ATIYAH–BOTT LOCALIZATION IN EQUIVARIANT COHOMOLOGY

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Abstract. In this paper, we explore the localization theorem in equivariant cohomology due to Atiyah and Bott. We introduce the concept of Borel equivariant cohomology, a generalization of cohomology for spaces with group actions. We then prove the localization theorem, which relates the equivariant cohomology of the space and the fixed points. Finally, we present examples of its applications.

Contents

1. Introduction

One of the ubiquitous tasks in mathematics is to compute integrals over spaces, that is to ask questions of the form

$$\int_X \phi = ?$$

for a space X. Depending on what ϕ means in different contexts, this integral can represent different concepts. To get the volume of the space, we integrate the volume form; to get topological invariants or numerical answers to enumerative problems, we may integrate certain characteristic classes; to compute some partition functions of physical systems (in particular, path integrals in field theories) we try to integrate over the states with a phase weight given by a prescribed action functional.

Such computation can be difficult, and there are many different strategies to tackle such a problem. There are two ideas that are particularly fruitful; namely, **symmetry** and **localization**.

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By symmetry, we mean we have a group G acting on our space X. The points of X in the same G-orbits are identified by the symmetry G, and we reduce ourselves to a smaller space X/G which is likely easier to work in. An elementary example can already be found in the theory of ODE's, where one seeks conserved quantities attached to the symmetry (the first integrals) to reduce the dimension of the dynamical system. The same idea generalizes to the theory of symplectic reductions, mathematical gauge theory and integrable systems.

Localization is another idea that is immensely useful. Informally, localization refers to the idea that we should try to represent global values as a combination of local contributions. An example of this idea would be the Poincaré–Hopf theorem in differential topology, which relates the Euler characteristic of a compact manifold X with the sum of indices at zeroes of a vector field on it:

$$\chi(X) = \int_X \mathbf{e}(TX) = \sum_{\mathbf{v}(p)=0} \mathrm{index}_{\mathbf{v}}(p).$$

Looking at a single point at a time is usually much easier than to deal with the whole global data. One may be familiar with the idea of localization in algebra, which is geometrically the same thing as looking at a single point (a prime) at a time. In all of such discussions the overarching idea is to understand a global object by understanding the local picture first and combining them in a coherent way.

These two ideas, symmetry and localization, intersect beautifully through the **Atiyah–Bott localization theorem**. The theorem states that in a situation where a smooth manifold X admits an action of a (compact connected Lie) group G then the integral on X localizes at the fixed points $i : F = X^G \hookrightarrow X$ of the group action. More precisely, if F is a set of isolated points p with associated normal bundles ν_p in X with Euler classes $\mathbf{e}(\nu_p)$, we have

$$\int_X \phi = \sum_{p \in F} \int_p \frac{i^*\phi}{\mathbf{e}(\nu_p)}.$$

(It must be noted that a formula of nearly the same flavor was obtained by Berline–Vergne in [4]. We focus on the Atiyah–Bott version because of its conceptual clarity and generality.)

The computational power of the integration formula is immense, and the idea has led to many applications in widely varying fields. The goal of this article is to provide an explanation of this theorem and a (necessarily inexhaustive) survey of its applications.

A considerable portion of the article will go in to setting up the necessary framework of **equivariant cohomology**, which is a generalization of ordinary cohomology to the setting where group actions are available. We will begin in Section 2 by defining equivariant cohomology and explain its basic properties. In Section 3 we provide a description of de Rham models for equivariant cohomology. The de Rham models allow us to interpret equivariant cohomology and its integration in terms of differential forms, which are much more familiar objects.

Once we are done with setting up the framework, we will prove the Atiyah–Bott localization theorem and the integration formula in Section 4. We will be following Atiyah–Bott's original paper [1]. In section 5, we conclude by explaining some selections of its applications in algebraic geometry and symplectic geometry.

We will assume some familiarity with algebraic topology and the theory of compact Lie groups. In particular, concepts such as ordinary and compactly supported cohomology, principal bundles, differential forms, the adjoint representation of a Lie group, maximal tori and Weyl groups are used without definition. All of these concepts are clearly explained in [6] for algebraic topology and in [8] for Lie theory. The final chapter on applications may introduce less familiar topics, and references will be provided.

2. Borel equivariant cohomology

In this section we introduce Borel equivariant cohomology and describe some of its basic properties. We first explain the Borel construction, also known as the homotopy orbit space. We then introduce the Borel equivariant cohomology as the ordinary cohomology of the Borel construction. Bott gave a more detailed version of the exposition to follow in his article [11] targeted towards a physics audience.

Let a compact Lie group G smoothly act on a smooth manifold X. We want our equivariant cohomology to serve as an invariant that remembers something about both the space and the group action. Our intuition is that the equivariant theory will consider the G-orbits as "equivariant points," and therefore the role of the points in the ordinary theory will be replaced by those of the G-orbits.

In this viewpoint, the first guess for the definition of equivariant cohomology would then be as the ordinary cohomology of the orbit space X/G. This is the right guess when G acts freely on X, but there are two severe limitations of this description in the general case.

The first reason not to use this definition is that the orbit space X/G itself does not remember about the stabilizers of the group action, which may vary along different orbits. Ideally we would want an object that knows about both the orbits and the stabilizers. The second and perhaps the more important reason is that equivariant homotopy equivalences $X \to Y$ do not necessarily induce homotopy equivalences $X/G \to Y/G$. For an example, consider $X = \mathbb{R}$, $Y = *$ with $\mathbb{Z}$-action on X by translation. So if we want to require the equivariant homotopy invariance axiom on our equivariant cohomology, we should not use this orbit space construction.

The fix to the idea comes from the Borel construction. Before we give the definition, we first introduce the necessarily preliminary; namely, the universal G-bundle.

Theorem 2.1. *For a topological group G, there exists a principal G-bundle $EG \to BG$ where the total space EG is contractible.*

Theorem 2.2. *Given a topological space X, there is a bijection between the homotopy classes of maps from X into BG and the isomorphism classes of principal G-bundles over X: that is, $[X, BG] \simeq \mathrm{Bun}_G(X)/\mathrm{iso}$.*

The proposed bijection is given by pulling back the universal bundle $EG \to BG$ along a chosen map $f : X \to BG$ to a bundle over X. The content is that this association is well-defined (that is, pulling back under homotopic maps give isomorphic bundles) and that it is indeed bijective. In this sense, the base space BG is called the **classifying space** and the "platonic" bundle $EG \to BG$ is called the **universal bundle**. The classifying space is well-defined up to homotopy type.

Example 2.3. Let $G = S^1$, the unit norm subgroup of the multiplicative group $\mathbb{C}^*$. Then the fiber bundle $S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$, obtained inductively from the fiber bundles $S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n$, is a principal S^1-bundle with a contractible total space S^∞. Therefore

$$ES^1 \simeq S^\infty \to BS^1 \simeq \mathbb{CP}^\infty$$

is a universal S^1-bundle.

For a more thorough treatment of the theory of classifying spaces see [7]. It in particular contains the proofs of Theorem 2.1 and 2.2. Now we are ready to define the Borel construction.

Definition 2.4. Let G be a topological group acting on a space X. The **Borel construction** or the **homotopy orbit space** of X by the action of G is $EG \times_G X$, the quotient of the product $EG \times X$ by the diagonal action: $(e, x) \sim (eg, g^{-1}x)$. We denote this space by X_{hG}. In particular, $*_{hG} = BG$.

In the Borel construction, we replace the naïve quotient of X by the action of G with the quotient of $EG \times X$ by the action of G. This space $EG \times X$ is homotopy equivalent to X, because EG is contractible. It has the advantage that G necessarily acts freely on it, since the G-action on EG is free. In particular, we can now return to our original guess for equivariant cohomology as the cohomology of the orbit space which worked fine for free actions, and make it into a definition. We simply replace the plain old orbit space with the homotopy orbit space.

Definition 2.5. The (Borel) equivariant cohomology of X for a G-action is defined as the singular cohomology of the homotopy quotient:

$$H^*_G(X) = H^*(X_{hG}).$$

The ring of coefficients can be chosen as we want. For convenience, we fix the coefficients to be $\mathbb{C}$. As stated, the definition depends on the choice of $EG \to BG$. However the homotopy type of X_{hG} is well-defined, so equivariant cohomology is well-defined up to isomorphism.

