THE POINCARÉ-HOPF THEOREM

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ABSTRACT. In this paper, we will introduce the reader to the field of topology given a background of Calculus and Analysis. To familiarize the reader with topological concepts, we will present a proof of Brouwer's Fixed Point Theorem. The end result of this paper will be a proof of the Poincaré-Hopf Theorem, an important theorem equating the index of a vector field on a manifold, and the Euler characteristic, an invariant of the manifold itself. We will conclude this paper with some useful applications of the Poincaré-Hopf Theorem.

Contents

1. INTRODUCTION

Topology is the study of the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing. These informal terms might be reminiscent to a child playing with Play-Doh, or crafting an origami crane. Indeed, a lot of the fun of topology stems from the fun and intricate visualizations of objects forming and deforming in some larger space. But how does one describe these objects in real terms that would lead to logical conclusions with rigorous proofs?

Section 3 will provide some preliminary definitions to introduce the reader to the study of Topology. It concludes with a proof of Brouwer's Fixed Point Theorem. Section 4 explores further the idea of indices and degrees. The final theorem presented in Section 4 equates the sum of indices to the Index

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of the vector field. This provides the first half of the proof of the Poincaré-Hopf Theorem. Section 5 will finish proving the theorem by equating the sum of indices to the Euler Characteristic. Finally, in Section 6, we will explore some consequences of the Poincaré-Hopf Theorem.

2. Preliminary Definitions and Brouwer's Fixed Point Theorem

For first-timers venturing into homology, the journey starts with the idea of a Manifold.

Definition 2.1 A manifold, M, is a topological space that is locally Euclidean.

Definition 2.2 A diffeomorphism is a map between manifolds which is differentiable and has a differentiable inverse. Two manifolds are diffeomorphic if there exists a diffeomorphism between them.

Definition 2.3 A subset $M \subset \mathbb{R}^k$ is called a smooth manifold of dimension m if for any x in M, there exists a neighborhood $W \subseteq M$ such that W is diffeomorphic to an open set U with U diffeomorphic to $\mathbb{R}^m$.

Definition 2.4 Define the closed half-space to be

$$H^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m | x_m \geq 0\}.$$

The boundary ∂H^m is defined to be the hyperplane $R^{m-1} \times 0 \subset \mathbb{R}^m$. A subset $M \subset \mathbb{R}^k$ is a smooth m-manifold with boundary if each $x \in M$ has a neighborhood $U \cap M$ diffeomorphic to an open subset $V \cap H^m$ or H^m. The boundary of M, ∂M, is the set of all points in X which correspond to points of ∂H^m under such a diffeomorphism.

Next we can impose vector fields onto manifolds. We will need some terminology to effectively describe and analyze properties of vector fields.

Definition 2.5 Let $f : M \to N$ be smooth with $dim(M) = dim(N)$. Then x in M is a regular point of f if df is non-singular (i.e. invertible). In addition, $f(x)$ is called a regular value.

Definition 2.6 An orientation for a finite dimensional real vector space is an equivalence class of ordered bases as follows: the ordered basis $(b_1, \ldots, b_n)$ preserves the orientation of the basis $(b'_1, \ldots, b'_n)$ if $b'_i = \sum a_{ij} b_j$ with $det(a_{ij}) > 0$. It reverses orientation if $det(a_{ij}) < 0$.

Definition 2.7 Consider a smooth map $f : N^n \to M^n$ between compact, n-dimensional, oriented manifolds with M connected. For a regular value

$p \in M$ and $q \in f^{-1}(p)$, the local index is defined as follows:

$$Ind(f,q) = \begin{cases} 1 & \text{if } D_q f : T_q N \to T_p M \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

The degree of such a function, f, is a number that represents the number of times that the domain manifold wraps around the range manifold under the mapping. The notion of degree was first defined by Brouwer. It is more formally defined as follows:

Definition 2.8 Let $x \in M$ be a regular point of f such that $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism between oriented vector spaces. Define $sign(df_x)$ to be $+1$ if it preserves orientation and -1 if it reverses orientation. Then, for any regular value $y \in N$,

$$deg(f;y) = \sum_{x \in f^{-1}(y)} sign(df_x)$$

To gain familiarity with these concepts introduced by Brouwer, we will prove Brouwer's Fixed Point Theorem. There exist a handful of fixed point theorems in topology. Brouwer's specifically claims that every continuous map from the unit disk to itself must have a fixed point.

