1}^{n} a^ib^i$.
(c) For $\mathfrak{so}_{2n}(\mathbb{C})$, $\bar{K}(\sum_{i=1}^{n} a^i\lambda_i, \sum_{i=1}^{n} b^i\lambda_i) = \frac{1}{4n-4}\sum_{i=1}^{n} a^ib^i$.
(d) For $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\bar{K}(\sum_{i=1}^{n} a^i\lambda_i, \sum_{i=1}^{n} b^i\lambda_i) = \frac{1}{4n}\sum_{i=1}^{n} a^ib^i$.

With the structure of an Euclidean space, the following corollary to the previous theorem will restrict the structure of the root system that sits inside this Euclidean space.

Corollary 6.13. *The space $\mathbb{E}$ is an Euclidean space with the dot product $\bar{K}$. Moreover, for $\alpha, \beta \in \mathbf{R}$, $0 \leq \frac{4 \cdot \bar{K}(\alpha,\beta)^2}{\bar{K}(\alpha,\alpha)\bar{K}(\beta,\beta)} \leq 4$ and $\frac{4 \cdot \bar{K}(\alpha,\beta)^2}{\bar{K}(\alpha,\alpha)\bar{K}(\beta,\beta)} \in \mathbb{Z}$. Hence, $\cos(\theta) = \frac{\cdot \bar{K}(\alpha,\beta)}{\sqrt{\bar{K}(\alpha,\alpha)\bar{K}(\beta,\beta)}} = 0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1$, where θ is the angle between α, β in the Euclidean space $\mathbb{E}$. Continuing from this, we arrive at the following conditions between roots:*

$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\frac{\|\|\beta\|\|}{\|\|\alpha\|\|}$	1	$\sqrt{3}$	$\sqrt{2}$	1	$*$	1	$\sqrt{2}$	$\sqrt{3}$	1

where $||\gamma|| = \sqrt{\bar{K}(\gamma, \gamma)}$ for $\gamma \in \mathbf{R}$.

Looking at Example 6.7, we can see the roots of semisimple algebras carrying some redundant information. For example, if $\alpha \in \mathfrak{h}^*$ is a root, then $-\alpha$ is also a root. Therefore, in order to understand Lie algebras and its roots, we can look at only 'simple' roots, which will be defined in the next part.

Definition 6.14. Choose a linear functional $l : \mathbb{E} \to \mathbb{R}$ such that $l(\lambda)$ is irrational for all $\lambda \in \mathbf{R}$. We denote $\mathbf{R}^+ = \mathbf{R} \cap \{\lambda \in \mathfrak{h}^* | l(\lambda) > 0\}$, and we call this the positive roots with respect to l. We call a root simple if it cannot be written as the sum of two positive roots.

Remark 6.15. l is not unique. We only need l to define the subset $R^+ \subset R$ such that:

(a) $\forall\ \lambda \in \mathbf{R}$, either $\lambda \in \mathbf{R}$ or $-\lambda \in \mathbf{R}$.
(b) If $\alpha, \beta \in \mathbf{R}^+$ and $\alpha + \beta \in \mathbf{R}$, then $\alpha + \beta \in \mathbf{R}^+$.

The following examples look at conventional choices of positive roots and simple roots of classical Lie algebras and how the form $\bar{K}$ interacts with them. This will provide us with an useful tool in the next chapter.

