Finite spaces and larger contexts

J. P. May

Contents

Introduction

A finite space is a topological space that has only finitely many points. At first glance, it seems ludicrous to think that such spaces can be of any interest. In fact, from the point of view of homotopy theory, they are equivalent to finite simplicial complexes. Therefore they support the entire range of invariants to be found in classical algebraic topology. For a striking example that sounds like nonsense, there is a space with six points and infinitely many non-zero homotopy groups. That is like magic: it sounds impossible until you know the trick, when it becomes obvious. We usually restrict attention to finite T_0-spaces, and those are precisely equivalent to finite posets (partially ordered sets). Therefore finite spaces are also of interest in combinatorics. In fact, there is a large and growing literature about finite spaces and their role in other areas of mathematics and science.

My own interest in the subject was aroused by 1966 papers by McCord [**32**] and Stong [**40**] that are the starting point of this book. However, I should admit that I came upon these papers while casting about for material to teach in Chicago's large scale REU, which I organize and run. I wanted something genuinely fascinating, genuinely deep, and genuinely accessible, with lots of open problems. Finite spaces provide a perfect REU topic for an algebraic topologist. Most experts in my field know nothing at all about finite spaces, so the material is new even to the experts, and yet it really is accessible to smart undergraduates. This book will feature several contributions made by undergraduates, some from Chicago's REU and some not.

When I first started talking about finite spaces, in the summer of 2003, my interest had nothing at all to do with my own areas of research, which seemed entirely disjoint. However, it has gradually become apparent that finite spaces can be integrated seamlessly into a global picture of how posets, simplicial complexes, simplicial sets, topological spaces, small categories, and groups are interrelated by a web of adjoint pairs of functors with homotopical meaning. The undergraduate may shudder at the stream of undefined terms!

The intention of this book is to introduce the algebraic topology of finite topological spaces and to integrate that topic into an exposition of a global view of a large swathe of modern algebraic topology that is accessible to undergraduates and yet has something new for the experts. A slogan of our REU is that "all concepts will be carefully defined", and we will follow that here. However, proofs will be selective. We aim to convey ideas, not all of the details. When the results are part of the mainstream of other subjects (group theory, combinatorics, point-set topology, and algebraic topology) we generally quote them. When they are particular to our main topics and not to be found on the textbook level, we give complete details.

These notes started out entirely concretely, without even a mention of things like categories or simplicial sets. Chicago students won't stand for oversimplification, and their questions always led me into deeper waters than I intended. They were also impatient with the restriction to finite spaces and finite simplicial complexes, one reason being that as soon as their questions forced me to raise the level of discourse, the restriction to finite things seemed entirely unnatural to them.

The infinite version of finite topological spaces is readily defined and goes back to a 1937 paper of Alexandroff [**1**]. We call these spaces Alexandroff spaces, and we use the abbreviation A-space for Alexandroff T_0-space. (The T_0 property means that the topology distinguishes points.) To go along with this, we also use the abbreviation F-space for finite T_0-space. Just as F-spaces are equivalent to finite posets, so A-spaces are equivalent to general posets. Similarly, from the point of view of homotopy theory, F-spaces are equivalent to finite simplicial complexes and A-spaces are equivalent to general simplicial complexes.

Roughly speaking, the first part of the book focuses on the homotopy theory of F-spaces and A-spaces. A central theme is the difference between weak homotopy equivalences and homotopy equivalences. A continuous map $f\colon X \longrightarrow Y$ is a homotopy equivalence if there is a map $g\colon Y \longrightarrow X$ such that the composite $g \circ f$ is homotopic to the identity map of X and the composite $f \circ g$ is homotopic to the identity map of Y. The map f is a weak homotopy equivalence (usually abbreviated to weak equivalence) if for every choice of basepoint $x \in X$ and every $n \geq 0$, the induced map $f_*\colon \pi_n(X,x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism (of sets if $n = 0$, of groups if $n = 1$, and of abelian groups if $n \geq 2$).

Every homotopy equivalence is a weak homotopy equivalence. A map between nice spaces, namely CW complexes, is a homotopy equivalence if it is a weak homotopy equivalence. All of the spaces that one encounters in standard introductions to algebraic topology are nice, so that the distinction seems parenthetical and of minor interest. It is by now very well understood by algebraic topologists that the definitively "right" notion of equivalence is weak equivalence, not homotopy equivalence. However, to get a feel for the strength of the distinction, one needs to see serious examples where the two notions are genuinely different.

The first half of the book offers just such a perspective. The work of Stong makes it very easy to understand homotopy equivalences of finite spaces. The work of McCord relates weak equivalences of Alexandroff spaces to weak equivalences, and therefore homotopy equivalences, of simplicial complexes. As we shall explain, a reinterpretation in terms of finite spaces of a conjecture of Quillen about the poset of non-trivial elementary subgroups of a finite group illuminates precisely this distinction between weak homotopy equivalences and actual homotopy equivalences. Another open problem also illuminates the distinction. The problem of enumerating homotopy equivalences of finite spaces combinatorially has been solved by a pair of Chicago undergraduates, Alex Fix and Stephan Patrias. The problem of enumerating weak homotopy equivalences combinatorially is still open.

The second half of the book guides the reader through the following oversimplified diagram of categories and functors between them.

$$\begin{tikzcd}
Groups \arrow[r, shift left, "{K(-,1)}"] \arrow[d, "i"'] & Spaces \arrow[l, shift left, "\pi_1"] \arrow[d, shift left, "S"] & \\
Categories \arrow[ur, "B"] \arrow[r, shift left, "N"] \arrow[d, shift left, "Sd^2"] & Simp.\ Sets \arrow[l, shift left, "\Pi"] \arrow[u, shift left, "T"] \arrow[r, dashed, shift left, "Sd^2"] & Ordered\ Simp.\ Complexes \arrow[l, shift left, "i"] \arrow[d, "U"] \\
Posets \arrow[u, shift left, "i"] \arrow[r, shift left, "\cong"] & A\text{-}Spaces \arrow[l, shift left, "\cong"] \arrow[ur, "\mathscr{K}"] & Abstract\ Simp.\ Complexes \arrow[l, "\mathscr{X}"]
\end{tikzcd}$$

The connections among these categories are remarkably close. It has been understood since the 1950's that topological spaces and simplicial sets can in principle be used interchangeably in the study of homotopy theory. In fact, except that groups only model very special spaces, called $K(\pi, 1)$'s, all of these categories can in principle be used interchangeably in the study of homotopy theory. We'd like people outside algebraic topology to become more aware of these interconnections.

One thing that is largely new is a careful combinatorial analysis of exactly how subdivision ties together the categories of simplicial sets, (small) categories, and posets, alias A-spaces. This is due in large part to Rina Foygel, a recent Chicago PhD and now faculty member in Statistics , and her work is included with her permission. In particular, we give a careful explanation of the classical result that the second subdivision of a suitably well-behaved simplicial set is a simplicial complex and the folklore result that the second subdivision of any (small) category is a poset. One striking result is that, when regarded as a simplicial set, any classical (ordered) simplicial complex is the nerve of a category. As far as I know, that has never before been noticed. We ask the novice not to be intimidated. We will go slow! We ask the expert to be patient. There will be new things along the way.

There are all sorts of possible choices of material and presentation for a book on this general topic, and I'll explain, but not justify, my choices rather flippantly. The main justification is that the REU is supposed to be fun, and so is this book.

It is a standard saying that one picture is worth a thousand words. It is a defect of the author that he is not good at drawing pictures, and there will not be as many as there should be. The reader should draw lots of them! In mathematics, it is perhaps fair to say that one good definition is worth a thousand calculations. The author likes to make up definitions and to see relations between seemingly unrelated concepts, so we will do lots of that.

However, to quote a slogan from a T-shirt worn by one of the author's students, "calculation is the way to the truth". There is a need for more calculational understanding of the subject here, and the author, being too old and lazy to compute himself, hopes that readers will be inspired.

In fact, the author's notes on this subject have been online since 2003, and a number of people have been inspired by them. In particular, Gabriel Minian, in Buenos Aires, and his students have followed up problems in my notes. His student Jonathan Barmak wrote a 2009 thesis, now a book [**5**], that has a good deal of overlap with the first half of this book.[1] I'll content myself with the basic theory and refer to Barmak's book for more recent advances made in Argentina.

Pedagogically, I've been using this material as a device to offer beginning undergraduates capsule introductions to point-set topology, algebraic topology, and category theory. I've also used the evolution of concepts as a means to help students

[1] I'll quote from his introduction. "In 2003, Peter May writes a series of unpublished notes in which he synthesizes the most important ideas on finite spaces until that time. In these articles, May also formulates some natural and interesting questions and conjectures which arise from his own research. May was one of the first to note that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack algebraic topology problems using finite spaces. Those notes came to the hands of my PhD advisor Gabriel Minian, who proposed me to work on this subject. May's notes and problems, jointly with Stong's and McCord's papers, were the starting point of our research on the Algebraic Topology of Finite Topological Spaces and Applications."

gain an intuition for abstraction and conceptualization in modern mathematics.[2] These twin purposes pervade and guide the exposition.

Add comments on REU paper contributions!

[2] An advertisement for just such a use of the subject of finite spaces as a pedagogical tool has been published by two students of a student of mine [**19**].

Part 1

Alexandroff spaces, posets, and simplicial complexes

CHAPTER 1

Alexandroff spaces and posets

1.1. The basic definitions of point set topology

The intuitive notion of a set in which there is a prescribed description of nearness of points is obvious. So is the intuitive notion of a function that takes nearby points to nearby points. However, formulating the "right" general abstract notion of what a "topology" on a set should be and what a "continuous map" between topological spaces should be is not so obvious. Since, intuitively, nearness is thought of in terms of distance, the most immediate way to make the intuition precise is to use distance functions. That leads to metric spaces and the ε-δ description of continuity, which is how we usually think of spaces and maps. Hausdorff came up with a much more abstract and general notion that is now universally accepted.

DEFINITION 1.1.1. A *topology* on a set X consists of a set $\mathscr{U}$ of subsets of X, called the "open sets of X in the topology $\mathscr{U}$", with the following properties.

(i) The empty set $\emptyset$ and the set X are in $\mathscr{U}$.
(ii) A finite intersection of sets in $\mathscr{U}$ is in $\mathscr{U}$.
(iii) An arbitrary union of sets in $\mathscr{U}$ is in $\mathscr{U}$.

A *neighborhood* of a point $x \in X$ is an open set U such that $x \in U$.

We write $(X, \mathscr{U})$ for the set X with the topology $\mathscr{U}$. More usually, when the topology $\mathscr{U}$ is understood, we just say that X is a topological space. We say that a topology $\mathscr{U}$ is *finer* than a topology $\mathscr{V}$ if every set in $\mathscr{V}$ is also in $\mathscr{U}$ ($\mathscr{U}$ has more open sets). We then say that $\mathscr{V}$ is *coarser* than $\mathscr{U}$. We have two obvious and uninteresting topologies on any set X.