Definition 2.9 Given a function $f : M \to M$ with $x \in M$, x is called a fixed point if $f(x) = x$.

Lemma 2.10 *Let M be a compact m-manifold with boundary. There is no smooth map $f : M \to \partial M$ such that for any x in ∂M, $f(x) = x$*

Proof Suppose such a map existed, and call it f. Let $y \in \partial M$ be a regular value for f. Then, y must be a regular value for $f|\partial M$ which, by the assumption, is an identity map. It follows that $f^{-1}(y)$ is a smooth 1-manifold, with

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M = \{y\}.$$

But $f^{-1}(y)$ is also compact. By a claim proved in the appendix of [1], every compact 1-manifold is a finite disjoint union of circles and segments and must consist of an even number of points. $\partial f^{-1}(y)$ is a 1-manifold consisting of one point. We have reached a contradiction, thus proving the lemma.

Let us introduce the standard notion of the unit disk and sphere:

$$D^n = \{x \in \mathbb{R}^n | ||x|| \leq 1\}$$

$$S^{n-1} = \{x \in \mathbb{R}^n | ||x|| = 1\}$$

The unit disk is a compact manifold bounded by the unit sphere. Hence as a special case we have proved that *the identity map of S^{n-1} cannot be extended to a smooth map $D^n \to S^{n-1}$.*

Lemma 2.11 Any smooth map $g : D^n \to D^n$ has a fixed point.

Proof Suppose g has no fixed point. For $x \in D^n$, let $f(x) \in S^{n-1}$ be the point nearer x on the line through x and $g(x)$. The $f : D^n \to S^{n-1}$ is a smooth map with $f(x) = x$ for $x \in S^{n-1}$, which is impossible by Lemma 2.10.

Finally, we are ready to prove the Brouwer Fixed Point Theorem. We will utilize a procedure that is often helpful in proofs regarding continuous mappings. That is to prove the theorem for smooth maps, then use an approximation theorem to generalize to continuous maps.

Brouwer's Fixed Point Theorem *Any continuous function $G : D^n \to D^n$ has a fixed point.*

Proof As suggested, we will start by approximating G by a smooth mapping. Given $\epsilon > 0$, according to the Weirstrass approximation theorem, there is a polynomial function $P_1 : \mathbb{R}^n \to \mathbb{R}^n$ with $||P_1(x) - G(x)|| < \epsilon$ for $x \in D^n$. However, P_1 may send points of D^n into points outside of D^n. To correct this we set

$$P(x) = \frac{P_1(x)}{(1+\epsilon)}.$$

Then clearly P maps D^n into D^n and $||P(x) - G(x)|| < 2\epsilon$ for $x \in D^n$. Suppose that $G(x) \neq x$ for all $x \in D^n$. Then the continuous function $||G(x) - x||$ must take on a minimum $\mu > 0$ on D^n. Choosing $P : D^n \to D^n$ as above, with $||P(x) - G(x)|| < \mu$ for all x, we clearly have $P(x) \neq x$. Thus P is a smooth map from D^n to itself without a fixed point. This contradicts Lemma 2.11, and completes the proof.

The remainder of this paper will focus on proving the Poincaré-Hopf theorem.

The Poincaré-Hopf Theorem *Let X be a smooth vector field on a compact manifold M. If X has only isolated zeros then*

$$Index(X) = \chi(M)$$

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3. Degrees and Indices of Vector Fields

Definition 3.1 A smooth real-valued function on a manifold M is a Morse function if it has no degenerate critical points.

Definition 3.2 Given a Morse function f on a manifold M, a gradient-like vector field V for the function f satisfies the following two conditions:

(1) for every non-critical point $p \in M$, $d_p f(X(p)) > 0$.

(2) around every critical point, there is a neighborhood on which f is given as in the Morse lemmas:

$$f(v) = f(b) - v_1^2 - \ldots - v_\alpha^2 + v_{\alpha+1}^2 + \ldots + v_n^2$$

and on which V equals the gradient of f.