Remark 2.6. There is a more general version of equivariant cohomology, known as Bredon equivariant cohomology. Bredon equivariant cohomology takes values in coefficient systems, which is an algebraic tool that allow us to assign different coefficients for different orbits in the picture. If one takes the constant coefficient system, i.e. assign the identical abelian group as the coefficients for all orbits, then Bredon equivariant cohomology of the space computes the ordinary cohomology of the orbit space (quotient by the group action) with the chosen coefficients. In particular, the Bredon cohomology of $EG \times X$ with constant coefficient systems recovers the Borel cohomology. For an accessible introduction to the more general theory of Bredon equivariant cohomology, see [19]. In this article, equivariant cohomology will always mean Borel equivariant cohomology.

Let us consider the extreme example where our space X is a point.

Example 2.7. Let G act on $X = \{\mathrm{pt}\}$ by the trivial action. Then

$$X_{hG} = EG \times_G \{\mathrm{pt}\} \cong EG/G = BG,$$

so

$$H^*_G(\{\mathrm{pt}\}) = H^*(BG).$$

That is, the equivariant cohomology of a point is the group cohomology of G. Let us denote this ring by H_G^*, following [1].

More generally, if G acts trivially on X we have

$$H_G^*(X) = H^*(EG \times_G X) = H^*(BG \times X) = H_G^* \otimes H^*(X)$$

by Künneth formula. The other extreme is when G acts freely on X.

Proposition 2.8. *Let G act on X freely. Then the natural projection*

$$\sigma : EG \times_G X \to X/G$$

given by collapsing the EG summand is a weak homotopy equivalence.

Proof. From the fact that the G-action on X is free, one can show that σ is a fiber bundle with a contractible fiber EG. Now the result follows from the homotopy long exact sequence. □

As a corollary, we have

$$H_G^*(X) = H^*(X_{hG}) \cong H^*(X/G)$$

in this situation. Therefore we recover the cohomology of the orbit space as the equivariant cohomology, as expected.

Remark 2.9. Using the map $X \to *$, one sees that $H_G^*(X)$ naturally carries a $H_G^* = H_G^*(\{pt\})$-module structure. Later in Section 4 we will consider the torsion of $H_G^*(X)$ as H_G^*-modules.

Many properties from ordinary cohomology transfer to the equivariant setting. For example, functoriality works perfectly well. The long exact sequence of a pair also works well, and the definition is as follows. If we are given a space Y with a G-action and a G-invariant subspace $X \hookrightarrow Y$, we obtain an inclusion $X_{hG} \hookrightarrow Y_{hG}$. This map allows us to define relative equivariant cohomology of pairs as the relative cohomology of the homotopy quotients:

$$H_G^*(Y, X) = H^*(Y_{hG}, X_{hG}).$$

The corresponding long exact sequence then works exactly as in the ordinary case. We can construct equivariant tubular neighborhoods, because the usual averaging techniques (we are assuming G to be compact) allow us to define G-invariant metrics to the spaces if we want. Such tubular neighborhoods allow us to define an equivariant Thom isomorphism; the construction is carefully outlined in [13].

If we are working with manifolds, we would want an equivariant version of the notion of pushforward (the "wrong way" maps) as well. There does not exist an obvious theory of Poincaré duality in the equivariant setting, so the situation is more difficult here; naïvely bringing over the notion from the non-equivariant setting does not immediately work.

The fix is to factor every map $f : X \to Y$ of compact, oriented manifolds into the composition

$$f : X \to X \times Y \to Y$$

where the first map is the graph map $\Gamma_f : X \to X \times Y$ and the second map is the projection map $\pi_Y : X \times Y \to Y$. If we can make sense of the pushforward for both Γ_f and π_Y, the pushforward of f can be declared as the composition $f_* = (\Gamma_f)_* \circ (\pi_Y)_*$. More generally, we would like to define pushforwards for inclusions and fiber bundles. The former corresponds to Thom isomorphism and the latter corresponds to integration along the fiber.

Equivariant integration can be understood quite intuitively once we introduce the de Rham-type models (that is, description in terms of differential forms) for equivariant cohomology, which is our excuse for introducing them now.

3. The de Rham models and integration in equivariant cohomology

In this section, we assume our space X to be a smooth manifold. In particular, the space comes with the de Rham complex $\Omega^*(X)$ of differential forms on X. To be consistent with our choice of complex coefficients, one may consider the complex valued differential forms $\mathbb{C} \otimes_{\mathbb{R}} \Omega^*(X)$. We will write this as $\Omega^*(X)$ anyways by abuse of notation.

The goal of this section is to obtain a de Rham model for equivariant cohomology, namely to associate a de Rham-type complex to the space X whose cohomology will compute the equivariant cohomology. We will introduce two models, the Weil model and the Cartan model. Using the Cartan model, we will define the pushforward in equivariant cohomology.

We begin by introducing the Weil model of equivariant cohomology. The intuition is as follows. Suppose we have a compact connected Lie group G and a principal G-bundle $G \hookrightarrow P \to X$ of smooth manifolds. The image of the pullback map

$$\Omega^*(X) \to \Omega^*(P)$$

is characterized as the *basic* forms, which we will define now. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra corresponding to G. For an element $v \in \mathfrak{g}$ we consider the corresponding vertical vector field $v^{\#}$ on P, given by the derivative of the G-action on P along the direction of v. More precisely,

$$v^{\#} : P \to T(G \times P) \to TP$$

where the first map is the inclusion $p \mapsto ((g,p),(v,0))$ and the latter map is the derivative of the G-action. By abuse of notation, let $\mathcal{L}_v, \iota_v$ represent the Lie derivative and the interior product with respect to the vector field $v^{\#}$, respectively.

Definition 3.1. A differential form $\omega \in \Omega^*(P)$ is **basic** if it satisfies $\mathcal{L}_{\mathbf{v}}\omega = 0$, $\iota_v\omega = 0$ for all $v \in \mathfrak{g} = \mathrm{Lie}(G)$.

The condition $\iota_v\omega = 0$ means that the form is *horizontal*, i.e. it does not have any components in the fiber (vertical) direction. The condition $\mathcal{L}_v\omega = 0$ means that the form is G-invariant. A form is basic if and only if it is a pullback of a form from the base space.

We want a complex that would serve as a model for differential forms on X_{hG}. The idea of the Weil model is to find a model of $EG \times X$ and declare the basic forms as the forms corresponding to X_{hG}. For the manifold X, the de Rham complex $\Omega^*(X)$ is already available so it suffices to describe the model for EG. This model will be obtained from the Weil algebra associated to the group.

Definition 3.2. The **Weil algebra** associated to the group G is the algebra

$$W = W(\mathfrak{g}) = \Lambda\left(\mathfrak{g}^*\right) \otimes S\left(\mathfrak{g}^*\right)$$

where $\Lambda(\mathfrak{g}^*)$ is the exterior algebra as a graded commutative algebra and $S(\mathfrak{g}^*)$ is the symmetric algebra as an algebra with all elements in even degree.

In particular, we have a copy of $\mathfrak{g}^*$ in degree 1 as odd elements and another copy in degree 2 as even elements in the Weil algebra. We would like to identify this algebra as a de Rham-type complex of EG. In particular, we would like to define derivations L, d, I on $W(\mathfrak{g})$ that resembles the Lie derivative $\mathcal{L}$, exterior derivative d, and interior product ι on differential forms, respectively.

Remark 3.3. The notions of such derivations and their relations are axiomatized by the language of G^*-modules, which we will not introduce. An interested reader may consult [13].

The canonical action of G on $\mathfrak{g}^*$ is through the coadjoint representation. This representation induces an action of G on W. For clarity let us choose a basis v_a for $\mathfrak{g}$, which in turn yields the generators θ^a, x^a of $\Lambda(\mathfrak{g}^*), S(\mathfrak{g}^*)$, respectively. In terms of the structure constants

$$[v_a, v_a] = f^c_{ab} v_c,$$

we define the degree 0 derivations L_a analogously to the Lie derivative, i.e.

$$L_a\theta^b = -f^b_{ac}\theta^c, \quad L_a x^b = -f^b_{ac}x^c$$

using the definition of coadjoint representation. We also have the **Koszul differential** $d = d_K$ acting on the generators by

$$d_K\theta^a = x^a, \quad d_K x^a = 0.$$

This map extends by the Leibniz rule to define a differential graded algebra structure on Weil algebra.