DEFINITION 1.1.2. The *discrete topology* on X is the topology in which all sets are open. It is the finest topology on X. The *trivial* or *coarse* or *indiscrete* topology on X is the topology in which $\emptyset$ and X are the only open sets. It is the coarsest topology on X. We write D_n and C_n for the discrete and coarse topologies on a set with n elements. These are the largest and the smallest possible topologies (in terms of the number of open subsets).

DEFINITION 1.1.3. Let X be a topological space. A subset of X is *closed* if its complement is open. The closed sets satisfy the following conditions.

(i) The empty set $\emptyset$ and the set X are closed.
(ii) An arbitrary intersection of closed sets is closed.
(iii) A finite union of closed sets is closed.

We shall make little or no use of the following definition, but it may help make clear how the abstract definitions correspond to common notions in calculus.

DEFINITION 1.1.4. Let A be a subset of a topological space X. The *interior* $\mathring{A}$ of A is the union of the open subsets of X contained in A. The *closure* $\bar{A}$ of A is the intersection of the closed sets containing A. A point $x \in X$ is a *limit point* of A if every neighborhood of x contains a point $a \neq x$ of A. A is *dense* in X if $\bar{A} = X$.

We shall omit proofs of many standard results that are part of basic point-set topology, such as the next one. While this result is not too hard and can safely be left as an exercise, other omitted proofs will be more substantial. This is not a textbook and we do not aspire to completeness.

PROPOSITION 1.1.5. *A point $x \in X$ is in $\bar{A}$ if and only if every neighborhood of x contains a point of A, and $\bar{A}$ is the union of A and the set of limit points of A. The set A is closed if and only if it contains all of its limit points.*

1.2. Alexandroff and finite spaces

It is very often interesting to see what happens when one takes a standard definition and tweaks it a bit. The following tweaking of the notion of a topology is due to Alexandroff [**1**], except that he used a different name for the notion[1].

DEFINITION 1.2.1. A topological space X is an *Alexandroff space* if the set $\mathscr{U}$ is closed under arbitrary intersections, not just finite ones.

REMARK 1.2.2. The notion of an Alexandroff space has a pleasing complementarity. If X is an Alexandroff space, then the closed subsets of X give it a new topology in which it is again an Alexandroff space. We write X^{op} for X with this opposite topology. Then $(X^{op})^{op}$ is the space X back again.

A space is *finite* if the set X is finite, and the following observation is immediate.

LEMMA 1.2.3. *A finite space is an Alexandroff space.*

It turns out that a great deal of what can be proven for finite spaces applies equally well more generally to Alexandroff spaces, with exactly the same proofs. When that is the case, we will prove the more general version. However, finite spaces have recently captured people's attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, finite topological spaces seem a natural tool in many such scientific applications. There are quite a few papers on the subject, although few of much mathematical depth, starting from the 1980's.

There was a brief early flurry of beautiful mathematical work on this subject. Two independent papers, by McCord and Stong [**32, 40**], both published in 1966, are especially interesting. We will work through them. We are especially interested in questions that are raised by the union of these papers but are answered in neither. These questions have only recently been pursued. We are also interested in calculational questions about the enumeration of finite topologies.

There is a hierarchy of "separation properties" on spaces, and intuition about finite spaces is impeded by too much habituation to the stronger of them.

DEFINITION 1.2.4. Let $(X, \mathscr{U})$ be a topological space.

(i) X is a T_0-*space* if for any two points of X, there is an open neighborhood of one that does not contain the other. That is, the topology distinguishes points.
(ii) X is a T_1-*space* if each point of X is a closed subset.

[1] His name was Diskrete Räume, which translates as discrete spaces.

(iii) X is a T_2-*space*, or *Hausdorff space*, if any two points of X have disjoint open neighborhoods.[2]

LEMMA 1.2.5. *If X is a T_2-space, then it is a T_1-space. If X is a T_1-space, then it is a T_0-space.*

There are still stronger separation properties, as summarized in §13.3 below. In most of topology, the spaces considered are at least Hausdorff. For example, metric spaces are Hausdorff. We discuss them briefly in §13.1. It is commonplace to use the following property.

PROPOSITION 1.2.6. *Let A be a subset of a Hausdorff space X and let $x \in X$. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points in A.*

Obviously, intuition gained from thinking about Hausdorff spaces is likely to be misleading when thinking about finite spaces! In fact, there are no interesting spaces that are both Alexandroff and T_1, let alone T_2.

LEMMA 1.2.7. *If an Alexandroff space is T_1, then it is discrete.*

PROOF. Every subset of any set is the union of its subsets with a single element. In an Alexandroff space, all unions of closed subsets are closed. In a T_1-space, all singleton subsets are closed. If both of these conditions hold, every subset is closed. Therefore every subset is open. □

In contrast, Alexandroff T_0-spaces are very interesting. The following warm-up problem might seem a bit difficult right now, but its solution will shortly become apparent.

EXERCISE 1.2.8. Show that a finite T_0-space has at least one point which is a closed subset.

NOTATION 1.2.9. As in the introduction, we define an F-space to be a finite T_0-space and an A-space to be an Alexandroff T_0-space.

1.3. Bases and subbases for topologies

Alexandroff spaces have canonical minimal bases, which we describe in this section. We first recall the notions of a basis and a subbasis for a topology. The idea is that one often has a preferred collection of "small" or canonical open sets, a"basis" from which all other open sets are generated.

DEFINITION 1.3.1. A *basis* for a topology on a set X is a set $\mathscr{B}$ of subsets of X such that

(i) For each $x \in X$, there is at least one $B \in \mathscr{B}$ such that $x \in B$.
(ii) If $x \in B' \cap B''$ where $B', B'' \in \mathscr{B}$, then there is at least one $B \in \mathscr{B}$ such that $x \in B \subset B' \cap B''$.

The topology $\mathscr{U}$ *generated* by the basis $\mathscr{B}$ is the set of subsets U such that, for every point $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. Equivalently, a set U is in $\mathscr{U}$ if and only if it is a union of sets in $\mathscr{B}$.

---

[2] The terminology is due to a 1935 paper of Alexandroff and Hopf [2]. The German word for separation is "Trennung", hence the letter T for the hierarchy of separation properties.

In the definition, we did not assume that we started with a topology on X. If we do start with a given topology $\mathscr{U}$, then it usually admits many different bases. We can easily characterize which subsets of $\mathscr{U}$ give bases.

LEMMA 1.3.2. *Let $(X,\mathscr{U})$ be a topological space. A subset $\mathscr{B}$ of $\mathscr{U}$ is a basis that generates $\mathscr{U}$ if and only if for every $U \in \mathscr{U}$ and every $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.*

We can generate bases for topologies from subbases.

DEFINITION 1.3.3. A *subbasis* for a topology on a set X is a set $\mathscr{S}$ of open subsets of X whose union is X; that is, $\mathscr{S}$ is a *cover* of X. The set of finite intersections of sets in $\mathscr{S}$ is the basis generated by $\mathscr{S}$. If $(X,\mathscr{U})$ is a topological space, a subbasis $\mathscr{S}$ for the topology $\mathscr{U}$ is a subset of $\mathscr{U}$ such that every set in $\mathscr{U}$ is a union of finite intersections of sets in $\mathscr{S}$.

EXAMPLE 1.3.4. The set of singleton sets $\{x\}$ is a basis for the discrete topology on X. The set of open balls $B(x,r) = \{y|d(x,y) < r\}$ is a basis for the topology on a metric space X.

Returning to Alexandroff spaces, we find that such a space has a canonical basis which is minimal in the strong sense that the open sets in the canonical basis are open sets in *any* basis for the topology on X.

DEFINITION 1.3.5. Let X be an Alexandroff space. For $x \in X$, define U_x to be the intersection of the open sets that contain x. Define a relation $\leq$ on the set X by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Write $x < y$ if the inclusion is proper.

LEMMA 1.3.6. *The set of open sets U_x is a basis $\mathscr{B}$ for X. If $\mathscr{C}$ is any other basis, then $\mathscr{B} \subset \mathscr{C}$. Therefore $\mathscr{B}$ is the unique minimal basis for X.*

PROOF. The first statement is clear from the definitions. If $\mathscr{C}$ is another basis and $x \in X$, then there is a $C \in \mathscr{C}$ such that $x \in C \subset U_x$. This implies that $C = U_x$, so that $U_x \in \mathscr{C}$. $\square$

We can detect whether or not an Alexandroff space is T_0 in terms of its minimal basis.

LEMMA 1.3.7. *Two points x and y in X have the same neighborhoods if and only if $U_x = U_y$. Therefore X is T_0 if and only if $U_x = U_y$ implies $x = y$.*

PROOF. If x and y have the same neighborhoods, then obviously $U_x = U_y$. Conversely, suppose that $U_x = U_y$. If $x \in U$ where U is open, then $U_y = U_x \subset U$ and therefore $y \in U$. Similarly if $y \in U$, then $x \in U$. Thus x and y have the same neighborhoods. $\square$

1.4. Operations on spaces

There are many standard operations on spaces that we shall have occasion to use. We record four of them now and will come back to others later.

DEFINITION 1.4.1. The *subspace topology* on $A \subset X$ is the set of all intersections $A \cap U$ for open sets U of X.

Subspace topologies are defined for injective functions. There is a perhaps less intuitive analogue for surjective functions.

Definition 1.4.2. Let X be a topological space and $q : X \longrightarrow Y$ be a surjective function. The *quotient topology* on Y is the set of subsets U such that $q^{-1}(U)$ is open in X.

Definition 1.4.3. The *topology of the union* on the disjoint union $X \amalg Y$ has as open sets the unions of an open set of X and an open set of Y. More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the topology of the union on the disjoint union $\coprod_{i\in I} X_i$ has as open sets the unions of open sets $U_i \subset X_i$. Note that a subset is closed if and only if it intersects each X_i in a closed subset.

Definition 1.4.4. The *product topology* on the cartesian product $X \times Y$ is the topology with basis the products $U \times V$ of an open set U in X and an open set V in Y. More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the product topology on the product set $\prod_{i\in I} X_i$ is the topology generated by the basis consisting of all products $\prod_{i\in I} U_i$ where U_i is open in X_i and $U_i = X_i$ for all but finitely many i.

There is a consistency observation relating the subspace and product topologies.

Proposition 1.4.5. *If $A \subset X$ and $B \subset Y$, then the subspace and product topologies on $A \times B \subset X \times Y$ coincide.*

For Hausdorff spaces, we have the following observations, which make good exercises.

Proposition 1.4.6. *A space X is Hausdorff if and only if the diagonal subspace $\{(x, x)\} \subset X \times X$ is closed.*

Proposition 1.4.7. *A subspace of a Hausdorff space is Hausdorff. A quotient of a Hausdorff space need* not *be Hausdorff. A disjoint union of Hausdorff spaces is Hausdorff. Any product of Hausdorff spaces is Hausdorff.*

We leave it as another good exercise to verify the following analogue for Alexandroff spaces.

Proposition 1.4.8. *A subspace of an Alexandroff space is an Alexandroff space. A quotient of an Alexandroff space is an Alexandroff space. A disjoint union of Alexandroff spaces is an Alexandroff space. A product of finitely many Alexandroff spaces is an Alexandroff space.*

Here is a thought exercise for the reader.