Example 3.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function $f(x) = c - x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$ where $c \in \mathbb{R}, \lambda \in \mathbb{Z}$ and $0 \leq \lambda \leq n$. Since $grad_x(f) = 2(-x_1, \ldots, -x_\lambda, -x_{\lambda+1}, \ldots, x_n)$, 0 is the only critical point of f. We find that

$$\frac{d^2 f}{dx_i dx_j}(0) = deg(-2, \ldots, -2, 2, \ldots, 2)$$

with exactly λ diagonal entries equal to -2. Thus the origin is non-degenerate of index λ. We note that the vector field $grad(f)$ has the origin as its only zero and thus it is non-degenerate of index $(-1)^\lambda$.

Lemma 3.4 *Let f be a Morse function on M and V a smooth tangent vector field such that $d_p f(V(p)) > 0$ for every $p \in M$ that is not a critical point for f. Let $p_0 \in M$ be a critical point for f of index λ. If $V(p_0) = 0$, then $i(V; p_0) = (-1)^\lambda$.*

Proof Let X be a gradient-like vector field of f. By Example 3.3, $i(X; p_0) = (-1)^\lambda$. Let U be an open neighborhood of p_0 that is diffeomorphic to $\mathbb{R}^n$ and chosen so small that p_0 is the only critical point in U. The inequalities $d_p f(V(p)) > 0$ and $d_p f(X(p)) > 0$ valid for $p \in U - p_0$, show that $V(p)$ and $X(p)$ belong to the same open half-space in $T_p M$. Thus

$$(1-t)V(p) + tX(p) \quad (0 \leq t \leq 1)$$

defines a homotopy between V and X considered as maps from $U - p_0$ to $\mathbb{R}^n - \{0\}$ and $i(V; p_0) = i(X; p_0)$.

Definition 3.5 The Gauss map is a function f from an oriented surface M in Euclidean space $\mathbb{R}^3$ to the unit sphere in $\mathbb{R}^3$.

Lemma 3.6 (Hopf) *If $V : X \to \mathbb{R}^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum $\sum i$ is equal to the degree of the Gauss mapping for ∂X to S^{m-1}. In particular, $\sum i$ does not depend on the choice of v.*

Proof Removing an ϵ-ball around each zero, we obtain a new manifold with boundary. The function $\bar{v} = \frac{v(x)}{||v(x)||}$ maps this manifold into S^{m-1}. Hence the sum of the degrees of $\bar{v}$ restricted to the various boundary components is zero. But $\bar{v}|\partial X$ is homotopic to g, and the degrees on the other boundary components add up to $-\sum i$. (The minus sign occurs since each small sphere gets the opposite orientation.) Therefore

$$deg(g) - \sum i = 0$$

as required.

Theorem 3.7 *Let $M \subset \mathbb{R}^k$ be compact and boundaryless, and N_ϵ denote the closed ϵ-neighborhood of M. Then for any vector field V on M with only non-degenerate zeros, the index sum $\sum i$ is equal to the degree of the Gauss mapping*

$$g : \partial N_\epsilon \to S^{k-1}.$$

In particular, this sum does not depend on the choice of vector field.

Proof Fix $x \in N_\epsilon$. Let $r(x) \in M$ denote the closest point of M to x. Note that the vector $x - r(x)$ must be perpendicular to the tangent space of M at $r(x)$, for $r(x)$ to be the closest point. If ϵ is sufficiently small, then the function $r(x)$ is smooth and well defined. Additionally, let

$$\phi(x) = ||x - r(x)||^2,$$

and note that

$$grad(\phi) = 2(x - r(x)).$$

Hence, for each point x of the level surface $\partial N_\epsilon = \phi^{-1}(\epsilon^2)$, the outward unit normal vector is given by

$$g(x) = grad\frac{grad(\phi)}{||grad(\phi)||} = \frac{x - r(x)}{\epsilon}.$$

Extend v to a vector field w on the neighborhood N_ϵ by setting

$$w(x) = (x - r(x)) + v(r(x)).$$

Then w points outward along the boundary, since the inner product $w(x) \cdot g(x)$ is equal to $\epsilon > 0$. Note that w can vanish only at the zeros of v in M; this is clear since the two summands $(x - r(x))$ and $v(r(x))$ are mutually orthogonal. Computing the derivative of w at a zero $z \in M$, we see that

$$dw_z(h) = dv_z(h) \text{ for all } h \in TM_z$$

$$dw_z(h) = h \text{ for } h \in TM_z^{\perp}.$$

Thus the determinant of dw_z is equal to the determinant of dv_z. Hence the index of w at the zero z is equal to the index i of v at z. Now according to Lemma 3.6 the index sum $\sum i$ is equal to the degree of g.