Proposition 3.4. *The Koszul complex is acyclic.*

Proof. We can also define a derivation $Q = Q_K$ so that $Q_K x^a = \theta^a$, $Q_K\theta^a = 0$ on the generators. Then $Q^2 = 0$, and $dQ + Qd = \mathrm{id}$ on generators. It follows that $dQ + Qd$ on $\Lambda^k \otimes S^l$ is $(k + l)\mathrm{id}$, and hence the cohomology is concentrated at $\Lambda^0 \otimes S^0 \cong \mathbb{C}$. □

Note that up to this point the analogue of an interior product is not defined on W. Let us keep in mind that we would like W to be an analogue of the de Rham complex for EG. From this perspective, interpreting the degree 1 generators θ^a's as the vertical forms in the bundle $EG \to BG$, it is natural to define

$$I_a\theta^b = \delta^b_a, \quad I_a x^b = I_a d\theta^b = (I_a d + dI_a)\theta^b = L_a\theta^b = -f^b_{ac}\theta^c.$$

Here the equation

$$L_a = I_a d + dI_a \tag{3.5}$$

is the **Cartan formula** which is (an infinitesimal version of) the claim that flowing by vector fields is homotopic to the identity. Of course, the interior product and Lie derivative are considered only as formal algebraic derivations on $W(\mathfrak{g})$ in our current setting, so this description should only be regarded as a geometric intuition.

Now we would like to describe the analogue of basic elements. Following Definition 3.1, these elements of W should be G-invariant and also horizontal in the sense that the interior product always vanishes for them.

If one makes the change of variables

$$u^a = x^a + \frac{1}{2} f^a_{bc}\theta^b\theta^c,$$

the definition of the interior product on W_G implies (after some computation)

$$I_a u^b = 0.$$

Since the θ's and x's generate W, so do the θ's and u's. From this description it is clear that the even part $S(\mathfrak{g}^*)$ generated by the u^a's is exactly the horizontal subcomplex of W. Taking the G-invariants, we see that the basic elements of W are exactly the G-invariant polynomials

$$(W(\mathfrak{g}))_{\text{basic}} = S(\mathfrak{g}^*)^G.$$

Remark 3.6. The possibly mysterious change of variables amounts to adding the Chevalley–Eilenberg differential in Lie algebra cohomology to the Koszul differential. The action of this operator on the generators is given by

$$d_{CE}\theta^a = -\frac{1}{2} f^a_{bc}\theta^b\theta^c, \quad d_{CE}u^a = -f^a_{bc}\theta^b u^c.$$

One can start by defining the differential on W as $d = d_K + d_{CE}$, but acyclicity is more difficult to see from such approach.

In sum, we now have the complex

$$W_G = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*),$$

with odd generators $\theta^a \otimes 1$ in degree 1 and even generators $1 \otimes u^a$ in degree 2, equipped with the differential $d = d_K$. The de Rham model for equivariant differential forms can now be obtained as

$$\text{the basic subcomplex of } W(\mathfrak{g}) \otimes \Omega^*(X),$$

namely those elements that are annihilated by both

$$I_a \otimes 1 + 1 \otimes \iota_a, \quad L_a \otimes 1 + 1 \otimes \mathcal{L}_a.$$

This basic subcomplex is the **Weil model** of equivariant de Rham complex. Its cohomology computes the G-equivariant cohomology of X.

Remark 3.7. In the case where X is a point, this construction reduces to the statement that the cohomology ring of BG is the cohomology of the G-invariant polynomials on $\mathfrak{g}$. One can show that d annihilates all basic elements, so taking cohomology preserves the complex. That is, we have

$$H^*_G = H^*(BG) = S(\mathfrak{g}^*)^G$$

which is at the heart of the Chern–Weil theory of characteristic classes.

The Weil model has an advantage that it is constructed in a conceptually natural manner. In practice, however, one prefers to work with a model that yields itself to easier computation. Mathai–Quillen give such a model by proving the following proposition in [18], which they attribute to Cartan:

Theorem 3.8. *Consider the algebra map*

$$\epsilon : W(\mathfrak{g}) \otimes \Omega^*(X) \to S(\mathfrak{g}^*) \otimes \Omega^*(X)$$

defined by $\epsilon(\theta^a) = 0, \epsilon(u^a) = u^a$ *on generators of* $W(\mathfrak{g})$ *and* $\epsilon(\alpha) = \alpha$ *for* $\alpha \in \Omega^*(X)$. *Then there is an algebra map*

$$\phi = \exp(-\theta^a \otimes \iota_a) : S(\mathfrak{g}^*) \otimes \Omega^*(X) \to W(\mathfrak{g}) \otimes \Omega^*(X)$$

such that ϵ *and* ϕ *are algebra isomorphisms when restricted to horizontal subcomplexes, i.e. to the subcomplexes of elements that are annihilated by* $I_a \otimes 1 + 1 \otimes \iota_a$.

The map ϕ is called the **Mathai–Quillen isomorphism**. It is moreover true that these maps are G-equivariant, so there is an induced isomorphism

$$\phi : (S(\mathfrak{g}^*) \otimes \Omega^*(X))^G \cong (W(\mathfrak{g}) \otimes \Omega^*(X))_{\text{basic}}$$

by taking G-invariants. We can define a differential on the former algebra by

$$d_G = \epsilon \circ d \circ \phi$$

which acts on generators by

$$d_G(u^a \otimes 1) = 0, \quad d_G(1 \otimes \alpha) = 1 \otimes d\alpha - u^a \otimes \iota_a \alpha.$$

The complex $(S(\mathfrak{g}^*) \otimes \Omega^*(X))^G$ equipped with d_G is called the **Cartan model** of equivariant de Rham complex. Equivariant differential forms in the Cartan complex that vanish under d_G are called **equivariantly closed forms**.

Remark 3.9. Note that the Cartan differential defined as it is is *not* a differential on the full complex $S(\mathfrak{g}^*) \otimes \Omega^*(X)$, i.e. $d_G^2 \neq 0$ in general. It is a differential when restricted to the invariant subcomplex $(S(\mathfrak{g}^*) \otimes \Omega^*(X))^G$, by Cartan formula.

Example 3.10. Consider the case $G = T = \left(S^1\right)^m$. The Lie algebra is $\mathbb{R}^m$ and the structure constants vanish as T is abelian. Therefore the adjoint (and hence the coadjoint) action is trivial and all of

$$S(\mathfrak{g}^*) = \mathbb{C}\left[u^1, \ldots, u^m\right]$$

are G-invariant. It follows that the Cartan model is exactly

$$(S(\mathfrak{g}^*) \otimes \Omega^*(X))^G = \Omega^*(X)^G \left[u^1, \ldots, u^m\right],$$

i.e. the polynomial ring on m generators with coefficients in G-invariant differential forms on X.

Using the Cartan complex it is now straightforward to define the pushforward in equivariant cohomology. Let X and Y be compact oriented manifolds with dimensions m and n, respectively. Let $d = n - m$ and consider a map $f : X \to Y$. We would like to construct a map in cohomology

$$f_* : H_G^*(X) \to H_G^{*+d}(Y).$$

In the case f is an inclusion of a submanifold, the equivariant version of Thom isomorphism can be applied. Let ν_X be the G-equivariant tubular neighborhood of X in Y.

Theorem 3.11. *There is a G-equivariant isomorphism $H_G^*(X) \to H_G^*(\nu_X)_c$ from the cohomology of X to the compactly supported cohomology of ν_X, given by multiplying the Thom class $\Phi \in H_G^d(\nu_X)_c$.*

A proof is more or less verbatim from the ordinary case. The ordinary version is covered in detail in [6], and Chapter 10 of [13] explains the modifications that need to be made in the equivariant version. Now there is a composition

$$H_G^*(X) \cong H_G^{*+d}(\nu_X)_c \to H_G^{*+d}(Y)_c = H_G^{*+d}(Y).$$

Recall our setup in the end of the previous section. Given a map $f : X \to Y$, there is an inclusion of X into $X \times Y$ by the graph Γ_f of f. Moreover, $f = \pi_Y \circ \Gamma_f$ for the projection $\pi_Y : X \times Y \to Y$. So it now suffices to define the pushforward along π_Y, and declare $f_* = (\pi_Y)_* \circ (\Gamma_f)_*$. In other words, we are reduced to the case where $f : X \to Y$ is a fiber bundle.