Problem 1.4.9. *Is the product of infinitely many Alexandroff spaces an Alexandroff space?*

1.5. Continuous functions and homeomorphisms

Definition 1.5.1. Let X and Y be spaces. A function $f : X \longrightarrow Y$ is *continuous* if $f^{-1}(U)$ is open in X for all open subsets U of Y. A continuous function is often called a *map*.

It suffices that $f^{-1}(U)$ be open for each U in a basis for the topology on Y, or even for each U in a subbasis. The reader is encouraged to use that to verify that the abstract definition of continuity just given coincides with the usual ε-δ definition of continuity on metric spaces; see §13.1. By passage to complements, a function f is continuous if and only if $f^{-1}(C)$ is closed in X for all closed subsets C of Y. This can be reinterpreted in terms of closures (and thus in terms of limit points).

LEMMA 1.5.2. *A function $f : X \longrightarrow Y$ is continuous if and only if, for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.*

LEMMA 1.5.3. *Let A be a subspace of a space X. A continuous function from A to a Hausdorff space Y admits at most one extension to a continuous map $\bar{A} \longrightarrow Y$.*

Identity functions and composites of continuous functions are continuous.

LEMMA 1.5.4. *Let X be a space, let $A \subset X$, and give A the subspace topology. Then the inclusion $i : A \longrightarrow X$ is a continuous function. If B is a space and $j : B \longrightarrow A$ is a function such that $i \circ j$ is continuous, then j is continuous.*

LEMMA 1.5.5. *Let X be a space, let $q : X \longrightarrow Y$ be a surjective function, and give Y the quotient topology. Then q is a continuous function. If Z is a space and $r : Y \longrightarrow Z$ is a function such that $r \circ q$ is continuous, then r is continuous.*

LEMMA 1.5.6. *Let X_i be spaces and let $\iota_i\colon X_i \longrightarrow \coprod X_i$ be the inclusion. Then ι_i is a continuous function. If Z is a space and $\eta_i\colon X_i \longrightarrow Z$ are continuous functions, then the unique function $\coprod X_i \longrightarrow Z$ that restricts to η_i on X_i is continuous.*

LEMMA 1.5.7. *Let X_i be spaces and let $\pi_i : \prod_i X_i \longrightarrow X_i$ be the projection. Then π_i is a continuous function. If Y is a space and $\rho_i : Y \longrightarrow X_i$ are continuous functions, then the unique function $Y \longrightarrow \prod X_i$ with i^{th} coordinate ρ_i is continuous.*

The four previous propositions state that the subspace, quotient, union, and product topologies satisfy certain "universal properties". In each of these results, the specified topology is the only topology for which the last statement is true.

Continuity is a local condition on a function.

LEMMA 1.5.8. *A function $f : X \longrightarrow Y$ is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

LEMMA 1.5.9. *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in an open cover of X is continuous.*

There is an analogue for finite closed covers.

LEMMA 1.5.10. *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in a finite closed cover of X is continuous.*

In particular, if $X = A \cup B$ where A and B are closed subsets of X, then continuous functions $A \longrightarrow Y$ and $B \longrightarrow Y$ that agree on $A \cap B$ induce a continuous function $X \longrightarrow Y$.

DEFINITION 1.5.11. A continuous bijection $f : X \longrightarrow Y$ is a *homeomorphism* if its inverse f^{-1} is also continuous. That is, a homeomorphism is a continuous bijection with a continuous inverse. Equivalently, a map $f\colon X \longrightarrow Y$ is a homeomorphism if there is a map $g\colon Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. An *inclusion* or *embedding* is a continuous injection that is a homeomorphism onto its image. We write $X \cong Y$ to indicate that X is homeomorphic to Y.

Intuitively, homeomorphism is the topological counterpart of the algebraic notion of isomorphism. Topologists are interested in properties of spaces that are invariant under homeomorphism. We shall later (Lemma 1.7.1, Theorem 13.2.7) give conditions on X and Y that ensure that a continuous bijection is a homeomorphism.

1.6. Alexandroff spaces, preorders, and partial orders

Here we relate Alexandroff spaces to the combinatorial notions of preorder and partial order.

DEFINITION 1.6.1. A *preorder* on a set X is a reflexive and transitive relation, denoted $\leq$. This means that $x \leq x$ and that $x \leq y$ and $y \leq z$ imply $x \leq z$. A preorder is a *partial order* if it is antisymmetric, which means that $x \leq y$ and $y \leq x$ imply $x = y$. Then $(X, \leq)$ is called a *poset*. A poset is totally ordered if for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Recall from Definition 1.3.5 that, in an Alexandroff space X, $x \leq y$ means that $U_x \subset U_y$.

LEMMA 1.6.2. *The relation $\leq$ on an Alexandroff space X is reflexive and transitive, so that the relation $\leq$ is a preorder. The relation is also antisymmetric, so that $(X, \leq)$ is a poset, if and only if the space X is T_0.*

PROOF. The first statement is clear and the second holds by Lemma 1.3.7. □

LEMMA 1.6.3. *A preorder $(X, \leq)$ determines a topology $\mathscr{U}$ on X with basis the set of all sets $U_x = \{y | y \leq x\}$. It is called the order topology on X. The space $(X, \mathscr{U})$ is an Alexandroff space. It is a T_0-space if and only if $(X, \leq)$ is a poset.*

PROOF. If $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$, hence $x \in U_x \subset U_y \cap U_z$. Therefore $\{U_x\}$ is a basis for a topology. The intersection U of a set $\{U_i\}$ of open subsets is open since if $x \in U$, then $U_x \subset U_i$ for each i and therefore U is the union of these U_x. Therefore $(X, \mathscr{U})$ is an Alexandroff space with minimal basis $\{U_x\}$. Since $U_x = U_y$ if and only if $x \leq y$ and $y \leq x$, Lemma 1.3.7 implies that $(X, \mathscr{U})$ is T_0 if and only if $(X, \leq)$ is a poset. □

We put things together to obtain the following conclusion.

PROPOSITION 1.6.4. *For a set X, the Alexandroff space topologies on X are in bijective correspondence with the preorders on X. The topology $\mathscr{U}$ corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.*

REMARK 1.6.5. If $\leq$ is a preorder on X, the opposite preorder is given by $x \leq^{op} y$ if and only if $y \leq x$. The corresponding Alexandroff space is X^{op}.

The real force of the comparison between Alexandroff spaces and preorders comes from the fact that continuous maps correspond precisely to order-preserving functions.

DEFINITION 1.6.6. Let X and Y be preorders. A function $f\colon X \longrightarrow Y$ is order-preserving if $w \leq x$ in X implies $f(w) \leq f(x)$ in Y.

LEMMA 1.6.7. *A function $f\colon X \longrightarrow Y$ between Alexandroff spaces is continuous if and only if it is order preserving.*

PROOF. Let f be continuous and suppose $w \leq x$. Then $w \in U_x \subset f^{-1}U_{f(x)}$ and thus $f(w) \in U_{f(x)}$. This means that $f(w) \leq f(x)$. For the converse, let f be order preserving and let V be open in Y. If $f(x) \in V$, then $U_{f(x)} \subset V$. If $w \in U_x$, then $w \leq x$ and thus $f(w) \leq f(x)$ and $f(w) \in U_{f(x)} \subset V$, so that $w \in f^{-1}(V)$. Thus $f^{-1}(V)$ is the union of these U_x and is therefore open. □

1.7. Finite spaces and homeomorphisms

In this section we specialize the theory above to finite spaces. Thus let X be a finite space and write $|X|$ for the number of points in X. One might think that finite spaces are uninteresting since they are just finite preorders in disguise, but that turns out to be far from the case.

Topologists are only interested in spaces up to homeomorphism, and we proceed to classify finite spaces up to homeomorphism.

LEMMA 1.7.1. *A map $f\colon X \longrightarrow X$ is a homeomorphism if and only if f is either one–to–one or onto.*

PROOF. By finiteness, one–to–one and onto are equivalent. Assume they hold. Then f induces a bijection 2^f from the set 2^X of subsets of X to itself. Since f is continuous, if $f(U)$ is open, then so is U. Therefore the bijection 2^f must restrict to a bijection from the topology $\mathscr{U}$ to itself. Alternatively, observe that the function f is a permutation of the set X and the set of permutations of X is a finite group. Therefore f^n is the identity for some n, and the continuous function f^{n-1} is f^{-1}. □

The previous lemma fails if we allow different topologies on X: there are continuous bijections between different topologies. We proceed to describe how to enumerate the distinct topologies up to homeomorphism. We say that two topologies $\mathscr{U}$ and $\mathscr{V}$ on X are equivalent if there is a homeomorphism $(X,\mathscr{U}) \longrightarrow (X,\mathscr{V})$. There are quite a few papers on this enumeration problem in the literature, although some of them focus on enumeration of all topologies, rather than homeomorphism classes of topologies [8, 9, 12, 12, 21, 22, 23, 25, 36, 37]. The difference already appears for two point spaces, where there are four distinct topologies but three inequivalent topologies, that is three non-homeomorphic two point spaces. Here is a table lifted straight from Wikipedia that gives an idea of the enumeration.

n	Distinct topologies	Distinct T_0-topologies	Inequivalent topologies	Inequivalent T_0-topologies
1	1	1	1	1
2	4	3	3	2
3	29	19	9	5
4	355	219	33	16
5	6942	4231	139	63
6	209,527	130,023	718	318
7	9,535,241	6,129,859	4,535	2,045
8	642,779,354	431,723,379	35,979	16,999
9	63,260,289,423	44,511,042,511	363,083	183,231
10	8,977,053,873,043	6,611,065,248,783	4,717,687	2,567,284

Through $n = 9$, a published source for the fourth column is [23]. However, this is not the kind of enumeration problem for which one expects to obtain a precise answer for all n. Rather, one expects bounds and asymptotics. There is a precise formula relating the second column to the first column, but we are really only interested in the last column. In fact, we are far more interested in refinements of the last column that shrink its still inordinately large numbers to smaller numbers of far greater interest to an algebraic topologist.

We shall explain how to reduce the determination of the third and fourth columns to a matrix computation, using minimal bases. For this purpose, it is convenient to describe minimal bases for a topology on X without reference to their enumeration by the elements $x \in X$, since the latter can give redundant information when the space is not T_0. The following sequence of lemmas applies to the study of general Alexandroff spaces, not necessarily finite.

LEMMA 1.7.2. *A set $\mathscr{B}$ of nonempty subsets of X is the minimal basis for an Alexandroff topology $\mathscr{U}$ if and only if the following conditions hold.*

(i) Every point of X is in some set B in $\mathscr{B}$.
(ii) The intersection of two sets in $\mathscr{B}$ is a union of sets in $\mathscr{B}$.
(iii) If a union of sets B_i in $\mathscr{B}$ is again in $\mathscr{B}$, then the union is equal to one of the B_i.