Theorem 3.8 *Let M^n be a compact differentiable manifold and X a smooth tangent vector field on M^n with isolated singularities. Let $f \in C^{\inf}(M, \mathbb{R})$ be a Morse functon and c_λ the number of critical points of index λ for f. Then we have that*

$$Index(X) = \sum_{\lambda=0}^{n} (-1)^\lambda c_\lambda.$$

Proof It is a consequence of Theorem 3.7 that any two tangent vector fields with isolated singularities have the same index. Thus we may assume that X is gradient-like for f. The zeros for X are exactly the critical points of f, and the claimed formula follows from Lemma 4.4.

4. MANIFOLDS

Definition 4.1 Let $M^n \subseteq \mathbb{R}^{n+k}$ denote a fixed smooth submanifold. If the cohomology of M^n is finite-dimensional (e.g. M^n is compact), then the ith Betti Number is given by $b_i(M) = dim_{\mathbb{R}} H^i(M^n)$.

Definition 4.2 The Euler characteristic of M^n is defined to be

$$\chi(M^n) = \sum_{i=0}^{n} (-1)^i b_i(M).$$

We will now show that the index sum $\sum i$ is also equal to the the Euler characteristic. We start by presenting two lemmas whose proofs involve methods from dynamical systems and ordinary differential equations beyond the scope of this paper, and thus will not be given.

First, fix a compact manifold M^n and a Morse function f on M. For $a \in \mathbb{R}$ we set

$$M(a) = \{p \in M | f(p) < a\}.$$

Lemma 4.3 *If there are no critical values in the interval $[a_1, a_2]$, then $M(a_1)$ and $M(a_2)$ are diffeomorphic.*

Lemma 4.4 *Suppose that a is a critical value and that $p_1, \ldots, p_r$ are the critical points in $f^{-1}(a)$. Let p_i have index λ_i. There exists an $\epsilon > 0$, and disjoint open neighborhoods U_i of p_i, such that*

(1) $p_1, \ldots, p_r$ *are the only critical points in* $f^{-1}([a-\epsilon, a+\epsilon])$.

(2) U_i *is diffeomorphic to an open contractible subset of* $\mathbb{R}^n$

(3) $U_i \cap M(a-\epsilon)$ *is diffeomorphic to* $S^{\lambda_i - 1} \times V_i$*, where* V_i *is an open contractible subset of* $\mathbb{R}^{n-\lambda_i+1}$ *(in particular* $U_i \cap M(a-\epsilon) = \emptyset$ *if* $\lambda_i = 0$*).*

(4) $M(a+\epsilon)$ *is diffeomorphic to* $U_1 \cup \ldots \cup U_r \cup M(a-\epsilon)$.

Lemma 4.5 *Let* U *and* V *be open subsets of a smooth manifold. If* U, V *and* $U \cap V$ *have finite dimensional de Rham cohomology, the same is true for* $U \cup V$*, and*

$$\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Proof We use the long exact Mayer-Vietoris sequence

$$\ldots \to H^{p-1}(U \cap V) \to H^p(U \cup V) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to \ldots$$

First we conclude that $dim H^p(U \cup V) < \infty$. Second, the alternating sum of the dimensions of the vector spaces in an exact sequence is equal to zero.

Lemma 4.6 *If* $U \subseteq \mathbb{R}^n$ *is an open contractible set, then* $H^p(U) = 0$ *when* $p > 0$ *and* $H^0(U) = \mathbb{R}$.