Let $\alpha = P^i \otimes \alpha_i$ be a form in the Cartan complex, where $P^i \in S(\mathfrak{g}^*)$ and $\alpha_i \in \Omega^*(X)$. Integration along the fiber gives a map $f_* : \Omega^*(X) \to \Omega^{*+d}(Y)$ (note that $d = n - m < 0$ is the negative of the fiber dimension), so we define

$$f_*\alpha = P^i \otimes f_*\alpha_i.$$

When Y is a point, this gives a definition of integration for equivariant forms.

Definition 3.12. The **equivariant integral** of an equivariant differential form $\alpha = P^i \otimes \alpha_i \in S(\mathfrak{g}^*) \otimes \Omega^*(X)$ is defined as

$$\int_X \alpha = \left(\int_X \alpha_i \right) P^i \in S(\mathfrak{g}^*)^G = H_G^*.$$

As usual, integrals of forms $\alpha_i \in \Omega^*(X)$ of degree not equal to the dimension of X are always zero.

Since P^i's also carry a degree (the generators $u^i \in S(\mathfrak{g}^*)$ are assigned degree 2), we can integrate equivariant differential forms of degree not equal to the dimension of X. The result is not a number but a polynomial in general.

In sum, via the Cartan model we have given a construction of pushforward in equivariant cohomology. By construction it factors through the Thom isomorphism. It also satisfies functoriality. It also satisfies the "projection formula"

$$f_*\left(\alpha f^*\beta\right) = \left(f_*\alpha\right)\beta$$

for $\alpha \in H_G^*(X)$, $\beta \in H_G^*(Y)$ which is the claim that f_* is a map of $H_G^*(Y)$-modules.

Quite importantly, we have the equivariant version of the Euler class. For an inclusion $f : X \to Y$ of compact oriented manifolds, we have

$$f^*f_*1 = \mathbf{e}\left(\nu_{X/Y}\right) \in H_G^*(X)$$

where $\nu_{X/Y}$ is the equivariant normal bundle of X in Y and $\mathbf{e}$ is its equivariant Euler class, which is the pullback of the Thom class in $H_G^*(Y)$. The upshot of the Atiyah–Bott localization theorem is that when one is given a action of G on a manifold X with fixed locus $F \subset X$, one can *invert* the Euler class after localizing. We are ready to explain the theorem.

4. Atiyah–Bott localization and the integration formula

We are now ready to state the Atiyah–Bott localization theorem. Recall that by functoriality the terminal map $X \to *$ induces a $H_G^* = H_G^*(\{\mathrm{pt}\})$-module structure on the equivariant cohomology rings $H_G^*(X)$. In particular, we can discuss the *torsion in $H_G^*(X)$ as H_G^*-algebras.* While our coefficients are fixed to be a field $\mathbb{C}$, there may indeed be elements in $H_G^*(X)$ that are annihilated by nontrivial elements in H_G^* (which are in the general case some subring of a polynomial ring). In this section, torsion will always mean torsion as a H_G^*-algebra.

The content of the theorem is that all information in the equivariant cohomology *modulo torsion as a H_G^*-algebra* is contained in the fixed point set of the action. From the localization theorem one can deduce the powerful abstract integration formula as a corollary. We will closely follow the original paper [1] by Atiyah–Bott.

Let $T \subset G$ be a maximal torus. The case $G = T$ is the most important case, essentially by the following proposition:

Proposition 4.1. *Let $T \subset G$ be a maximal torus, and $W = N(T)/Z(T) = N(T)/T$ be the Weyl group. Then*

$$H_G^*(X) \cong H_T^*(X)^W,$$

where $H_T^*(X)^W$ is the W-invariant elements of the equivariant cohomology.

Proof. (Sketch; see [21] for a slightly different proof.) Consider the principal G/T-bundle

$$G/T \hookrightarrow X_{hT} \to X_{hG}.$$

In [5], Bott–Samelson shows that $H^*(G/T)$ is concentrated in even degrees and that the Euler characteristic (in this case, the dimension of the cohomology ring as a vector space) is exactly $|W|$. Now we allude to the Lefschetz fixed point theorem which states that given a map $f : Z \to Z$ of compact spaces, if

$$\tau(f) = \sum_k (-1)^k \mathrm{tr}\left(f^* : H^k(Z) \to H^k(Z)\right)$$

is non-zero then f has a fixed point. The action of W on $H^*(G/T)$ is induced by the canonical action of W on G/T, which acts without fixed points for $1 \neq w \in W$ and fixes everything for $1 = w$. In particular applying the fixed point theorem to $Z = G/T$, the trace of w^* for $1 \neq w \in W$ is zero and $|W|$ for $1 = w$. We conclude that W acts on $H^*(G/T)$ by the regular representation. The proposition then follows from inspecting the Serre spectral sequence associated to the fiber bundle and taking W-invariants. □

In the case where X is a point, we recover the splitting principle. Another proof can be found in [13] which directly compares the relevant spectral sequences associated to the Cartan models for G-equivariant and $N(T)$-equivariant cohomology. Their proof is via Chern–Weil theory and uses a nontrivial theorem of Chevalley in the theory of Lie algebras.

Now we are justified to restrict to the case $G = T$. We now proceed to the proof of the localization theorem. We first state the preliminary definitions.

Definition 4.2. Let A be a ring. A **support** of a (finitely generated) A-module M is the set of prime ideals

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$$

where $M_{\mathfrak{p}}$ is the localization of the module M at $\mathfrak{p}$.

In sheaf-theoretic terms, we interpret M as a sheaf over $\mathrm{Spec}(A)$ and the support is where the stalk of this sheaf is not zero. This is a closed subset of $\mathrm{Spec}(A)$.

In the case we are interested in, $A = H_T^* = \mathbb{C}[u^1, \ldots, u^m]$ is the polynomial ring on $m = \dim T$ generators. The (complex points of the) spectrum of A is the vector space $\mathbb{C}^m$. The support of a H_T^*-module M is the closed (in Zariski topology) subset

$$\mathrm{Supp}(M) = \bigcap_f \{x \in \mathbb{C}^m \mid f(x) = 0\}$$

where $f \in \mathbb{C}[u]$ ranges over the annihilators of M, i.e. the elements such that $fM = 0$. In particular, we consider the support of H_T^*-modules naturally as subsets of $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^m$, where the generators $u^i \in H_T^*$ should be thought of as coordinates on this Lie algebra.

The key lemma to the localization theorem is the following.

Lemma 4.3. *If there is a T-equivariant map $V \to T/K$ for a closed subgroup $K \le T$, then*

$$\operatorname{Supp}(H_T^*(V)) \subset \mathfrak{k}_{\mathbb{C}}.$$

Proof. The maps $V \to T/K \to *$ induce the maps $H_T^* \to H_T^*(T/K) \to H_T^*(X)$ of H_T^*-modules. Now $H_T^*(T/K) = H_K^*$, which shows that the H_T^*-module structure on $H_T^*(V)$ factors through the $H_K^* = H_{K_0}^*$-module structure ($K_0 \le K$ is the identity component, a subtorus of T). In particular, the support of $H_T^*(V)$ naturally lies in $\mathfrak{k}_{\mathbb{C}}$. □

The key idea to the localization theorem is as follows. The assumption of the lemma holds when $V \subset X$ is a T-orbit of the action of T on X with isotropy group K. In particular, if $K \neq T$ (i.e., the orbit is not a fixed point of T) then the support of $H_T^*(X)$ lies in a positive codimension subspace of $\mathfrak{t}_{\mathbb{C}}$. This implies in turn that the module $H_T^*(V)$ is torsion, because free modules necessarily have the whole space as their supports.

In sum, all the information in $H_T^*(X)$ from the T-orbits that are not from the fixed points is torsion. The non-torsion part is completely governed by the data at the fixed points. This is the content of the **Atiyah–Bott localization theorem**.