Example 6.16. (a) For $\mathfrak{sl}_n(\mathbb{C})$, we obtain $\mathbb{E} = \{\sum_{i=1}^{n} a^i\lambda_i | a^i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a^i = 0\}$. Thus we can choose $l : \mathbb{E} \to \mathbb{R}$ such that $l(\sum_{i=1}^{n} a^i\lambda_i) = \sum_{i=1}^{n} c_i a^i$ where $c_i \in \mathbb{R}$, $\sum_{i=1}^{n} c_i = 0$ and $c_i \geq c_{i+1}$ $\forall i = 1, 2...n-1$. Hence we obtain $\mathbf{R}^+ = \{\lambda_i - \lambda_j | i < j\}$. The set of simple roots of $\mathfrak{sl}_n(\mathbb{C})$ is $\{\lambda_i - \lambda_{i+1} | i = 1, 2, ...n-1\}$.
Doing similar processes, we obtain:
(b) For $\mathfrak{sp}_{2n}(\mathbb{C})$, $\mathbf{R}^+ = \{\lambda_i + \lambda_j | i \leq j\} \cup \{\lambda_i - \lambda_j | i < j\}$. The set of simple roots of $\mathfrak{sp}_{2n}$ is $\{\lambda_i - \lambda_{i+1} | i = 1, 2..., n-1\} \cup \{2\lambda_n\}$.
(c) For $\mathfrak{so}_{2n}(\mathbb{C})$, $\mathbf{R}^+ = \{\lambda_i + \lambda_j | i < j\} \cup \{\lambda_i - \lambda_j | i < j\}$. The set of simple roots of $\mathfrak{sl}_{2n}$ is $\{\lambda_i - \lambda_{i+1} | i = 1, 2..., n-1\} \cup \{\lambda_{n-1} + \lambda_n\}$.
(d) For $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathbf{R}^+ = \{\lambda_i + \lambda_j | i < j\} \cup \{\lambda_i - \lambda_j | i < j\} \cup \{\lambda_i\}$. The set of simple roots of $\mathfrak{sp}_{2n+1}$ is $\{\lambda_i - \lambda_{i+1} | i = 1, 2..., n-1\} \cup \{\lambda_n\}$.

Example 6.17. (a) For $\mathfrak{sl}_n(\mathbb{C})$, $\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_j - \lambda_{j+1}) = \begin{cases} \frac{1}{n} & if\ i = j \\ -\frac{1}{2n} & if\ |i-j| = 1 \\ 0 & if\ |i-j| \geq 2 \end{cases}$

(b) For $\mathfrak{sp}_{2n}(\mathbb{C})$, $\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_j - \lambda_{j+1}) = \begin{cases} \frac{1}{2n+2} & if\ i = j \\ -\frac{1}{4n+4} & if\ |i-j| = 1 \\ 0 & if\ |i-j| \geq 2 \end{cases}$

$\bar{K}(\lambda_i - \lambda_{i+1}, 2\lambda_n) = \begin{cases} -\frac{1}{2n+2} & if\ i = n-1 \\ 0 & otherwise \end{cases}$ and $\bar{K}(2\lambda_n, 2\lambda_n) = \frac{1}{n+1}$

(c) For $\mathfrak{so}_{2n}(\mathbb{C})$, $\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_j - \lambda_{j+1}) = \begin{cases} \frac{1}{2n-2} & if\ i = j \\ -\frac{1}{4n-4} & if\ |i-j| = 1 \\ 0 & if\ |i-j| \geq 2 \end{cases}$

$\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_{n-1} + \lambda_n) = \begin{cases} -\frac{1}{4n-4} & if\ i = n-2 \\ 0 & otherwise \end{cases}$ and $\bar{K}(\lambda_{n-1} + \lambda_n, \lambda_{n-1} + \lambda_n) = \frac{1}{2n-2}$

(d) For $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_j - \lambda_{j+1}) = \begin{cases} \frac{1}{2n} & if\ i = j \\ -\frac{1}{4n} & if\ |i-j| = 1 \\ 0 & if\ |i-j| \geq 2 \end{cases}$

$\bar{K}(\lambda_i - \lambda_{i+1}, \lambda_n) = \begin{cases} -\frac{1}{2n} & if\ i = n-1 \\ 0 & otherwise \end{cases}$ and $\bar{K}(\lambda_n, \lambda_n) = \frac{1}{4n}$

At this point, one may ask what information do simple roots carry? Are they enough to reconstruct the whole root system? We already have Theorem 6.9 to help us construct the roots from its simple roots. However, in this form, the process is proven to be diffcult. Hence, we will define the following symmetry that will provide us with more structure of the roots sitting in $\mathbb{E}$ and help us with reconstructing the root system from simple roots.