PROOF. Conditions (i) and (ii) are equivalent to saying that $\mathscr{B}$ is a basis for a topology, which we call $\mathscr{U}$. We suppose this topology is Alexandroff. Then each B in $\mathscr{B}$ must be a union of sets of the form U_x and each U_x must be in $\mathscr{B}$ by Lemma 1.3.6. If (iii) holds, then B must be one of the U_x and thus $\mathscr{B}$ is the minimal basis. Conversely, suppose that $\mathscr{B}$ is the minimal basis. Each given set B_i in (iii) must then be U_y for some $y \in X$. If the union of these U_y is also in $\mathscr{B}$, then the union must be U_x for some $x \in X$. But then x is in U_y for some y and thus $U_x = U_y$, so that (iii) holds. □

This result implies the following relationships between minimal bases and subspaces, quotients, disjoint unions, and products of Alexandroff spaces.

LEMMA 1.7.3. *If A is a subspace of X, the minimal basis of A consists of the intersections $A \cap U$, where U is in the minimal basis of X.*

LEMMA 1.7.4. *If Y is a quotient space of X with quotient map $q\colon X \longrightarrow Y$, the minimal basis of Y consists of the subsets U of Y such that $q^{-1}(U)$ is in the minimal basis of X.*

LEMMA 1.7.5. *The minimal basis of $X \amalg Y$ is the union of the minimal basis of X and the minimal basis of Y.*

LEMMA 1.7.6. *The minimal basis of $X \times Y$ is the set of products $U \times V$, where U and V are in the minimal bases of X and Y.*

Returning to finite spaces X, we shall show how to enumerate the homeomorphism classes of spaces with finitely many elements. This is meant only to illustrate how such an enumeration problem can be reduced to computationally accessible form. To allow spaces that are not T_0, the finite number to focus on is not $|X|$ but rather the number of elements in the minimal basis for the topology on X. These numbers are equal if and only if X is a T_0-space.

DEFINITION 1.7.7. Consider square matrixes $M = (a_{i,j})$ with integer entries that satisfy the following properties.

(i) $a_{i,i} \geq 1$.
(ii) $a_{i,j}$ is -1, 0, or 1 if $i \neq j$.
(iii) $a_{i,j} = -a_{j,i}$ if $i \neq j$.
(iv) $a_{i_1,i_s} = 0$ if there is a sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s > 2$ and $a_{i_k,i_{k+1}} = 1$ for $1 \leq k \leq s-1$.

Say that two such matrices M and N are equivalent if there is a permutation matrix T such that $T^{-1}MT = N$ and let $\mathscr{M}$ denote the set of equivalence classes of such matrices.

THEOREM 1.7.8. *The homeomorphism classes of finite spaces are in bijective correspondence with $\mathscr{M}$. If the homeomorphism class of X corresponds to the equivalence class of an $r \times r$ matrix M, then r is the number of sets in a minimal basis for X, and the trace of M is the number of elements of X. Moreover, X is a T_0-space if and only if the diagonal entries of M are all one.*

PROOF. We work with minimal bases for the topologies rather than with elements of the set. For a minimal basis $U_1, \cdots, U_r$ of a topology $\mathscr{U}$ on a finite set X, define an $r \times r$ matrix $M = (a_{i,j})$ as follows. If $i = j$, let $a_{i,i}$ be the number of elements $x \in X$ such that $U_x = U_i$. Define $a_{i,j} = 1$ and $a_{j,i} = -1$ if $U_i \subset U_j$ and there is no k (other than i or j) such that $U_i \subset U_k \subset U_j$. Define $a_{i,j} = 0$ otherwise. Clearly (i)–(iv) hold, and a reordering of the basis results in a permutation matrix that conjugates M into the matrix determined by the reordered basis. Thus X determines an element of $\mathscr{M}$.

If $f\colon X \longrightarrow Y$ is a homeomorphism, then f determines a bijection from the basis for X to the basis for Y. This bijection preserves inclusions and the number of elements that determine corresponding basic sets, hence X and Y determine the same element of $\mathscr{M}$. Conversely, suppose that X and Y have minimal bases $\{U_1, \cdots, U_r\}$ and $\{V_1, \cdots, V_r\}$ that give rise to the same element of $\mathscr{M}$. Reordering bases if necessary, we can assume that they give rise to the same matrix. For each i, choose a bijection f_i from the set of elements $x \in X$ such that $U_x = U_i$ and the set of elements $y \in Y$ such that $V_y = V_i$. We read off from the matrix that the f_i together specify a homeomorphism $f\colon X \longrightarrow Y$. Therefore our mapping from homeomorphism classes to $\mathscr{M}$ is one–to–one.

To see that our mapping is onto, consider an $r \times r$-matrix M of the sort under consideration and let X be the set of pairs of integers (u, v) with $1 \le u \le r$ and $1 \le v \le a_{u,u}$. Define subsets U_i of X by letting U_i have elements those $(u, v) \in X$ such that either $u = i$ or $u \ne i$ but $u = i_1$ for some sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s \ge 2$, $a_{i_k,i_{k+1}} = 1$ for $1 \le k \le s - 1$, and $i_s = i$. We see that the U_i give a minimal basis for a topology on X by verifying the conditions specified in Lemma 1.3.6.

Condition (i) is clear since $(u, v) \in U_u$. To verify (ii) and (iii), we observe that if $(u, v) \in U_i$ and $u \ne i$, then $U_u \subset U_i$. Indeed, we certainly have $(u, v) \in U_i$ for all v, and if $(k, v) \in U_u$ with $k \ne u$, then we must have a sequence connecting k to u and a sequence connecting u to i. These can be concatenated to give a sequence connecting k to i, which shows that (k, v) is in U_i. To see (ii), if $(u, v) \in U_i \cap U_j$, then $U_u \subset U_i \cap U_j$, which implies that $U_i \cap U_j$ is a union of sets U_u. To see (iii), if a union of sets U_i is a set U_j, there is an element of U_j in some U_i and then $U_j \subset U_i$, so that $U_j = U_i$. A counting argument for the diagonal entries and consideration of chains of inclusions show that the matrix associated to the topology whose minimal basis is $\{U_i\}$ is the matrix M that we started with. □

1.8. Spaces with at most four points

We describe the homeomorphism classes of spaces with at most four points, with just a start on taxonomy. Recall Definition 1.1.2.

There is a unique space with one point, namely $C_1 = D_1$.

There are three spaces with two points, namely C_2, $P_2 = \mathbb{C}D_1$, and D_2.

Proper subsets of X are those not of the form $\emptyset$ or X. Since $\emptyset$ and X are in any topology, we often restrict to proper subsets when specifying topologies. The following definitions prescribe the two names for the second space in the short list just given.

DEFINITION 1.8.1. We define certain topologies on a set S_n with n elements. Let $P_n = P_{1,n}$ be the space (unique up to homeomorphism) which has only one proper open set, containing only one point $s \in S_n$; for $1 < m < n$, let $P_{m,n}$ be the space whose proper open subsets are all of the non-empty subsets of a given subset S_m of S_n with m elements.

DEFINITION 1.8.2. For a space X define the *non-Hausdorff cone* $\mathbb{C}X$ by adjoining a new point $+$ and letting the proper open subsets of $\mathbb{C}X$ be the non-empty open subsets of X. For example, $\mathbb{C}D_{n-1}$ is homeomorphic to $P_{n-1,n}$ as we see by identifying D_{n-1} with $S_{n-1} \subset S_n$ and identifying the cone point $+$ with the point of S_n not in S_{n-1}.

We shall see that $\mathbb{C}X$ is contractible in Lemma 2.3.2 below. This means that it is a point to the eyes of homotopy theory or algebraic topology.

Here is a table of the nine homeomorphism classes of topologies on a three point set $X = \{a, b, c\}$. All of these spaces are disjoint unions of contractible spaces. A space that is not the disjoint union of proper open and closed subspaces is *connected*.

Proper open sets	Name	T_0?	connected?
all	D_3	yes	no
a, b, (a,b), (b,c)	$D_1 \amalg P_2$	yes	no
a, b, (a,b)	$P_{2,3} \cong \mathbb{C}D_2$	yes	yes
a	P_3	no	yes
a, (a,b)	$\mathbb{C}P_2 \cong (\mathbb{C}P_2)^{op}$	yes	yes
a, (b,c)	$D_1 \amalg C_2$	no	no
a, (a,b), (a,c)	$(\mathbb{C}D_2)^{op}$	yes	yes
(a,b)	$\mathbb{C}C_2 \cong P_3^{op}$	no	yes
none	$C_3 = D_3^{op}$	no	yes

It is a perhaps instructive exercise to check that the spaces said to be homeomorphic in the above list are in fact homeomorphic.

We tabulate the proper open subsets of the thirty-three homeomorphism classes of topologies on a four point space $X = \{a, b, c, d\}$. That is, these topologies are obtained by adding in the empty set and the whole set. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by decreasing number of two-point subsets and then by decreasing number of three-point subsets.[3]

[3] I thank Mark Bowron for sending me a correction and suggesting a reordering.

1	all
2	a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
3	a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d)
4	a, b, c, (a,b), (a,c), (b,c), (a,b,c)
5	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
6	a, b, (a,b), (a,c), (b,d), (a,b,c), (a,b,d)
7	a, b, (a,b), (a,c), (a,b,c), (a,b,d)
8	a, b, (a,b), (a,c), (a,b,c), (a,c,d)
9	a, b, (a,b), (c,d), (a,c,d), (b,c,d)
10	a, b, (a,b), (a,c), (a,b,c)
11	a, b, (a,b), (a,b,c), (a,b,d)
12	a, b, (a,b), (a,b,c)
13	a, b, (a,b), (a,c,d)
14	a, b, (a,b)
15	a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
16	a, (a,b), (a,c), (a,b,c), (a,b,d)
17	a, (a,b), (a,c), (a,b,c)
18	a, (a,b), (c,d), (a,c,d)
19	a, (a,b), (a,b,c), (a,b,d)
20	a, (b,c), (a,b,c), (b,c,d)
21	a, (a,b), (a,b,c)
22	a, (a,b), (a,c,d)
23	a, (b,c), (a,b,c)
24	a, (a,b)
25	a, (a,b,c)
26	a, (b,c,d)
27	a
28	(a,b), (c,d)
29	(a,b), (a,b,c), (a,b,d)
30	(a,b), (a,b,c)
31	(a,b)
32	(a,b,c)
33	none

PROBLEM 1.8.3. *Determine which of these spaces are T_0 and which are connected. Give a taxonomy in terms of explicit general constructions that accounts for all of these topologies. That is, determine appropriate "names" for all of these spaces. How many are not contractible spaces or disjoint unions of contractible spaces? (Hint: there is a connected 4-point space that is not contractible; which one of the 33 is it?)*

CHAPTER 2

Homotopy equivalences of Alexandroff and finite spaces

2.1. Connectivity and path connectivity

We begin the exploration of homotopy properties of Alexandroff spaces by discussing connectivity and path connectivity. We recall the general definitions. We let $I = [0,1]$ denote the unit interval with its usual metric topology as a subspace of $\mathbb{R}$. A *path* in a space X is a map $f : I \longrightarrow X$; it is said to connect the points $f(0)$ and $f(1)$.