Theorem 4.4. *(Atiyah–Bott, 1984) Let a torus $T = (S^1)^m$ act on a smooth manifold X. Let $i : F \to X$ denote the inclusion of fixed points of the T-action. Then*

$$i^* : H_T^*(X) \to H_T^*(F), \quad i_* : H_T^*(F) \to H_T^*(X)$$

have torsion kernels and cokernels. More precisely, the support of the kernels and cokernels lie in $\cup_K \mathfrak{k}_{\mathbb{C}}$ where K ranges for all proper isotropy subgroups $K < T$.

Proof. The proof is by Mayer–Vietoris. We stratify X by T-orbits of varying isotropy groups. By the compactness of T we can construct T-invariant tubular neighborhoods of these orbits. Take U to be the neighborhood of F in X, and $X - U$ by compactness is covered by finitely many such neighborhoods of orbits.

It is a fact that a short exact sequence of modules $L \to M \to N$ gives $\operatorname{Supp}(M) \subset \operatorname{Supp}(L) \cup \operatorname{Supp}(N)$. By a combination of this fact with Mayer–Vietoris on the finite cover of $X - U$ by neighborhoods of orbits, we see that $H_T^*(X - U)$ is torsion by Lemma 4.3. Now since U has a T-equivariantly deformation retracts onto F, $H_T^*(X - F)$ is torsion. The same proof shows that the result holds for all T-invariant subspaces of $X - F$, and consequently for all pairs of such subspaces in $X - F$.

In particular, $H_T^*(X, F) \cong H_T^*(X - U, \partial(X - U))$ (by excision) is torsion. The long exact sequence for the pair (X, F) now shows the desired result for the pullback map $i^* : H_T^*(X) \to H_T^*(F)$.

Similarly, the pushforward map $i_* : H_T^*(F) \to H_T^{*+d}(X)$ factors through

$$H_T^*(F) \cong H_T^{*+d}(\nu_F, \nu_F - F) \cong H_T^{*+d}(X, X - F) \to H_T^{*+d}(X)$$

as equivariant pushforward was defined via Thom isomorphism. Now the fact that $H_T^*(X - F)$ is torsion with the long exact sequence for the pair $(X, X - F)$ implies the desired result.

□

The theorem says that once we invert certain polynomials in the ring H_T^*, the equivariant cohomology of X becomes isomorphic to the equivariant cohomology of the fixed point set F. More precisely, we must invert all polynomials that vanish on all of $\mathfrak{k}_{\mathbb{C}}$ for the proper isotropy subgroups $K < T$. This fact allows us to reduce the computations of equivariant cohomology classes of X to computations at the fixed points, which in many cases simplifies the computation greatly. This powerful idea is formalized as follows.

Corollary 4.5. *The maps i^* and i_* are isomorphisms, modulo torsion. Moreover, for a class $\phi \in \mathrm{Frac}(H_T^*)(X)$ we have*

$$\int_X \phi = \sum_P \frac{1}{\mathbf{e}(\nu_P)} \int_P i_P^* \phi$$

in the field of fractions $\mathrm{Frac}(H_T^*)$, where the index P ranges over the connected components of F.

Note that with

$$i^* i_* 1 = \mathbf{e}(\nu_F)$$

we know that $\mathbf{e}(\nu_F)$ is invertible in some ring by the localization theorem. This is always true if we localize at the generic point (that is, tensor everything with the field of fractions $\mathbb{C}(u^i)$), although it suffices in general to invert only those polynomials that occur in the H_T^0-component of the relevant Euler classes (see [1]). In particular, the condition on the Corollary 4.5 that ϕ should lie in the equivariant cohomology ring fully localized to the field of fractions can be relaxed.

Remark 4.6. As an example of the calculation of the Euler class, consider the case where F is a finite set of isolated fixed points. The normal bundle at a point $p \in F$ is just the restriction of the tangent bundle $TX|_p = T_pX$. This vector space is T-invariant and therefore splits as a direct sum of irreducible representations. (Note that the existence of such structure is crucially dependent on the fact that p is a fixed point.) These weights can be identified by the characters $\exp(2\pi i a_j) : T \to S^1$, and the Euler class of the normal bundle is exactly the product of the linear forms

$$\mathbf{e}(\nu_p) = \prod_j a_j \in H_T^*.$$

It suffices to invert these polynomials in u to obtain the desired isomorphism.

Corollary 4.5 now follows from the fact that i^* and $i_*/\mathbf{e}(\nu_F)$ are inverses to each other, i.e.

$$\phi = \sum_P \frac{1}{\mathbf{e}(\nu_P)} (i_P)_* (i_P)^* \phi.$$

One applies $(\pi_X)_*$ to both sides and use that $\pi_F = \pi_X \circ i_F$ to obtain the final formula in Corollary 4.5, which is the **Atiyah-Bott integration formula**.

Remark 4.7. Note that the formula applies to equivariant cohomology classes $\phi \in H_T^*(X)$. If we want to use the formula for ordinary cohomology classes $\phi \in H^*(X)$, we must find an equivariant class $\widetilde{\phi} \in H^*(X_T)$ such that its image under the pull-back along the inclusion of fiber $X \hookrightarrow X_T \to BT$ is equal to ϕ. From the Cartan model, in the form level we know that equivariant forms are just polynomials in u^i's with coefficients in the ordinary forms, and that $\widetilde{\phi}|_{u=0} = \phi$ must hold. Such $\widetilde{\phi}$ is called an **equivariantly closed extension** of the closed form ϕ. Equivariantly closed extensions are guaranteed to always exist for certain classes of spaces known as equivariantly formal spaces; these include all manifolds with cohomology concentrated in even degrees. They also exist for characteristic classes of vector bundles when the action on the base space lifts to an action on the bundle: see [22].

The usefulness of the integration formula cannot be understated and in the next section we will explore its applications in wide range of fields. Before we dive into these applications, we end this section by quickly showing how a special case of the Gauss–Bonnet–Hopf theorem can be deduced immediately from the integration formula.

Let the circle $T = S^1$ act on a manifold X by diffeomorphisms. There is a natural action on the tangent bundle TX which induces a bundle on X_{hT}. Suppose the action on X has finitely many isolated fixed points F under T. These are the zeroes of the vector field generated by the circle action. From the fact that TX restricted to a fixed point is exactly the normal bundle of the fixed point in X, the integration formula yields

$$\chi(X) = \int_X \mathbf{e}(TX) = \sum_F \int_F \frac{\mathbf{e}(\nu_{F/X})}{\mathbf{e}(\nu_{F/X})} = \sum_F 1 = |F|.$$

The first equality is the Gauss–Bonnet theorem, and the second equality can be interpreted as the Hopf theorem. For a visual example, imagine a 2-sphere rotating along a fixed axis.

5. Applications

In this section, we explore three applications of the Atiyah–Bott localization theorem. As noted in [22], the theorem has found many applications in fields as diverse as enumerative algebraic geometry, symplectic geometry, and mathematical physics. In this section we explore some selected flavors of such applications. This section may not be self-contained and results will mostly be provided without proofs, but pointers to references will be provided when necessary.

5.1. **Enumerative algebraic geometry.** The idea of Atiyah–Bott localization has great applications in enumerative algebraic geometry and intersection theory. A first approximation to the goal of enumerative algebraic geometry is to count the number of geometric objects. In a lot of cases such questions reduce to computing characteristic classes on a variety.

Example 5.1. In how many points do two lines in a (projective) plane intersect? The answer is obviously one, which can also be computed as the integral

$$\int_{\mathbb{P}^2} h^2 = 1$$

where $h \in H^2(\mathbb{P}^2)$ denotes the generator, which is the Poincaré dual of a hyperplane (line) in $\mathbb{P}^2$.

The above example generalizes to higher dimensional projective spaces. It gives us the first hands-on example of equivariant cohomology so we might as well work out the example. The treatment follows [14], and rigorous proofs of the statements we make are provided in [25].