Definition 6.18. Let Λ_R be the span $\mathbf{R}$ over $\mathbb{Z}$. For $\alpha \in \mathbf{R}$, we define $W_\alpha : \Lambda_R \to \Lambda_R$ such that $W_\alpha(\beta) = \beta - \frac{2 \cdot \bar{K}(\alpha,\beta)}{\bar{K}(\alpha,\alpha)}\alpha$.

It is clear that W_α is just a reflection about the axis α.

Theorem 6.19. $\mathbf{R}$ *is invariant under* W_α $\forall \alpha \in \mathbf{R}$.

A detailed proof can be found in section 9.2 in [3].

Definition 6.20. We will call the group $\mathfrak{M}$ generated by W_α for $\alpha \in \mathbf{R}$ the Weyl group of the root system.

Theorem 6.19 shows us that the root system is invariant under the Weyl group.

7. Classifying simple complex Lie algebras

In this final section, we will move to prove the big result of the whole paper: Except the classical Lie algebras as in example 2.4, there are only 5 other complex simple Lie algebras. As we see in Theorem 3.6, with this classification of simple complex Lie algebras, we have a better understanding of semisimple complex Lie algebras. However, as much as we want to, the construction of Lie algebras is too technical for this paper. Therefore, we will prove an important result that leads to it: There are only 5 possible root systems of simple complex Lie algebras other than those of classical Lie algebras. In doing so, first, we summarize the results we obtain from previous sections on how to reconstruct the root system from the simple roots. Then, we will try to figure out which configuration of the simple roots that is possible for a simple complex Lie algebra by introducing the notion of a Coxeter graph and a Dynkin graph. Finally, we will prove the big theorem of this paper.
In this section, we will the notation $(\alpha, \beta) := \bar{K}(\alpha, \beta)$ for all $\alpha, \beta \in \mathfrak{h}^*$.
We will summarize the results we have obtain in the previous sections:

(1) $\forall \alpha \in \mathbf{R}, n\alpha \in \mathbf{R}$ if and only if $n = \pm 1$.
(2) W_α preserves $\mathbf{R}$ $\forall \alpha \in \mathbf{R}$.
(3) $\frac{2\cdot(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ $\forall \alpha, \beta \in \mathbf{R}$.
(4) If $\alpha, \beta \in \mathbf{R}$ and $\alpha \neq \pm\beta$, then $\{\beta + n\alpha | n \in \mathbb{Z}\} \cap (\mathbf{R} \cup \{0\})$ is a finite connected string $\{\beta - p\alpha, \beta - (p-1)\alpha, ..., \beta, ..., \beta + q\alpha\}$ where $q, p \in \mathbb{Z}_{\geq 0}\}$ and $p - q = \frac{2\bar{K}(\alpha,\beta)}{\bar{K}(\alpha,\alpha)}$.

Remark 7.1. (a) Given $\alpha, \beta \in \mathbf{R}$, $\alpha \neq \pm\beta$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \mathbf{R}$. This follows directly from the fact that $p - q = \frac{2\cdot(\alpha,\beta)}{(\alpha,\alpha)} < 0$, which implies $q \geq 1$ and $\alpha + \beta \in \mathbf{R}$.
(b) If $\alpha, \beta \in \mathbf{R}$ are distinct simple roots, then $(\alpha, \beta) \leq 0$. This is because if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \mathbf{R}$ by part (a). Without loss of generality, we assume $\alpha - \beta \in \mathbf{R}^+$. This means that α can be written as $\beta + (\alpha - \beta)$, which contradicts the fact that α is a simple root.
(c) Simple roots are linear independent. Otherwise, there exists $c^i \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^{s} c^i\alpha_i = \sum_{j=s+1}^{k} c^j\alpha_j$. Note that both sides are non-empty since α_i are all positive roots. Hence $0 < (\sum_{i=1}^{s} c^i\alpha_i, \sum_{i=1}^{s} c^i\alpha_i) = (\sum_{i=1}^{s} c^i\alpha_i, \sum_{j=s+1}^{k} c^j\alpha_j) = \sum_{i=1}^{s}\sum_{j=s+1}^{k}(\alpha_i, \alpha_j) \leq 0$.
(d) There are precisely n simple roots where $n = \dim \mathfrak{g}$. This follows directly from part (c) and the fact that the simple roots span $\mathbb{E}$.