DEFINITION 2.1.1. Let X be a space.

(i) X is *connected* if the only subspaces of X that are both open and closed are $\emptyset$ and X.
(ii) X is *path connected* if any two points of X can be connected by a path.

A path connected space is connected, but not conversely. The following results can be found in any text in point-set topology, such as [**33**]. They also make good exercises.

LEMMA 2.1.2. *Let Y be a subspace of a space X and let $Y = A \cup B$. Then A and B are both open and closed in Y if and only if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty or, equivalently, A contains no limit point of B and B contains no limit point of A. We then say that $Y = A \cup B$ is a* separation *of Y. Thus Y is connected if and only if it has no separation.*

The following consequence is used very frequently.

PROPOSITION 2.1.3. *Let $X = A \cup B$ be a separation. If $Y \subset X$ is connected, then Y is contained in either A or B.*

PROPOSITION 2.1.4. *A union of connected or path connected spaces that have a point in common is connected or path connected.*

PROPOSITION 2.1.5. *If $f : X \longrightarrow Y$ is a continuous map and X is connected or path connected, then the image of f is connected or path connected.*

For example, I is a connected space, hence the image of a path in X is a connected subspace of X.

PROPOSITION 2.1.6. *Any product of connected or path connected spaces is connected or path connected.*

DEFINITION 2.1.7. Define two equivalence relations $\sim$ and $\approx$ on X.

(i) $x \sim y$ if x and y are both in some connected subspace of X. A *component* of X is an equivalence class of points under $\sim$. Let $\pi_0'(X)$ denote the set of components of X.

(ii) $x \approx y$ if there is a path connecting x and y. A *path component* of X is an equivalence class of points under $\approx$. Let $\pi_0(X)$ denote the set of path components of X.

If $x \approx y$, then $x \sim y$ since the image of a path connecting x and y is a connected subspace. Therefore each component of X is the union of some of its path components. For nice spaces, components and path components are the same.

DEFINITION 2.1.8. Let X be a space.

(i) X is *locally connected* if for each $x \in X$ and each neighborhood U of x, there is a connected neighborhood V of x contained in U.
(ii) X is *locally path connected* if for each $x \in X$ and each neighborhood U of x, there is a path connected neighborhood V of x contained in U.

PROPOSITION 2.1.9. *Let X be a space.*

(i) X is locally connected if and only if every component of an open subset U is open in X.
(ii) X is locally path connected if and only if every path component of an open subset U is open in X.
(iii) If X is locally path connected, then the components and path components of X coincide.

Now return to a finite or, more generally, Alexandroff space X. At first sight, one might imagine that there are no continuous maps from I to a finite space, but that is far from the case. The most important feature of finite spaces is that they are surprisingly richly related to the "real" spaces that algebraic topologists care about.

LEMMA 2.1.10. *Let X be an Alexandroff space. Then each U_x is connected. If X is connected and $x, y \in X$, there is a finite sequence of points z_i, $1 \le i \le q$, such that $z_1 = x$, $z_q = y$ and either $z_i \le z_{i+1}$ or $z_{i+1} \le z_i$ for $i < q$.*

PROOF. Suppose that $U_x = A \amalg B$, where A and B are open and disjoint. We may as well assume that x is in A. Then $U_x \subset A$ and therefore $B = \emptyset$ and $U_x = A$. Therefore U_x is connected. Now assume that X is connected. Fix x and consider the set A of points y that are connected to x by some sequence of points z_i, as in the statement. We see that A is open since if z is in A then the open set U_z of points $w \le z$ is contained in A. We see that A is closed since if y is not connected to x by a finite sequence of points, then neither is any point of U_y, so that the complement of A is open. Since X is connected, it follows that $A = X$. □

LEMMA 2.1.11. *If $x \le y$ in an Alexandroff space X, then there is a path $p\colon I \longrightarrow X$ connecting x and y.*

PROOF. Define $p(t) = x$ if $t < 1$ and $p(1) = y$. We claim that p is continuous. Let V be an open set of X. If neither x nor y is in V, then $p^{-1}(V) = \emptyset$. If x is in V and y is not in V, then $p^{-1}(V) = [0, 1)$. If y is in V, then x is in $U_y \subset V$ since $x \le y$, hence $p^{-1}(V) = I$. In all cases, $p^{-1}(V)$ is open. □

PROPOSITION 2.1.12. *An Alexandroff space is connected if and only if it is path connected.*

PROOF. The previous two lemmas, the second generalized by concatenation of paths to finite sequences as in the first, imply that $x \sim y$ if and only if $x \approx y$. □

2.2. Function spaces and homotopies

An open cover of a space X is any set of open subsets whose union is all of X. The following notion is fundamental to point-set topology. It is discussed in more detail in §13.2.

DEFINITION 2.2.1. A space is *compact* if every open cover has a finite subcover.

For example, a classical result called the Heine-Borel theorem says that a subspace of $\mathbb{R}^n$ is compact if and only if it closed and bounded.

DEFINITION 2.2.2. Let X and Y be spaces and consider the set Y^X of maps $X \longrightarrow Y$. The *compact–open topology* on Y^X is the topology in which a subset is open if and only if it is a union of finite intersections of sets

$$W(C,U) = \{f | f(C) \subset U\},$$

where C is compact in X and U is open in Y. This means that the set of all $W(C,U)$ is a subbasis for the topology.

Ignoring topology, for sets X, Y, and Z, functions $f\colon X \times Y \longrightarrow Z$ are in bijective correspondence with functions $\hat{f}\colon X \longrightarrow Z^Y$ via the relation

$$f(x,y) = \hat{f}(x)(y).$$

Returning to topology, and so restricting Z^Y to consist only of the continuous functions $Y \longrightarrow Z$, one would like to have that f is continuous if and only if $\hat{f}$ is continuous. The compact-open topology, which at first sight seems to be unmotivated, is designed to minimize conditions on X, Y, and Z which force this conclusion. In fact, there are several different criteria which guarantee the conclusion. We recall one due to Fox [**13**] which applies to both Alexandroff spaces and metric spaces.

DEFINITION 2.2.3. A space is *first countable* if every point x has a countable neighborhood basis $\mathscr{B}_x$. This means that if U is a neighborhood of x, then there is a $B \in \mathscr{B}_x$ such that $x \in B \subset U$.

EXAMPLE 2.2.4. An Alexandroff space X is first countable since the singleton set $\{U_x\}$ is a neighborhood basis for x. A metric space is first countable since the ε-neighborhoods $B(x,\varepsilon) = \{y | d(x,y) < \varepsilon\}$ for positive rational numbers ε form a countable neighborhood basis.

PROPOSITION 2.2.5. *Let X and Y be first countable spaces. Then a function $f\colon X \times Y \longrightarrow Z$ is continuous if and only if $\hat{f}\colon X \longrightarrow Z^Y$ is continuous.*

We shall use function spaces to study the notion of homotopy.

DEFINITION 2.2.6. A *homotopy* $h\colon f \simeq g$ is a map $h\colon X \times I \longrightarrow Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Two maps are homotopic, written $f \simeq g$, if there is a homotopy between them.

It is impossible to overstate the importance of this notion. We will be studying the homotopy theory of finite topological spaces. For finite spaces, the use of function spaces allows us to recognize homotopic maps in a very simple way. The first statement of the following result is clear, and the reader should check the second statement from the definitions. The conclusion reduces the determination of whether or not two maps are homotopic to the determination of whether or not they are in the same path component of Y^X.

COROLLARY 2.2.7. *If X is first countable, then homotopies $h\colon X \times I \longrightarrow Y$ correspond bijectively to paths $j\colon I \longrightarrow Y^X$ via $h \leftrightarrow j$ if $h(x,t) = j(t)(x)$. Therefore the homotopy classes of maps $X \longrightarrow Y$ are in canonical bijective correspondence with the path components of Y^X.*

When Y is Alexandroff, we can use its preorder to compare maps $X \longrightarrow Y$ for any space X.

DEFINITION 2.2.8. If Y is Alexandroff, define the pointwise ordering of maps $X \longrightarrow Y$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

PROPOSITION 2.2.9. *If Y is Alexandroff, then the intersection V_g of the open sets in Y^X that contain a map g is $\{f | f \leq g\}$.*

PROOF. Let $f \in V_g$ and $x \in X$. Since $g \in W(\{x\}, U_{g(x)})$ and $\{x\}$ is compact, $f \in W(\{x\}, U_{g(x)})$, so $f(x) \in U_{g(x)}$ and $f(x) \leq g(x)$. Since x was arbitrary, $f \leq g$. Conversely, let $f \leq g$. Consider any $W(C, U)$ that contains g and let $x \in C$. Then $g(x) \in U$ and, since $f(x) \leq g(x)$, $f(x) \in U_{g(x)} \subset U$. Therefore $f \in W(C, U)$ and f is in all open subsets of Y^X that contain g. □

Unfortunately, however, V_g need not be open in Y^X in general. This problem is addressed in work of Kukiela [**26**]. Since our primary interest is in finite spaces, we shall not go into detail, but the following remarks indicate the subtleties here.

REMARK 2.2.10. Michal Kukiela [**26**] studied the behavior of the compact open topology on Y^X when X and Y are possibly infinite Alexandroff spaces.[1] He showed that Y^X is rarely an Alexandroff space. In particular X^X is never an Alexandroff space if X is infinite, which contradicts an assumption made by Arenas [**3**]. However, Kukiela proved that Y^X is Alexandroff if X is finite. For any X we have an ordering on the set Y^X, hence we have the Alexandroff topology on Y^X that it determines. However the Alexandroff topology is generally finer (has more open sets) than the compact open topology.

When X and Y are both finite, so is Y^X, and then Proposition 2.2.9 has the following interpretation.

COROLLARY 2.2.11. *If X and Y are finite, then the pointwise ordering on Y^X coincides with the preordering associated to its compact open topology.*

Here, finally, is our easy way to recognize homotopic maps between finite spaces. Part of the result holds for all Alexandroff spaces.

Relate to natural transformations later

PROPOSITION 2.2.12. *If X and Y are Alexandroff spaces and $f \leq g$, then $f \simeq g$ by a homotopy h such that $h(x,t) = f(x)$ for all t and all points $x \in X$ such that $f(x) = g(x)$. Conversely, if X and Y are finite and $f \simeq g$, then there is a sequence of maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that either $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i < q$.*

PROOF. For the first statement, we have the path p connecting f to g in Y^X that is specified by $p(t) = f$ if $t < 1$ and $p(1) = g$. By Lemma 2.1.11, it is continuous if we give Y^X the Alexandroff topology associated to $\leq$. Since that topology has more open sets than the compact open topology, by Kukiela's result

[1]Kukiela made his contribution as an undergraduate at Nicolaus Copernicus University, in Toru'n, Poland. Quoting from an email from him, "my study of Alexandroff spaces was in a great degree inspired by your notes on finite spaces".

just mentioned, it is also continuous if we give Y^X the compact open topology. By Proposition 2.2.9, the corresponding function $X \times I \longrightarrow Y$ is also continuous, giving us the claimed homotopy. For the second statement, Corollary 2.2.7 shows that homotopies between maps $X \longrightarrow Y$ are paths in Y^X, hence two maps are homotopic if and only if they are in the same path component. Now Lemma 2.1.10 and Corollary 2.2.11 give the conclusion. $\square$

2.3. Homotopy equivalences

We have seen that enumeration of finite sets with reflexive and transitive relations $\leq$ amounts to enumeration of the topologies on finite sets. We have refined this to consideration of homeomorphism classes of finite spaces. We are much more interested in the enumeration of the homotopy types of finite spaces. We will come to a still weaker and even more interesting enumeration problem later, one which is still unsolved.