Example 5.2. We consider the complex projective space $\mathbb{P}^m$, thought of as a projectivization of a $(m+1)$-dimensional vector space $V = \mathbb{C}^{m+1}$. There is a natural action of $T = (S^1)^{m+1}$ on V, given by

$$(t_0, \dots, t_m) \cdot (z_0, \dots, z_m) = (t_0 z_0, \dots, t_m z_m).$$

On BT, we can consider the vector bundle $V_{hT} = ET \times_T V$ obtained by gluing in the representation V to the principal T-bundle $ET \to BT$. One can show that this vector bundle is isomorphic to the direct sum of line bundles $L_0 \oplus \cdots \oplus L_m$, where L_i is the pullback of the tautological bundle $\mathcal{O}(-1)$ over $BS^1 = \mathbb{P}^\infty$ along the projection $BT = (\mathbb{P}^\infty)^{m+1} \to BS^1$ to the ith factor.

The action on V induces an action on $\mathbb{P}^m$, with fixed points $p_0, \dots, p_m$ where $p_i = [0, \dots, 1, \dots, 0]$ are the coordinate axes of V (1 in the ith projective coordinate). The Borel construction of $\mathbb{P}^m$ with this action is now identified with the projectivization of this vector bundle over BT:

$$\mathbb{P}^m_{hT} = ET \times_T \mathbb{P}^m = ET \times_T \mathbb{P}V = \mathbb{P}\left(ET \times_T V\right) = \mathbb{P}\left(L_0 \oplus \cdots \oplus L_m\right).$$

By the splitting principle (see [6]) the equivariant cohomology of $\mathbb{P}^m$ can now be computed as

$$H_T^*\left(\mathbb{P}^m\right) = H^*\left(\mathbb{P}^m_{hT}\right) = \mathbb{C}[\mathbf{x}, u^0, \dots, u^m] / \left(\prod_{i=0}^m (\mathbf{x} - u^i)\right)$$

where $u^i = -\mathbf{e}(L_i)$ is the Chern class of the dual of L_i and $\mathbf{x} = \mathbf{e}(\mathcal{O}(1))$ is the hyperplane class, i.e. the Chern class of the dual of the tautological line bundle $\mathcal{O}_{\mathbb{P}^m_{hT}}(-1)$ over $\mathbb{P}^m_{hT}$.

Note that the pullback

$$\iota_i^* : H_T^*(\mathbb{P}^m) \to H_T^*(\{p_i\}) = H_T^* = \mathbb{C}[u^0, \dots, u^m]$$

under the inclusion of the ith fixed point $p_i \hookrightarrow \mathbb{P}^m$ sends x to u^i by observing that the tautological line bundle over $\mathbb{P}^m_{hT}$ restricts to L_i over $\{p_i\}_{hT} = BT$.

The normal bundle of p_i in $\mathbb{P}^m$ is the quotient of the tangent bundle at $\mathbb{P}^m$ restricted at p_i by the tangent bundle of the point p_i. In particular, it is isomorphic to the restriction of the tangent bundle of $\mathbb{P}^m$ at p_i because the tangent bundle of a point is trivial. The weights of the T-action on the tangent space are $(u^i - u^j)$, as one sees by identifying the cotangent vectors to p_i with linear forms on the affine coordinates z^j/z^i. Therefore the Euler class is $\prod_{j \neq i}(u^i - u^j)$.

With the information above, Atiyah–Bott integration formula now specializes to the **Bott residue formula**:

$$\int_{\mathbb{P}^m} f(\mathbf{x}, u) = \sum_{i=0}^m \mathrm{Res}_{\mathbf{x}=u^i} \frac{f(\mathbf{x}, u)}{\prod_{j=0}^m (\mathbf{x} - u^j)} = \frac{1}{2\pi i} \oint d\mathbf{x} \frac{f(\mathbf{x}, u)}{\prod_{j=0}^m (\mathbf{x} - u^j)}. \tag{5.3}$$

In particular taking $f(\mathbf{x}, u) \in H_T^*(\mathbb{P}^m)$ in (5.3) as $\mathbf{x}^m$, we obtain the enumerative result

$$\int_{\mathbb{P}^m} h^m = \sum_{i=0}^m \frac{(u^i)^m}{\prod_{j \neq i}(u^i - u^j)} = 1$$

which says that m generic hyperplanes in $\mathbb{P}^m$ intersect at a point.

Remark 5.4. In the previous example the final equality depends on seemingly miraculous cancellations. However, note that one can take different representatives of the hyperplane class $h \in H^2(\mathbb{P}^m)$ in the equivariant cohomology. If we take $\mathbf{x} - u^i$ to represent the ith copy in the product h^m, the residue formula yields

$$\int_{\mathbb{P}^m} h^m = \int_{\mathbb{P}^m} \prod_{i=1}^{m} (\mathbf{x} - u^i) = \frac{\prod_{j\neq 0}(u^0 - u^j)}{\prod_{j\neq 0}(u^0 - u^j)} + \sum_{i=1}^{m} 0 = 1.$$

Choosing $\mathbf{x} - u^i$ instead of $\mathbf{x}$ as a representative for the hyperplane class h corresponds to twisting the T-action induced on $\mathcal{O}(1)$ (whose Euler class is $\mathbf{x}$) by a one-dimensional representation V_i defined by $(t_0, \ldots, t_m) \cdot z = t_i z$. We have chosen different V_i's for different copies of the hyperplane class h in the product h^m.

For a more involved and interesting application, let us show how one can use the tool of equivariant integration to recover the classical enumerative result that there are 27 straight lines on a cubic surface.

Recall that a cubic surface is a zero set in a projective space $\mathbb{P}^3_k$ over a field k of a cubic homogeneous polynomial $f \in k[x_0, x_1, x_2, x_3]$. There is a general notion of smoothness for algebraic surfaces, but we will fix $k = \mathbb{C}$ so that a surface being smooth is equivalent to being a complex manifold.

Theorem 5.5. *(Cayley, 1849) A generic smooth cubic complex surface contains exactly* 27 *lines.*

Proof. We will see how equivariant integration can be used to obtain this result. We should first reduce the problem of finding the number of lines in the surface into a problem of integration, in a similar spirit to Example 5.1. Once we have an integral, the equivariant integration formula will allow an easy computation.

Let $S \subset \mathbb{P}^3$ be a smooth cubic complex surface. It is cut out by a cubic homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$.

Consider the set $\mathbf{Gr}(2, 4)$ of 2-planes in $\mathbb{C}^4$, or equivalently the set of lines in the projective space $\mathbb{P}^3$. This set, known as the Grassmannian, in fact carries the structure of a smooth complex projective variety. On a point $\ell \in \mathbf{Gr}(2, 4)$ (which is a line in a projective 3-space), we can consider the vector space

$$E_\ell = \{\text{homogeneous cubic polynomials on } \ell\}.$$

For example, if ℓ is the zero set of x_2 and x_3, then E_ℓ is generated by cubic monomials in x_0 and x_1. For every ℓ, E_ℓ is a vector space of dimension 4, and in fact fits into a rank 4 complex vector bundle E over $\mathbf{Gr}(2, 4)$.

The homogeneous cubic polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ that cuts out our surface $S \subset \mathbb{P}^3$ can then naturally be thought of as a section of E. A line $\ell \in \mathbf{Gr}(2, 4)$ is contained in S if and only if this section vanishes at ℓ. So the question of finding the number of lines contained in S is equivalent to the question of determining the vanishing locus of a section of E.

The Grassmannian $\mathbf{Gr}(2, 4)$ has complex dimension 4. To see this, observe that generically $2 \cdot 4$ complex numbers determine a 2-plane in 4-space but this is only unique up to 2×2 invertible matrices corresponding to the change of basis. Hence $2 \cdot 4 - 2 \cdot 2 = 4$ is the dimension of the Grassmannian. Since E is a rank 4 bundle over a 4-dimensional manifold, the vanishing locus of a generic section is expected to be of dimension 0, i.e. a finite set of points. In particular, the number of lines contained in S is expected to be finite.

The actual number can be computed by the following integral of the Chern class (cf. Example 5.1):

$$\int_{\mathbf{Gr}(2,4)} c_4(E).$$

From here, the classical approach is to exploit the geometry of the Grassmannian to compute this integral. This calculation scheme goes by the name of Schubert calculus and a computation of this flavor is outlined in [17]. Instead of following this approach, we would like to apply the Atiyah–Bott integration formula.