Definition 7.2. $\mathbf{R}$ is called a root system if it is finite and it satisfies (1), (2), and (3).

Definition 7.3. A root system $\mathbf{R}$ in $\mathbb{E}$ is reducible if it can be decomposed into root systems $\mathbf{R}_1, \mathbf{R}_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \mathbf{R}_1, \beta \in \mathbf{R}_2$. Otherwise, it is an irreducible root system.

Now we will represent the simple roots of $\mathfrak{g}$ with the Dynkin diagram. For each simple root, we will represent it with a node. For two simple roots α, β, we will connect them with lines depending on the angle θ between them, and if there is at least a line between them, we will add an arrow on the line pointing towards the shorter edge.

$$\theta = \frac{\pi}{2} : \qquad \underset{\alpha}{\circ} \quad \underset{\beta}{\circ}$$

$$\theta = \frac{2\pi}{3} : \qquad \underset{\alpha}{\circ} - \underset{\beta}{\circ}$$

$$\theta = \frac{3\pi}{4} : \qquad \underset{\alpha}{\circ} = \underset{\beta}{\circ}$$

$$\theta = \frac{5\pi}{6} : \qquad \underset{\alpha}{\circ} \equiv \underset{\beta}{\circ}$$

Example 7.4. From example 4.17, we obtain the followings Dynkin diagrams:

$$(A_n) \qquad \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} - \underset{\alpha_n}{\circ}$$

$$(B_n) \qquad \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} \Rightarrow \underset{\alpha_n}{\circ}$$

$$(C_n) \qquad \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} \Leftarrow \underset{\alpha_n}{\circ}$$

$$(D_n) \qquad \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-2}}{\overset{\overset{\circ\,\alpha_n}{|}}{\circ}} - \underset{\alpha_{n-1}}{\circ}$$

where (A_n) is Dynkin diagram of $\mathfrak{sl}_{n+1}(\mathbb{C})$ (where $n \geq 1$), (B_n) is the Dynkin diagram of $\mathfrak{so}_{2n+1}(\mathbb{C})$ (where $n \geq 2$), (C_n) is the Dynkin diagram of $\mathfrak{sp}_{2n}(\mathbb{C})$ (where $n \geq 3$), and (D_n) is the Dynkin diagram of $\mathfrak{so}_{2n}(\mathbb{C})$ (where $n \geq 4$).

Remark 7.5. (a) For $\mathfrak{sl}_n(\mathbb{C})$, $\alpha_i = \lambda_i - \lambda_{i+1}$ $\forall i = 1, 2, ..., n$.
(b) For $\mathfrak{so}_{2n+1}$, $\alpha_i = \lambda_i - \lambda_{i+1}$ $\forall i = 1, 2, ..., n-1$ and $\alpha_n = \lambda_n$.
(c) For $\mathfrak{sp}_{2n}$, $\alpha_i = \lambda_i - \lambda_{i+1}$ $\forall i = 1, 2, ..., n-1$ and $\alpha_n = 2\lambda_n$.
(d) For $\mathfrak{so}_{2n}$, $\alpha_i = \lambda_i - \lambda_{i+1}$ $\forall i = 1, 2, ..., n-1$ and $\alpha_n = \lambda_{n-1} + \lambda_n$.