DEFINITION 2.3.1. Two spaces X and Y are *homotopy equivalent* if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. A space is *contractible* if it is homotopy equivalent to a point.

This relationship can change the number of points. We have a first example.

LEMMA 2.3.2. *If X is a space containing a point y such that the only open (or only closed) subset of X containing y is X itself, then X is contractible. In particular, the non-Hausdorff cone $\mathbb{C}X$ is contractible for any X.*

PROOF. This is a variation on a theme we have already seen twice. Let $*$ denote a space with a single point, also denoted $*$. Define $r\colon X \longrightarrow *$ by $r(x) = *$ for all x and define $i\colon * \longrightarrow X$ by $i(*) = y$. Clearly $r \circ i = \mathrm{id}$. Define $h\colon X \times I \longrightarrow X$ by $h(x,t) = x$ if $t < 1$ and $h(x,1) = y$. Then h is continuous. Indeed, let U be open in X. If $y \in U$, then $U = X$ and $h^{-1}(U) = X \times I$, while if $y \notin U$, then $h^{-1}(U) = U \times [0,1)$. The argument when X is the only closed subset containing y is the same. Clearly h is a homotopy $\mathrm{id} \simeq i \circ r$. $\square$

DEFINITION 2.3.3. A point x of an Alexandroff space X is *maximal* if there is no $y > x$ in X; minimal points are defined similarly.

COROLLARY 2.3.4. *If X is an Alexandroff space and $x \in X$, then U_x is contractible. In particular, if X is finite and has a unique maximal point or a unique minimal point, then X is contractible.*

PROOF. The only open subset of U_x that contains x is U_x itself. If X is finite and x is the unique maximal point in X, then $X = U_x$. If x is the unique minimal point in X, then the only closed set containing x is X. $\square$

A result of McCord [**32**, Thm. 4] says that, when studying finite or, more generally, Alexandroff spaces up to homotopy type, there is no loss of generality if we restrict attention to T_0-spaces, that is, to posets. The proof is based on use of the Kolmogorov quotient of a space.

DEFINITION 2.3.5. Let X be any space. Define an equivalence relation $\sim$ on X by $x \sim y$ if x and y have the same open neighborhoods. The Kolmogorov quotient X_0 of X is the quotient space $X/(\sim)$ obtained by identifying equivalent points. It is a T_0 space. Let $q_X\colon X \longrightarrow X_0$ be the quotient map.

The Kolmogorov quotient satisfies a universal property.

LEMMA 2.3.6. *Let Z be a T_0-space and $f\colon X \longrightarrow Z$ be a map. Then there is a unique map $f_0\colon X_0 \longrightarrow Z$ such that $f_0 \circ q_X = f$. Therefore, if $f\colon X \longrightarrow Y$ is any map, there is a unique map $f_0\colon X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$.*

PROOF. Since the topology on Z separates points, f must take equivalent points to the same point. Therefore f factors through a function $f_0\colon X_0 \longrightarrow Y_0$, and f_0 is continuous by the universal property of the quotient topology. $\square$

THEOREM 2.3.7. *For an Alexandroff space X, the quotient map $q_X\colon X \longrightarrow X_0$ is a homotopy equivalence.*

PROOF. The equivalence relation $\sim$ on X is given by $x \sim y$ if $U_x = U_y$, or, equivalently, if $x \le y$ and $y \le x$. The relation $\le$ on X induces a relation $\le$ on X_0. We claim that $q(U_x) = U_{q(x)}$ for all $x \in X$. To see this, observe first that $q^{-1}q(U_x) = U_x$ since if $q(y) = q(z)$ where $z \in U_x$, then $y \in U_y = U_z \subset U_x$. Therefore $q(U_x)$ is open, hence it contains $U_{q(x)}$. Conversely, $U_x \subset q^{-1}(U_{q(x)})$ by continuity and thus $q(U_x) \subset U_{q(x)}$.

We conclude that the quotient topology on X_0 agrees with the topology determined by $\le$. It follows that $q(x) \le q(y)$ if and only if $x \le y$. Indeed, $q(x) \le q(y)$ implies $q(x) \in U_{q(y)} = q(U_y)$. Thus $q(x) = q(z)$ for some $z \in U_y$ and $U_x = U_z \subset U_y$, so that $x \le y$. Conversely, if $x \le y$, then $U_x \subset U_y$ and therefore $U_{q(x)} \subset U_{q(y)}$, so that $q(x) \le q(y)$.

To prove that q is a homotopy equivalence, let $f\colon X_0 \longrightarrow X$ be any function such that $q \circ f = \mathrm{id}$. That is, we choose a point from each equivalence class. By what we have just proven, f preserves $\le$ and is therefore continuous.[2] Let $g = f \circ q$. We must show that g is homotopic to the identity. We see that g is obtained by first choosing one x_u with $U_{x_u} = U$ for each U in the minimal basis for X and then letting $g(x) = x_u$ if $U_x = U$. Thus $U_{g(x)} = U_x$ and $g(x) \in U_x$, which means that $g \le \mathrm{id}$. Now Proposition 2.2.12 gives the required homotopy $h\colon \mathrm{id} \simeq g$. Note that $h(g(x), t) = g(x)$ for all t. $\square$

We conclude that to classify Alexandroff spaces up to homotopy equivalence, it suffices to classify A-spaces up to homotopy equivalence.

2.4. Cores of finite spaces

Stong [**40**, §4] has given an interesting way of studying homotopy types of finite spaces. An attempt to extend his results to Alexandroff spaces was made by Arenas [**3**], but his work had a mistake that was noticed and corrected by Kukiela [**26**]; see Remark 2.2.10. Since the generalization is not an immediate one, we give proofs for the finite space case in this section, turning to Alexandroff spaces in the next. However, we give the basic definitions in full generality. We change Stong's language a bit in the following exposition. We first single out an especially nice class of homotopy equivalences.

Or Fix-Patrias first

DEFINITION 2.4.1. Let Y be a subspace of a space X, with inclusion denoted by $i\colon Y \longrightarrow X$. We say that Y is a *deformation retract* of X if there is map $r\colon X \longrightarrow Y$

[2] I have seen it claimed in an undergraduate thesis that Theorem 2.3.7 holds for any space X, not necessarily Alexandroff. However, there need not be a continuous function $f\colon X_0 \longrightarrow X$ such that $q \circ f = \mathrm{id}$.

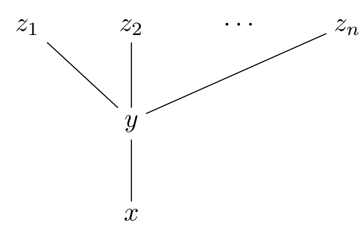
such that $r \circ i$ is the identity map of Y and there is a homotopy $h\colon X \times I \longrightarrow X$ from the identity map of X to $i \circ r$ such that $h(y,t) = y$ for all $y \in Y$ and $t \in I$.

Definition 2.4.2. Let X be a finite space.

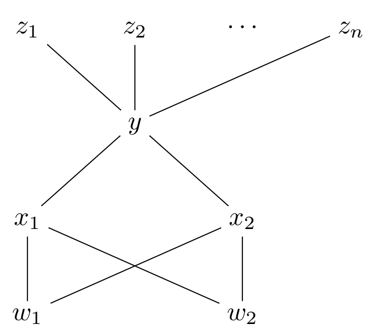
(a) A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$.
(b) A point $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$.
(c) A point $x \in X$ is a *beat point* if it is either an upbeat point or a downbeat point.

X is a *minimal finite space* if it is a T_0-space and has no beat points. A *core* of a finite space X is a subspace Y that is a minimal finite space and a deformation retract of X.

Remark 2.4.3. If we draw a graph of a poset by drawing a line downwards from y to x if $x < y$, we see that, above an upbeat point x, the graph of those edges with y as a vertex looks like



For a more complicated example, both x_1 and x_2 are upbeat points in the poset



Turning the pictures upside down, we see what the graphs below downbeat points look like. The essential point is that a beat point has either exactly one edge connecting to it from above or exactly one edge connecting to it from below.

Intuitively, identifying x and y and erasing the line between them should not change the homotopy type. We say this another way in the proof of the following result, looking at inclusions rather than quotients in accordance with our definition of a core.

Theorem 2.4.4. *Any finite space X has a core.*

Proof. With the notations of the proof of Theorem 2.3.7, identify X_0 with its image $f(X_0) \subset X$. The proof of Theorem 2.3.7 shows that X_0, so interpreted, is a deformation retract of X. Thus we may as well assume that X is T_0. Suppose that X has an upbeat point x. We claim that the subspace $X - \{x\}$ is a deformation retract of X. To see this define $f\colon X \longrightarrow X - \{x\} \subset X$ by $f(z) = z$ if $z \neq x$ and $f(x) = y$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \mathrm{id}$. We claim that f preserves order and is therefore continuous. Thus suppose that $u \leq v$. We must show that $f(u) \leq f(v)$. If $u = v = x$ or if neither u nor v is x, there is nothing to prove. When $u = x < v$, $f(u) = y$ and $f(v) = v \geq y$. When $u < x = v$, $f(u) = u < x < y = f(v)$. Now Proposition 2.2.12 gives the required deformation. A similar argument applies to show that $X - \{x\}$ is a deformation retract of X if x is a downbeat point. Starting with X_0, define X_i from X_{i-1} by deleting one upbeat or downbeat point. After finitely many stages, there are no more upbeat or downbeat points left, and we arrive at the required core. □

THEOREM 2.4.5. *If X is a minimal finite space and $f\colon X \longrightarrow X$ is homotopic to the identity, then f is the identity.*

PROOF. First suppose that $f \geq \mathrm{id}$. For all x, $f(x) \geq x$. If x is a maximal point, then $f(x) = x$. Let x be any point of X and suppose inductively that $f(z) = z$ for all $z > x$. Then, by continuity, $z > x$ implies $z = f(z) \geq f(x) \geq x$. If $f(x) \neq x$, this implies that x is an upbeat point, contradicting the minimality of X. Therefore $f(x) = x$. By induction, $f(x) = x$ for all x. A similar argument shows that $f \leq \mathrm{id}$ implies $f = \mathrm{id}$. By Proposition 2.2.12, it now follows that the component of the identity map in the finite space X^X consists only of the identity map. That is, any map homotopic to the identity is the identity. □

COROLLARY 2.4.6. *If $f\colon X \longrightarrow Y$ is a homotopy equivalence of minimal finite spaces, then f is a homeomorphism.*

PROOF. If $g\colon Y \longrightarrow X$ is a homotopy inverse, then $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$. By the theorem, $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. □

COROLLARY 2.4.7. *Finite spaces X and Y are homotopy equivalent if and only if they have homeomorphic cores. In particular, the core of X is unique up to homeomorphism.*

PROOF. This is immediate since the cores of X and Y are minimal finite spaces that are homotopy equivalent to X and Y. □

REMARK 2.4.8. In any homotopy class of finite spaces, there is a representative with the least possible number of points. This representative must be a minimal finite space, since its core is a homotopy equivalent subspace. The minimal representative is homeomorphic to a core of any finite space in the given homotopy class.

revisit

2.5. Cores of Alexandroff spaces

Not yet written (and probably never will be). The key reference is [**26**]. But see Cathy Chen REU paper.

revisit

2.6. Hasse diagrams and homotopy equivalence

Should definitely be added. The key reference is Fix and Patrias.