Note that $T = (S^1)^4$ acts on $\mathbb{C}^4$ by componentwise multiplication

$$(t_0, t_1, t_2, t_3) \cdot (x_0, x_1, x_2, x_3) = (t_0 x_0, t_1 x_1, t_2 x_2, t_3 x_3)$$

as in Example 5.2. This action induces an action on $\mathbb{P}^3$. Moreover, since lines in $\mathbb{P}^3$ are cut out by two linear forms, this action takes lines in $\mathbb{P}^3$ to lines in $\mathbb{P}^3$. In other words, there is also an induced action on $\mathbf{Gr}(2,4)$.

The torus action on $\mathbf{Gr}(2,4)$ has six isolated fixed points, which we denote by

$$\ell_{ij} = \{x \in \mathbb{C}^4 \mid x_i = 0, x_j = 0\} \in \mathbf{Gr}(2,4), \quad 0 \le i < j \le 3.$$

By the Atiyah–Bott formula, we have

$$\int_{\mathbf{Gr}(2,4)} c_4(E) = \int_{\mathbf{Gr}(2,4)} \mathbf{e}(E) = \sum_{i<j} \frac{\mathbf{e}(E|_{\ell_{ij}})}{\mathbf{e}(T\mathbf{Gr}(2,4)|_{\ell_{ij}})}, \tag{5.6}$$

where we identified the normal bundle of ℓ_{ij} in $\mathbf{Gr}(2,4)$ with the restriction of the tangent bundle at that point (cf. Example 5.2). It remains to compute the Euler classes, and this reduces to computing the T-weights of the action on the fiber of the vector bundles at the fixed points.

Without loss of generality, consider ℓ_{01}. On ℓ_{01} we have $x_0 = x_1 = 0$, and the remaining coordinates x_2, x_3 define the projective coordinates on this line. The fiber $E_{\ell_{01}}$ is then generated by

$$\{x_2^3, x_2^2 x_3^1, x_2^1 x_3^2, x_3^3\}$$

and the corresponding weights are

$$(3u_2), (2u_2 + u_3), (u_2 + 2u_3), (3u_2).$$

The Euler class is therefore

$$\mathbf{e}\left(E|_{\ell_{01}}\right) = (3u_2)(2u_2 + u_3)(u_2 + 2u_3)(3u_2) = 18u_2^3 u_3 + 45u_2^2 u_3^2 + 18u_2 u_3^3 \in H_T^*.$$

The Euler class of the normal bundle can also be computed. It is known that the tangent space at a point ℓ of a Grassmannian (a 2-plane in a 4-space) can be described as the space of linear transformations from that point:

$$T\mathbf{Gr}(2,4)|_\ell \cong \operatorname{Hom}\left(\ell, \mathbb{C}^4/\ell\right).$$

In particular, at ℓ_{01} this is the vector space of linear transformations from the 2-space spanned by x_2, x_3 to the 2-space spanned by x_0, x_1. The weights of the induced torus action on the tangent space are then

$$(u_0 - u_2), (u_1 - u_2), (u_0 - u_3), (u_1 - u_3)$$

and the Euler class is given by the product

$$\mathbf{e}\left(T\mathbf{Gr}(2,4)|_{\ell_{01}}\right) = (u_0 - u_2)(u_1 - u_2)(u_0 - u_3)(u_1 - u_3)..$$

In sum, (5.6) reduces to the following sum over the choice of indices i, j, where k, l denotes the other two indices:

$$\int_{\mathbf{Gr}(2,4)} \mathbf{e}(E) = \sum_{0 \le i < j \le 3} \frac{18u_k^3 u_l + 45u_k^2 u_l^2 + 18u_k u_l^3}{(u_i - u_k)(u_j - u_k)(u_i - u_l)(u_j - u_l)} = 27.$$

The computation is straightforward and indeed recovers the desired result.

□

As the examples we have explored demonstrate, many enumerative questions in intersection theory can be reduced to computations of certain integrals. In such cases, T-equivariant localization is often available as a powerful tool. Torus actions abound mainly because one works with projective spaces. Applications of equivariant localization in enumerative algebraic geometry (in particular, Gromov–Witten theory) are carefully explained in [14], [17], and [25].

We now turn to a different branch of mathematics where group actions and equivariant integration often appear: symplectic geometry.

5.2. **Symplectic geometry.** As already noted by Atiyah and Bott in [1], equivariant cohomology has deep and natural connections to symplectic geometry. The connection comes from the fact that equivariantly closed extensions (cf. Remark 4.7) of the symplectic form are in one-to-one correspondence with Hamiltonian maps. In this subsection we review this correspondence. As a result we see how the Atiyah–Bott integration formula can be specialized to the Duistermaat–Heckman formula, which characterizes the variation of the symplectic volume for symplectic quotients.

We begin by reviewing the relevant notions in symplectic geometry. A thorough treatment is given in [10].

A symplectic manifold is an even dimensional manifold X equipped with a non-degenerate closed 2-form ω called the symplectic form. The symplectic form has the property that the top exterior power $\omega^n/n!$ is nowhere vanishing where $2n = \dim X$; it is the Liouville volume form. A map of symplectic manifolds that preserve the symplectic form is called a symplectomorphism.

Example 5.7. A canonical example of a symplectic manifold is the phase space, which is mathematically just a cotangent bundle T^*X over some manifold X. In local coordinates $(q^1, \ldots, q^n)$ of the manifold for which there are corresponding cotangent coordinates $(p^1, \ldots, p^n)$ (called the conjugate momenta in physics), the symplectic form is locally written as $\omega = dp^i \wedge dq^i$.

We are interested in the case where a compact Lie group G acts on a symplectic manifold X by symplectomorphisms. The goal is to be able to construct what is essentially a quotient of X by G.

Definition 5.8. An action $\psi : G \times X \to X$ by symplectomorphisms is **Hamiltonian** if there exists a **moment map**

$$\mu : X \to \mathfrak{g}^*$$

that satisfies the following constraints.

- For $v \in \mathfrak{g}$, $\mu(v) : X \to \mathbb{R}$ is a function on X. For the vector field $v^{\#}$ on X that the one-parameter flow in the direction of v generates on X, we have $d\left(\mu(v)\right)(\cdot) = \omega(v^{\#}, \cdot)$.

- The map μ is G-equivariant with $\mathfrak{g}^*$ given the coadjoint action.

The function $\mu(v) : X \to \mathbb{R}$ associated to each $v \in \mathfrak{g}$ is called the **Hamiltonian map** for the vector field $v^{\#}$.

This definition is quite technical. The geometric picture is as follows. Given a G-action on X, one-parameter subgroups $\{\exp tv : t \in \mathbb{R}\} \subset G$ generated by $v \in \mathfrak{g}$ generate corresponding vector fields $v^{\#}$ on X by taking the derivative of the flow by the one-parameter subgroups. Then the fact that G-action on X is symplectic implies $\iota_{v^{\#}}\omega$ is a closed 1-form. To see this, note that

$$d\iota_{v^{\#}}\omega = (d\iota_{v^{\#}} + \iota_{v^{\#}}d)\,\omega = \mathcal{L}_{v^{\#}}\omega = 0$$

by the Cartan formula (3.5). The last equality comes from the fact that the action is symplectic and hence flowing by the action leaves the symplectic form invariant. *If $\iota_{v^{\#}}\omega$ is not only closed but also exact*, there is some function μ_v such that $d\mu_v = \iota_{v^{\#}}\omega$. We are asking that these functions μ_v must satisfy some sort of compatibility. Namely, the association $v \mapsto \mu_v$ must be a map of Lie algebras, where $C^\infty(X)$ is given the Lie algebra structure by the Poisson bracket: $\{f, g\} = \omega(\chi_f, \chi_g)$ for $\omega(\chi_f, \cdot) = df(\cdot)$. This description defines a map from the Lie algebra $\mathfrak{g}$ to $C^\infty(X)$, and dualizing it defines a map $X \to \mathfrak{g}^*$, just as in Definition 5.6. If we further ask that the map $\mathfrak{g} \to C^\infty(X)$ is G-equivariant with G acting on $\mathfrak{g}$ by the adjoint representation, then we exactly recover the definition of moment maps if we assume that G is connected.