Lemma 7.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then, $\mathfrak{g}$ is a simple Lie algebra if and only if its root system is irreducible.*

Proof. (necessity) Assume that the root system $\mathbf{R}$ of $\mathfrak{g}$ is reducible i.e. $\mathbf{R} = \mathbf{R}_1 \cup \mathbf{R}_2$ such that $\mathbf{R}_1$ and $\mathbf{R}_2$ are orthogonal. Then, choose $\alpha \in \mathbf{R}_1$, and $\beta \in \mathbf{R}_2$. Since $(\alpha + \beta, \alpha), (\alpha + \beta, \beta) \neq 0$, $\alpha + \beta$ is not a root. This means $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. Let $\mathfrak{k}$ be the Lie algebra generated by $\mathfrak{g}_\alpha$ for all $\alpha \in \mathbf{R}_1$ and $\mathfrak{t}$ be defined similarly for $\mathbf{R}_2$. It follows that if there exists $Y \in \mathfrak{g}_\beta$ for some $\beta \in \mathbf{R}_2$, then $[Y, X] = 0$ $\forall X \in \mathfrak{k}$, which means $Z(\mathfrak{k}) \neq 0$. Hence, $\mathfrak{k} \subsetneq \mathfrak{g}$. Moreover, if $X \in \mathfrak{k}$, then for all $Y \in \mathfrak{g}$, there exists $Z \in \mathfrak{k}, T \in \mathfrak{t}$ such that $Y = Z + T$. This implies $[X, Y] = [X, Z] \in \mathfrak{k}$. Hence, $\mathfrak{k}$ is a proper ideal of $\mathfrak{g}$, which means $\mathfrak{g}$ is not simple (contradiction).
(sufficiency) Assume that there exists a proper nonzero ideal of $\mathfrak{g}$. Let $\mathfrak{i}$ be a

maximal proper nonzero ideal of $\mathfrak{g}$. Since we have the decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbf{R}} \mathfrak{g}_\alpha$$

we can also decompose $\mathfrak{i}$ as:

$$\mathfrak{i} = \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in \mathbf{R}_1} \mathfrak{g}_\alpha$$

where $\mathfrak{h}_1$ is a vector subspace of $\mathfrak{h}$. Let $\mathfrak{k} = \{X \in \mathfrak{g} | K(X, Y) = 0 \, \forall Y \in \mathfrak{i}\}$. Moreover, for $Y \in \mathfrak{k}$, $Z \in \mathfrak{g}$:

$$K(X, [Z, Y]) = K([X, Z], Y) = 0$$

for all $X \in \mathfrak{i}$, since $\mathfrak{i}$ is an ideal. Therefore, $[Z, Y]$ is in $\mathfrak{k}$ for all $Z \in \mathfrak{g}$ and $Y \in \mathfrak{k}$ i.e. $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Therefore, we have a similar decomposition as before:

$$\mathfrak{k} = \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \mathbf{R}_2} \mathfrak{g}_\alpha$$

Let $\mathfrak{l} = \mathfrak{i} + \mathfrak{k} = \{X + Y | X \in \mathfrak{i}, Y \in \mathfrak{k}\}$. Note that $\mathfrak{l}$ is an ideal because $[X + Y, Z] = [X, Z] + [Y, Z] \in \mathfrak{l}$ for all $X \in \mathfrak{i}, Y \in \mathfrak{k}, Z \in \mathfrak{g}$. Then, either $\mathfrak{k} \subset \mathfrak{i}$ or $\mathfrak{k} + \mathfrak{i} = \mathfrak{g}$ since $\mathfrak{i}$ is a maximal proper nonzero ideal.

Let $\mathfrak{j}$ be $\mathfrak{k} \cap \mathfrak{i}$. Note that $\mathfrak{j}$ is an ideal. Let $K_\mathfrak{j}$ be the Killing form of the Lie algebra $\mathfrak{j}$. Then $K_\mathfrak{j}(\mathfrak{j}, [\mathfrak{j}, \mathfrak{j}]) = K(\mathfrak{j}, [\mathfrak{j}, \mathfrak{j}]) = 0$ by definition of $\mathfrak{j}$. This means that $\mathfrak{j}$ is a solvable ideal of $\mathfrak{g}$ by Cartan's Criterion, which implies that $\mathfrak{j} = \{0\}$ since $\mathfrak{g}$ is semisimple.