CHAPTER 3

Homotopy groups and weak homotopy equivalences

3.1. Homotopy groups

We recall the definition of the homotopy groups $\pi_n(X, x)$ of a space X at $x \in X$. We shall not give adequate motivation here. This is the first of several places where the author will advertise his book [30] as a source for a more complete treatment, but in fact all standard textbooks in algebraic topology treat these definitions. For $n = 0$, we define $\pi_0(X)$ to be the set of path components of X, with the component of x taken as a basepoint (and there is no group structure). When $n = 1$, we define $\pi_1(X, x)$, or $\pi_1(X)$ when the basepoint is assumed, to be the fundamental group of X at the point x.

For all $n \geq 0$, $\pi_n(X)$ can be described most simply by considering the standard sphere S^n with a chosen basepoint $*$. One considers all maps $\alpha\colon S^n \longrightarrow X$ such that $f(*) = x$. One says that two such maps α and β are based homotopic if there is a based homotopy $h\colon \alpha \simeq \beta$. Here a homotopy h is based if $h(*, t) = x$ for all $t \in I$. If $n = 1$, the map α is a loop at x, and we can compose loops to obtain a product which makes $\pi_1(X, x)$ a group. The homotopy class of the constant loop at x gives the identity element, and the loop $\alpha^{-1}(t) = \alpha(1 - t)$ represents the inverse of the homotopy class of α. There is a similar product on the higher homotopy groups, but, in contrast to the fundamental group, the higher homotopy groups are abelian.

A path p from x to x' induces an isomorphism $\pi_n(X, x) \longrightarrow \pi_n(X, x')$. On the fundamental group, it maps a loop α to the composite $p \circ \alpha \circ p^{-1}$, where p^{-1} is the reverse path $p^{-1}(t) = p(1 - t)$ from x' to x.

A map $f\colon X \longrightarrow Y$ induces a function $f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$. One just composes maps α and homotopies h as above with the map f. If $n \geq 1$, f_* is a homomorphism.

3.2. Weak homotopy equivalences

Definition 3.2.1. A map $f\colon X \longrightarrow Y$ is a *weak homotopy equivalence* if

$$f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $x \in X$ and all $n \geq 0$. If $n = 0$, this means that components are mapped bijectively. Two spaces X and Y are weakly homotopy equivalent if there is a finite chain of weak homotopy equivalences $Z_i \longrightarrow Z_{i+1}$ or $Z_{i+1} \longrightarrow Z_i$ starting at $X = Z_1$ and ending at $Z_q = Y$.

The definition may seem strange at first sight, but it has gradually become apparent that the notion of a weak homotopy equivalence is even more important in algebraic topology than the notion of a homotopy equivalence. The notions are related. We state some theorems that the reader can take as reference points. Proofs can be found in [**30**]. We mention CW complexes in the following result because they give the appropriate level of generality. They will be defined later, in Definition 9.6.1. However, all the reader needs to know here is that the geometric realizations of simplicial complexes, which will be defined in Definition 4.1.6, are special cases of CW complexes.

THEOREM 3.2.2. *A homotopy equivalence is a weak homotopy equivalence. Conversely, a weak homotopy equivalence between CW complexes (for example, between simplicial complexes) is a homotopy equivalence.*

THEOREM 3.2.3. *Spaces X and Y are weakly homotopy equivalent if and only if there is a space Z and weak homotopy equivalences $Z \longrightarrow X$ and $Z \longrightarrow Y$. When this holds, there is such a Z which is a CW complex.*

That is, the chains that appear in the definition need only have length two. For those who know about homology and cohomology, we record the following result.

THEOREM 3.2.4. *A weak homotopy equivalence induces isomorphisms of all singular homology and cohomology groups.*

3.3. A local characterization of weak equivalences

NO: FIND A QUICK PROOF; maybe Quillen Thm A analogue of Barmak Or look at my paper and Gray

Expository paper topic?

An essential point in our work, which we will take for granted, is that weak homotopy equivalence is a local notion in the sense of the following theorem. McCord [**32**] relies on point-by-point comparison with arguments in the early paper [**10**], which doesn't prove the result but comes close. More modern references are [**29, 43**].

THEOREM 3.3.1. *Let $p\colon E \longrightarrow B$ be a continuous map. Suppose that B has an open cover $\mathscr{O}$ with either of the first two and the third of the following properties.*

(i) If x is in the intersection of sets U and V in $\mathscr{O}$, then there is some $W \in \mathscr{O}$ with $x \in W \subset U \cap V$.
(ii) For each $U \in \mathscr{O}$, the restriction $p\colon p^{-1}U \longrightarrow U$ is a weak homotopy equivalence.

Then p is a weak homotopy equivalence.

3.4. The non-Hausdorff suspension

The suspension is one of the most basic constructions in all of topology. Following McCord [**32**], we show that it comes in two weakly equivalent versions, the classical one and a non-Hausdorff analogue that preserves finite spaces. For the purposes of these notes, we shall use the following unbased variant of the classical suspension.

DEFINITION 3.4.1. Define the *cone* CX of a topological space X to be the quotient space $X \times I/X \times \{1\}$ obtained by identifying $X \times \{1\}$ to a single point, denoted $+$. Define the *suspension* SX of X to be the quotient space obtained from $X \times [-1,1]$ by identifying $X \times \{1\}$ to a single point $+$ and identifying $X \times \{-1\}$ to another single point, denoted $-$. Thus SX can be thought of as obtained by gluing together the bases of two cones on X. For a map $f\colon X \longrightarrow Y$, define $Sf\colon SX \longrightarrow SY$ by $(Sf)(x,t) = (f(x),t)$.

It should be clear that CX is contractible to its cone point $+$. We defined the non-Hausdorff cone $\mathbb{C}X$ by adjoining a new cone point $*$ and letting the proper open subsets of $\mathbb{C}X$ be all of the open subsets of X, and we saw that $\mathbb{C}X$ is contractible. We now change notation and call the added point $+$.

Definition 3.4.2. Define the *non-Hausdorff suspension* $\mathbb{S}X$ by adjoining two points, $+$ and $-$ to X, and topologizing $\mathbb{S}X$ as the union of two copies of $\mathbb{C}X$ glued along X. Thus the proper open subsets are the open subsets of X and the two copies of $\mathbb{C}X$. When X is an A-space, $x < +$ and $x < -$ for all $x \in X$. For a map $f\colon X \longrightarrow Y$, define $\mathbb{S}f\colon \mathbb{S}X \longrightarrow \mathbb{S}Y$ by sending $+$ to $+$, $-$ to $-$, and letting $\mathbb{S}f$ restrict to f on X.

Observe that if X is a T_0-space, then so are $\mathbb{C}X$ and $\mathbb{S}X$.

Definition 3.4.3. Define a comparison map

$$\gamma = \gamma_X \colon SX \longrightarrow \mathbb{S}X$$

by $\gamma(x,t) = x$ if $-1 < t < 1$, $\gamma(+) = +$ and $\gamma(-) = -$. It is an easy exercise to check that γ is continuous. Observe that, for a map $f\colon X \longrightarrow Y$, $\gamma_Y \circ Sf = \mathbb{S}f \circ \gamma_X$. Inductively, define $S^nX = SS^{n-1}X$ and $\mathbb{S}^nX = \mathbb{S}\mathbb{S}^{n-1}X$ and let $\gamma^n\colon S^nX \longrightarrow \mathbb{S}^nX$ be the common composite displayed in the commutative diagram

$$\begin{tikzcd}
S^nX \arrow[r, "S\gamma^{n-1}"] \arrow[d, "\gamma"'] \arrow[dr, "\gamma^n"] & S\mathbb{S}^{n-1}X \arrow[d, "\gamma"] \\
\mathbb{S}S^{n-1}X \arrow[r, "\mathbb{S}\gamma^{n-1}"'] & \mathbb{S}^nX
\end{tikzcd}$$

Theorem 3.4.4. *For any space X, the map $\gamma\colon SX \longrightarrow \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f\colon X \longrightarrow Y$, the maps $Sf\colon SX \longrightarrow SY$ and $\mathbb{S}f\colon \mathbb{S}X \longrightarrow \mathbb{S}Y$ are weak homotopy equivalences. Therefore $\gamma^n\colon S^nX \longrightarrow \mathbb{S}^nX$ is a weak homotopy equivalence for any space X.*

Proof. This is an application, or rather several applications, of Theorem 3.3.1. Take the three subspaces X, $X \cup \{+\}$, and $X \cup \{-\}$ as our open cover of $\mathbb{S}X$ and observe that the latter two subspaces are copies of $\mathbb{C}X$ and are therefore contractible. The respective inverse images under γ of these three subsets are the images in SX of $X \times (-1,1)$, $X \times (-1,1]$, and $X \times [-1,1)$. The restrictions of γ on these three subspaces are homotopy equivalences, hence weak homotopy equivalences.

Similarly, taking the three subspaces Y, $Y \cup \{+\}$, and $Y \cup \{-\}$ as our open cover of $\mathbb{S}Y$, their inverse images under $\mathbb{S}f$ are X, $X \cup \{+\}$, and $X \cup \{-\}$, and the restrictions of $\mathbb{S}f$ on these three subspaces are weak homotopy equivalences. Finally, take the images in SY of $Y \times (-1/2, 1/2)$, $Y \times [-1, 1/2)$, and $Y \times (-1/2, 1]$ as our open cover of SY. Their inverse images under Sf are the corresponding subspaces of SX, and the restrictions of Sf to these subspaces are weak homotopy equivalences. □

Example 3.4.5. Consider the discrete space D_3. We have the five-point space $\mathbb{S}D_3$ and the weak equivalence $SD_3 \longrightarrow \mathbb{S}D_3$. The space SD_3 is homotopy equivalent to the wedge, or 1-point union, of two circles as the reader should check. We can also form the opposite space $(\mathbb{S}D_3)^{op}$ corresponding to the opposite partial order, so that we now have two minimal points. A moment's reflection will convince the reader that we also have a weak equivalence $SD_3 \longrightarrow (\mathbb{S}D_3)^{op}$, and it will later become clear that X and X^{op} have the same weak homotopy type for any finite space X. Already in our five point example, this gives two weakly homotopy equivalent minimal finite spaces with the same number of points that are not homotopy equivalent. Moreover, there is no direct weak homotopy equivalence from one to the other: one needs a chain, like $\mathbb{S}D_3 \longleftarrow SD_3 \longrightarrow (\mathbb{S}D_3)^{op}$.