Proposition 4.7 *Suppose* $M(a-\epsilon)$ *has finite-dimensional cohomology. Then the same will be true for* $M(a+\epsilon)$*, and*

$$\chi(M(a+\epsilon)) = \chi(M(a-\epsilon)) + \sum_{i=1}^{r} (-1)^{\lambda_i}.$$

Proof For $U = U_1 \cup \ldots \cup U_r$, Lemma 4.4 and Lemma 4.6 suggest that

$$H^p(U_i) \simeq \begin{cases} 0 & \text{if } p \neq 0 \\ \mathbb{R} & \text{if } p = 0. \end{cases}$$

This gives $\chi(U) = r$ since the U_i are disjoint. Lemma 4.3 shows that $U_i \cap M(a-\epsilon)$ is homotopy equivalent to $S^{\lambda_i - 1}$, and

$$\chi(U_i \cap M(a-\epsilon)) = 1 + (-1)^{\lambda_i - 1}.$$

Since $U \cap M(a-\epsilon)$ is a disjoint union of the sets $U_i \cap M(a-\epsilon)$, it has a finite-dimensional de Rham cohomology, and

$$\chi(U \cap M(a-\epsilon)) = \sum_{i=1}^{r} (1 + (-1)^{\lambda_i - 1}) = r - \sum_{i=1}^{r} (-1)^{\lambda_i} = \chi(U) - \sum_{i=1}^{r} (-1)^{\lambda_i}.$$

The claimed formula now follows from Lemma 4.3 and Lemma 4.5 applied to U and $V = M(a-\epsilon)$.

Theorem 4.8 *If f is a Morse function on the compact manifold M^n, then*

$$\chi(M^n) = \sum_{\lambda=0}^{n} (-1)^{\lambda} c_{\lambda},$$

where c_λ denotes the number of critical points for f of index λ.

Proof Let $a_1 < a_2 < \ldots < a_{k-1} < a_k$ be the critical values. Choose real numbers $b_0 < a_1$, $b_j \in (a_j, a_{j+1})$ for $1 \leq j \leq k-1$ and $b_k > a_k$. Lemma 4.3 shows that the dimensions of $H^d(M(b_j))$ are independent of the choice of b_j from the relevant interval. If $M(b_{j-1})$ has finite-dimensional de Rham cohomology, the same will be true for $M(b_j)$ according to Proposition 5.7, and

$$\chi(M(b_j)) - \chi(M(b_{j-1})) = \sum_{p \in f^{-1}(a_j)} (-1)^{\lambda(p)} \tag{1}$$

Here the sum runs over the critical points $p \in f^{-1}(a_j)$ and $\lambda(p)$ denotes the index of p. We can start from $M(b_0) = \emptyset$. An induction shows that $dim H^d(M(b_j)) < \infty$ for all j and d. The sum of the formulas of (1) for $1 \leq j \leq k$ gives

$$\chi(M) = \chi(M(b_k)) \sum_{p} (-1)^{\lambda(p)}$$

where p runs over the critical points.

The Poincaré-Hopf Theorem follows immediately follows from Theorem 3.8 and Theorem 4.8.

5. Applications and Consequences

Theorem 5.1 *If M^n is compact and of odd dimension n then $\chi(M^n) = 0$.*

Proof Let f be a Morse function on M. Then $-f$ is also a Morse function, and $-f$ has the same critical points as f. If a critical point p has index λ with respect to f, then p has index $n - \lambda$ with respect to $-f$. Theorem 4.8 applied to both f and $-f$ gives

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^{\lambda} c_{\lambda} = \sum_{\lambda=0}^{n} (-1)^{n-\lambda} c_{\lambda}.$$

The two sums differ by the factor $(-1)^n$, and the assertion follows.

The next theorem is commonly referred to as The Hairy Ball Theorem. It was given this comical name because it is often introduced through the analogy of combing a hairy ball. More specifically, it is impossible to comb a hairy ball such that there is no hair sticking straight up.

Theorem 5.2 *A smooth vector field on an even dimensional sphere must have at least one zero vector.*

Proof This follows directly from the Poincaré-Hopf Theorem. If there were no zero vectors, index sum would equal 0. However, the Euler Characteristic of an even dimensional sphere is 2. By the Poincaré-Hopf Theorem, the index sum must also equal zero. Thus, there exists at least one zero vector.

Note that Theorem 5.2 specifies the sphere must be of even dimension. This is consistent with Theorem 5.1 which implies that the sum of indices must also be zero. This makes a non-zero vector field possible on a sphere of odd dimension.

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