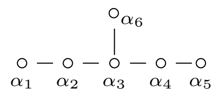
Recall a result from linear algebra: Given a finite-dimensional complex vector space V and a symmetric bilinear form f on V. Then, for all subspace W of V, $\dim\ W + \dim\ W^\perp \geq \dim\ V$, where $\dim\ W^\perp = \{y \in V | f(x, y) = 0 \ \forall x \in W\}$.

If $\mathfrak{k} \subset \mathfrak{i}$, then $\mathfrak{k} = \mathfrak{j} = \{0\}$, which is a contradiction since $\dim\ \mathfrak{i} + \dim\ \mathfrak{k} \geq \dim\ \mathfrak{g}$, by theorem above.

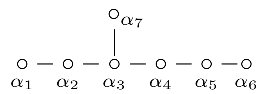
In the other case, $\mathfrak{k} \oplus \mathfrak{i} = \mathfrak{g}$ since $\mathfrak{k} \cap \mathfrak{i} = \{0\}$. This makes $\mathbf{R}_1 \cap \mathbf{R}_2 = \emptyset$, $\mathbf{R}_1 \cup \mathbf{R}_2 = \mathbf{R}$, and $\mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{h}$. Note that $\mathbf{R}_1$ and $\mathbf{R}_2$ are non-empty. Moreover, for all $\alpha \in \mathbf{R}_1$ and $\beta \in \mathbf{R}_2$, $(\alpha, \beta) = 0$. Indeed, choose non-zero $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $Z \in \mathfrak{g}_{-\beta}$ such that $K(Y, Z) = \frac{2}{K(\beta, \beta)}$, and $H = [Y, Z]$. Then $(\alpha, \beta) = \alpha(\nu^{-1}(\beta)) = \alpha(\frac{[Y,Z]}{K(Y,Z)}) = \frac{2}{K(\beta,\beta)}\alpha(H)$. However, $[H, X] = \alpha(H)X \in \mathfrak{g}_\alpha$. Moreover, since $Y \in \mathfrak{g}_\beta \subset \mathfrak{k}$, $H = [Y, Z] \in \mathfrak{k}$ as $\mathfrak{k}$ is an ideal, which makes $[H, X] \in \mathfrak{k}$. Hence $[H, X] \in \mathfrak{g}_\alpha \cap \mathfrak{k} \subset \mathfrak{i} \cap \mathfrak{k} = \{0\}$. Therefore, $\alpha(H) = 0$ i.e. $(\alpha, \beta) = 0$ for all $\alpha \in \mathfrak{i}$, $\beta \in \mathfrak{k}$. Therefore, the root system $\mathbf{R}$ of $\mathfrak{g}$ is reducible, which is a contradiction. □

Theorem 7.7. *Except for the above Dynkin diagrams, there are only 5 other diagrams of irreducible root systems:*

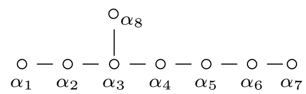
(E_6)



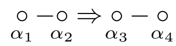
(E_7)



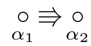
(E_8)



(F_4)

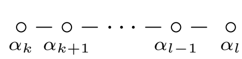


(G_2)



Proof. We first note that the Dynkin diagram of an irreducible root system is connected. We call a Dynkin diagram without the arrows a Coxeter diagram. We will prove that the above diagrams (without arrows) are the only Coxeter diagrams of simple complex Lie algebras. We call a graph of n nodes admissible if there exists n linear independent unit vectors spanning a Euclidean space $\mathbb{E}$ such that the angle between any two of them is $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ according to the number of lines connecting them in the Coxeter graph. Note that $(\alpha_i, \alpha_j) = 0, \frac{-1}{2}, \frac{-\sqrt{2}}{2}, \frac{-\sqrt{3}}{2}$, corresponding to the angles $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$. Hence $4(\alpha_i, \alpha_j)^2 =$ the number of lines between them. We have the following remarks:

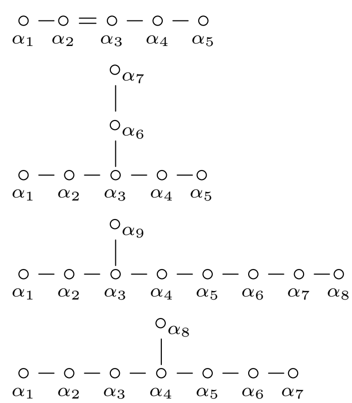
- A subdiagram of an admissible graph is an admissible graph.
- Moreover, since α_i are linear independent, $0 < (\sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \alpha_i) = n + 2\sum_{i<j}(\alpha_i, \alpha_j)$. However, if there is at least a line between the pair α_i, α_j, $(e_i, e_j) \le \frac{-1}{2}$; hence, the number of pairs of nodes that is connected by at least one line is strictly less than n, or, equivalently, less than or equal to $n - 1$ (1).
- If the graph has a loop, then its loop is an admissible subdiagram having at least as many lines as nodes, contradicting the above argument. Hence, an admissible diagram has no loop (2).
- We will show that each node has at most 3 lines connected to it (3): Without loss of generalization, we will prove the result for α_1. By (1), we can assume that every node is connected to α_1 by choosing a subdiagram of it and its neighbors. With this assumption, any two of the other nodes are not connected as there are no loops by (2). Since α_1 is not in the span of $\alpha_2, ...\alpha_n$ by linear independency, $1 = (\alpha_1, \alpha_1)^2 > (\alpha_1, P(\alpha_1))^2 = \sum_{k=2}^{n}(\alpha_1, \alpha_k)^2$ (by pairwise orthogonality of vectors other than α_1) where $P(\alpha_1)$ is the projection of α_1 onto the span of $\alpha_2, ..., \alpha_n$. Therefore, $4 > \sum_{k=2}^{n} 4(\alpha_1, \alpha_k)^2 =$ the number of lines connected to α_1.
- In an admissible graph, a string of nodes that is connected in the following way:



, where each α_j is connected only to α_{j-1} and α_{j+1} $\forall j = k+1, ...l-1$, can be collapsed into one node such that the resulting diagram is still admissible.

This is because if we replace the string by a single node $\beta = \sum_{s=k}^{l} \alpha_s$, β still satisfies $(\beta,\beta) = (\sum_{s=k}^{l} \alpha_s, \sum_{s=k}^{l} \alpha_s) = (l-s+1) + 2\sum_{s=k}^{l-1} (\alpha_s, \alpha_{s+1}) = (l-s+1) - (l-s) = 1$. Moreover, if γ is connected to α_s, then γ is not connected to $\alpha_{s+1}, ...\alpha_l$ by (2); hence $(\gamma, \beta) = (\gamma, \alpha_s)$. We have the similar result for γ connected to α_l. Hence, the resulting diagram is admissible.

Now, we can classify all connected admissible diagrams. The only connected diagrams with 2 nodes connected by 3 lines is (G_2). By (1), (3), (4), an admissible diagram cannot possess 2 double lines, 2 triple nodes, or 1 triple node and 1 double line. Finally, we will rule out the inadmissible cases (see Chapter 21 in [1] for proof):



After ruling out these cases, the only diagrams left are $(A_n), (B_n), (C_n), (D_n), (E_6)$, $(E_7), (E_8), (F_4)$, and (G_2). To complete this, we will need to show explicitly the root systems of $(A_n), (B_n), (C_n)$, $(D_n), (E_6), (E_7), (E_8), (F_4)$, and (G_2). We have already shown the root system of $(A_n), (B_n), (C_n), (D_n)$ in Example 6.7. The rest is shown in Chapter 21 in [1]. □

Remark 7.8. In order to make the claim that there are only 5 exceptional Lie algebras, we still need to show that each of these root systems yields a unique simple complex Lie algebra, which will not be shown here. See Chapter 21 in [1] for more information on the uniqueness theorem and Chapter 22 for the construction of the Lie algebra $\mathfrak{g}_2$ derived from the Dynkin diagram (G_2).

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9. Bibliography

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