Example 5.9. Let $G = \mathrm{SO}(3)$ act on $\mathbb{R}^3$ by rotations. There is a lift of this action on the phase space $T^*\mathbb{R}^3 = \mathbb{R}^6$ (which is a symplectic manifold; cf. Example 5.5) as a symplectic action. The corresponding moment map is

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3 = \mathrm{Lie}(\mathrm{SO}(3))^*, \quad \mu : (\mathbf{q}, \mathbf{p}) \to (\mathbf{a} \mapsto (\mathbf{q} \times \mathbf{p}) \cdot \mathbf{a}),$$

and it is called the angular momentum in physics.

The physical picture comes from Noether's principle in classical mechanics, which states that when there is a symmetry of a system there is a corresponding conserved quantity (the Hamiltonian map of the symmetry). For each conserved quantity the degree of freedom of the dynamical system decreases by two. The idea is formalized in the framework of symplectic geometry in terms of symplectic quotients.

Theorem 5.10. *Let (X, ω, G, μ) denote the data of a Hamiltonian action. If G acts freely on the preimage $i^* : \mu^{-1}(0) \hookrightarrow X$, then $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle. Moreover, $\mu^{-1}(0)/G$ can be given a canonical symplectic structure ω_0 such that $i^*\omega = \pi^*\omega_0$.*

The space $X_0 = \mu^{-1}(0)/G$ with its canonical symplectic structure is the **symplectic quotient** and has dimension $\dim X - 2\dim G$, as expected by classical mechanics. The rigorous construction of symplectic quotients can be found in [10].

Remark 5.11. When G is a torus, moment maps are only unique up to a constant $t \in \mathfrak{g}^*$ (see [10]). So in particular the construction makes sense by replacing $\mu^{-1}(0)$ with $\mu^{-1}(t)$ as long as G acts freely on $\mu^{-1}(t)$ (that is, $t \in \mathfrak{g}^*$ is a regular value of μ). The corresponding symplectic quotient will be denoted (X_t, ω_t).

The theory of Hamiltonian torus actions is well studied. One theorem in the area is the **Duistermaat–Heckman theorem** [12], which characterizes the pushforward of the Liouville measure under the moment map as a piecewise polynomial.

Let $G = T$ be a torus acting on a symplectic manifold (X, ω) and let this torus action be Hamiltonian.

Definition 5.12. The **Duistermaat–Heckman measure** m_{DH} on $\mathfrak{t}^*$ is the push-forward of the Liouville measure $\omega^n/n!$ under the moment map $\mu : (X, \omega) \to \mathfrak{t}^*$. That is,

$$\mathrm{m}_{DH}(U) = \int_{\mu^{-1}(U)} \frac{\omega^n}{n!}.$$

Theorem 5.13. *(Duistermaat–Heckman, 1982) The density obtained as the Radon–Nikodym derivative of the Duistermaat–Heckman measure with respect to the Lebesgue measure*

$$f(t) = \frac{d\mathrm{m}_{DH}}{d\lambda}$$

on $\mathfrak{t}^* \cong \mathbb{R}^d$ *is piecewise polynomial on* $t \in \mathfrak{t}^*$*, with breaks occuring at critical values of* μ*.*

Duistermaat–Heckman in [12] note that this property of the density f being piecewise polynomial implies—in fact, is equivalent to—that the inverse Fourier transform is exactly equal to the sum of contributions from the fixed points of the action. In other terms, the oscillatory integral is equal to its stationary phase approximation. Their formula writes

$$\int_X e^{it\langle v, \mu(x)\rangle} \frac{\omega^n}{n!} = \sum_j \frac{\mathrm{vol}(X_j) e^{it\langle v, \mu(X_j)\rangle}}{\left(\frac{t}{2\pi i}\right)^{n_j} \prod_{k=1}^{n_j} \langle v, \omega_{jk}\rangle}. \tag{5.14}$$

Here t is just a real parameter and $v \in \mathfrak{t}$. Moreover, $F = \{X_j\}$ is the fixed point set of the T-action on X, with connected components X_j of codimensions $2n_j$. Finally, $\omega_{jk} \in \mathfrak{t}^*$ is the coefficient of the quadratic term in the Taylor expansion of μ at a fixed point $x_j \in X_j$ in the coordinates in which the T-action is linear (this is always possible):

$$\mu_v(x) = \mu_v(x_j) + \sum_{k=1}^{m} \omega_{jk}(v) \frac{p_k^2 + q_k^2}{2}.$$

We relabel the k's so that the first n_j coefficients ω_{jk} with $k = 1, \ldots, n_j$ at x_j are the nonzero coefficients. In particular, the product in the denominator of the right hand side in (5.14) makes sense.

In the case where T is a circle and the fixed points F are isolated points, the formula (5.14) reduces to

$$\int_X e^{it\mu_v} \frac{\omega^n}{n!} = \sum_{p \in F} \frac{e^{it\mu_v(p)}}{(t/i)^n E_p}. \tag{5.15}$$

Here $\mu_v = \mu(v) \in C^\infty(X)$ is the evaluation of the moment map at a chosen $v \in \mathfrak{t}$, and E_p is the integer multiple that appears in the equivariant Euler class of the normal bundle of $p \in X$:

$$\mathbf{e}(\nu_p) = E_p u^n \in H^*_{S^1}(\{p\}) = \mathbb{C}[u].$$

Now (5.15) is a special case of Atiyah–Bott integration formula; this observation was the initial stimulus of Atiyah and Bott for [1]. The connection comes from the following proposition (also proved in [1], essentially by computation).

Lemma 5.16. *For any choice of $v \in \mathfrak{g}$, the Hamiltonian map $\mu_v = \mu(v)$ gives an equivariantly closed extension (in the Cartan model) of the symplectic form:*

$$\omega^v = \omega - \mu_v u \qquad \Longrightarrow \qquad d_{S^1}\omega^v = 0.$$

Given Lemma 5.16, one now applies Atiyah–Bott integration formula to each term in the power series

$$\exp(\omega^v) = \exp(\omega)\exp(-\mu_v u) \in H_T^*(X) \otimes_{H_T^*} \mathbb{C}[[u]]$$

to get

$$\int_X e^{-\mu_v u} e^{\omega} = \sum_{p \in F} \frac{\int_p e^{-\mu_v u} e^{\omega}}{\mathbf{e}(\nu_p)}. \tag{5.17}$$

The only term that survives on the left hand side of (5.17) after integrating over X is the Liouville form $\omega^n/n!$ by dimension reasons. That is, the only term in the power series $e^\omega = \sum_k \omega^k/k!$ that has the same degree as the dimension of the manifold X is the Liouville form $\omega^n/n!$. Similarly on the right hand side only the constant term is picked up. It follows that

$$\int_X e^{-\mu_v u} \frac{\omega^n}{n!} = \sum_{p \in F} \frac{e^{-\mu_v(p)u}}{E_p u^n}$$

and (5.15) follows from replacing u by $-it$.

Hamiltonian torus actions are particularly nice with many interesting properties; Duistermaat–Heckman theorem is only one of them. They have deep connections to the theory of integrable systems as well. In fact, the primary example of an integrable system is indeed a phase space with given an effective Hamiltonian torus action. A thorough treatment of torus actions and their connections to equivariant cohomology/integrable systems is given in [3].

Nonabelian generalizations of the Duistermaat-Heckman theorem were found and extensively used by Witten in [24]. Later, Jeffrey and Kirwan in [15] used Witten's ideas to prove a residue formula type result that is closer to the expression of the Atiyah–Bott integration formula.

5.3. **Further remarks.** We note that the idea of localization and equivariant cohomology have been used in far more extensive situations than what was presented here. Tu gives a list of some examples in [22], and has also used equivariant cohomology in his original research [21] to compute the characteristic numbers of homogenous spaces. Vergne also gave a survey of the applications in ICM 2006 [23]. Analogous results in K-theory were discussed by Atiyah himself with Segal in [2] and used extensively in the study of equivariant index theory. In quantum physics, equivariant cohomology is also studied extensively in the context of supersymmetric quantum field theories, cohomological field theories, and BRST quantization of gauge theories. For a survey of the applications in physics, [20] and [9] are recommended. Ideas from physics has also influenced the study of equivariant cohomology in mathematics. For one example, Kalkman in [16] showed that the construction from physics can be used to interpolate between the Weil model and the Cartan model of equivariant de Rham theory. I hope that the article has convinced the reader that localization in equivariant cohomology is an interesting and important topic that appears in a wide range of subjects.

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