Example 3.4.6. Let $X = S^0$, a two-point discrete space. Then $S^n X$ is homeomorphic to the n-sphere S^n, while $\mathbb{S}^n X$ is a T_0-space with $2n+2$ points. Thus we have a weak homotopy equivalence γ^n from S^n to a finite space with $2n+2$ points.

Proposition 3.4.7. *Each $\mathbb{S}^n S^0$, $n \geq 1$, is a minimal finite space.*

Proof. Certainly $\mathbb{S}^n S^0$ is T_0, and it has no upbeat or downbeat points since each point has incomparable points above it or below it in the partial ordering. □

Example 3.4.8. There are minimal finite spaces with more than $2n+2$ points that are also weakly homotopy equivalent to S^n. For example, there is a six point finite space weakly equivalent to a circle, with three minimal points and three maximal points. You can draw it yourself, or you can look later at the finite space associated to the barycentric subdivision of the boundary of a 2-simplex. As an exercise, construct a weak homotopy equivalence from this 6-point circle to the 4-point circle; the map cannot be a homotopy equivalence, since both of these finite models for the circle are minimal finite spaces.

3.5. 6-point spaces and height

Up to homeomorphism, the only minimal connected spaces with at most five points are the one point space, the 4-point circle, and the two 5-point minimal spaces described in Example 3.4.5.

Proposition 3.5.1. *Up to homeomorphism, there are seven connected minimal 6-point spaces X, and none of them are weakly contractible. One is the six point two sphere $\mathbb{S}^2 S^0$, two are $\mathbb{S}D_4$ and its opposite. The remaining four have three maximal and three minimal points.*

Proof. We must have at least two minimal and at least two maximal points. Indeed, if we have just one intermediate point y, any point greater or less than it is upbeat or downbeat. If we have two intermediate points, they cannot be comparable without again contradicting minimality, and if they are incomparable we arrive by minimality at $\mathbb{S}^2 S^0$, which is homeomorphic to its opposite. The only remaining cases have all points either minimal or maximal. By the minimality of X, each minimal point must be less than at least two maximal points and each maximal point must be greater than at least two minimal points. There is only one example with two minimal points, and its opposite is the only example with four minimal points. We are left with the case when there are three minimal and three maximal points. Here each minimal point must be less than at least two maximal points and zero, one, two, or all three of them can be less than all three maximal points. In all four cases, the resulting space is homeomorphic to its opposite. □

Remark 3.5.2. In the next chapter we will define polytopes $|\mathscr{K}(X)|$ associated to finite spaces. The polytope assigned to $\mathbb{S}^2 S^0$ is homeomorphic to S^2. The polytopes assigned to the remaining connected minimal 6-point spaces are graphs that are homotopy equivalent to the wedge (or 1-point union) of one, two, three, or four circles.

The height $h(X)$ of a poset X is the maximal length h of a chain $x_1 < \cdots < x_h$ in X. It is one more than the dimension $d(X)$ of the space $|\mathscr{K}(X)|$. In the analysis just given, we noticed that if X has six elements then $h(X)$ is 2 or 3. Barmak and Minian [6] observed the following related inequality.

PROPOSITION 3.5.3. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$ and therefore weakly homotopy equivalent to $S^{h(X)-1}$.*

PROOF. Let $x_1 < \cdots < x_h$ be a maximal chain in X. Since X cannot have a minimimum point, there is a y_1 which is not greater than x_1. Since no x_i is an upbeat point, $1 \leq i < h$, there must be some $y_{i+1} > x_i$ such that y_{i+1} is not greater than x_{i+1}. The points y_i are easily checked to be distinct from each other and from the x_j. Now suppose that X has exactly these $2h$ points. By the maximality of our chain, the x_i and y_j are incomparable. For $i < j$, we started with $x_i < x_j$, and we check by cases from the absence of upbeat and downbeat points that $y_i < x_j$, $y_i < y_j$, and $x_i < y_j$. Comparing with the iterated suspension, we see that this implies that X is homeomorphic to $\mathbb{S}^{h-1}S^0$. □

When I first taught finite spaces in the REU, in 2003, I asked if $2n + 2$ was the least number of points in a finite space of the weak homotopy type of S^n. Barmak and Minian [6] followed up by proving the previous result. Their proof uses homology, but we have just seen that is easy to give a direct elementary proof.

Drawing posets, and thinking about them, leads to lots of eliminations from the list of F-spaces that might not be contractible or weakly contractible (weakly homotopy equivalent to a point). There is a well-known example of an 11-point space that is weakly contractible but not contractible. Find it

PROBLEM 3.5.4. *What is the smallest number n that there is an n-point weakly contractible space that is not contractible* answer 9; CianciOttina Expository paper topic?

CHAPTER 4

Simplicial complexes

4.1. A quick introduction to simplicial complexes

Simplicial complexes provide a general class of spaces that is sufficient for most purposes of basic algebraic topology. There are more general classes of spaces, in particular the CW complexes, that are more central to the modern development of the subject, but they give exactly the same collection of homotopy types, as we shall recall. We shall give basic material on simplicial complexes here, but largely restricting ourselves to what we shall use later. More detail can be found in many textbooks in algebraic topology (although not in my own book [**30**]). However, it is hard to find as precise a demarkation between simplicial complexes and ordered simplicial complexes as is needed for conceptual understanding, and this will become increasingly important as we go on.

DEFINITION 4.1.1. An *abstract simplicial complex* K is a set $V = V(K)$, whose elements are called *vertices*, together with a set K of (non-empty) finite subsets of V, whose elements are called *simplices*, such that every vertex is an element of some simplex and every subset of a simplex is a simplex; such a subset is called a *face* of the given simplex. We say that K is finite if V is a finite set. The *dimension* of a simplex is one less than the number of vertices in it.

DEFINITION 4.1.2. A map $g\colon K \longrightarrow L$ of abstract simplicial complexes is a function $g\colon V(K) \longrightarrow V(L)$ that takes simplices to simplices. We say that K is a *subcomplex* of L if the vertices and simplices of K are some of the vertices and simplices of L. We say that K is a *full subcomplex* of L if, further, every simplex of L whose vertices are in K is a simplex of K.

As already said, there is a very important distinction to be made between simplicial complexes as we have just defined them and *ordered* simplicial complexes.

DEFINITION 4.1.3. An *ordering* of an abstract simplicial complex K is a partial order on the vertices of K that restricts to a total order on the vertices of each simplex of K. A map of ordered simplicial complexes is a map of simplicial complexes that is given by an order preserving map on its poset of vertices.

While imposition of an ordering may seem artificial, since we have no canonical choice, it is essential to a serious calculational theory. We shall later introduce simplicial sets, which generalize simplicial complexes and elegantly systematize orderings. Many of the definitions below have evident ordered variants. We shall not belabor the point. However, orderings will be essential to understanding the relationship between simplicial complexes and finite spaces. Of course, this is not surprising since finite spaces are essentially the same as finite posets.

DEFINITION 4.1.4. A set $\{v_0, \cdots, v_n\}$ of points of $\mathbb{R}^N$ is *geometrically independent* if the vectors $v_i - v_0$, $1 \leq i \leq n$, are linearly independent. An equivalent characterization that gives none of the v_i a privileged role is that the equations $\sum_{t=0}^{t=n} t_i v_i = 0$ and $\sum_{t=0}^{t=n} t_i = 0$ for real numbers t_i imply $t_0 = \cdots = t_n = 0$. The *n-simplex* σ spanned by $\{v_0, \cdots, v_n\}$ is then the set of all points $x = \sum_{t=0}^{t=n} t_i v_i$, where $0 \leq t_i \leq 1$ and $\sum t_i = 1$. The t_i are called the *barycentric coordinates* of the point x. When each $t_i = 1/(n+1)$, the point x is called the *barycenter* of σ. The points v_i are the *vertices* of σ. A simplex spanned by a subset of the vertices is a *face* of σ; it is a *proper face* if the subset is proper. The *standard* n-simplex $\Delta[n]$ is the n-simplex spanned by the standard basis of $\mathbb{R}^{n+1}$. Thus the standard 0-simplex is the point $1 \in \mathbb{R}$, the standard 1-simplex is the line$\{t, 1-t\} \subset \mathbb{R}^2$, and so forth. Later, when necessary for clarity, we will sometimes denote these topological n-simplices by $\Delta[n]^t$ to distinguish them from other kinds of n-simplices that will appear.

Definition 4.1.5. A *simplicial complex*, or *geometric simplicial complex*, K is a set of simplices in some $\mathbb{R}^N$ such that every face of a simplex in K is a simplex in K and the intersection of two simplices in K is a simplex in K. The set of vertices of K is the union of the sets of vertices of its simplexes. Note that although we require all vertices to lie in some $\mathbb{R}^N$ and we require each set of vertices that spans a simplex of K to be geometrically independent, we do not require the entire set of vertices to be geometrically independent. For example, we can have three vertices on a single line in $\mathbb{R}^N$, as long as the two vertices furthest apart do not span a 1-simplex of K. A subcomplex L of a simplicial complex K is a simplicial complex whose simplices are some of the simplices of K. It is a full subcomplex if every simplex of K with vertices in L is in L.

Definition 4.1.6. The *geometric realization* $|K|$ of a simplicial complex K is the union of the simplices of K, each regarded as a subspace of $\mathbb{R}^N$, with the topology whose closed sets are the sets that intersect each simplex in a closed subset. If K is finite, but not in general otherwise, this is the same as the topology of $|K|$ as a subspace of $\mathbb{R}^N$. The open simplices of $|K|$ are the interiors of its simplices (where a vertex is an interior point of its 0-simplex), and every point of $|K|$ is an interior point of a unique simplex. The *boundary* $\partial\sigma$ of a simplex σ is the subcomplex given by the union of its proper faces. The *closure* of a simplex is the union of its interior and its boundary. A space homeomorphic to $|K|$ for some K is called a *polytope.*

The *dimension* of a simplicial complex is the maximal dimension of its simplices, and that of course corresponds to our geometric intuition.

Definition 4.1.7. A map $g\colon K \longrightarrow L$ of simplicial complexes is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that, for each subset S of $V(K)$ that spans a simplex of K, the set $g(S)$ is the set of vertices of a simplex of L. Then g determines the continuous map $|g|\colon |K| \longrightarrow |L|$ that sends $\sum t_i v_i$ to $\sum t_i g(v_i)$. Note that although we do not require g to be one–to–one on vertices, $|g|$ is nevertheless well-defined and continuous. If g is a bijection on vertices and simplices, we say that it is an isomorphism, and then $|g|$ is a homeomorphism.

It is usual to abbreviate $|g|$ to g and to refer to it as a simplicial map, but for now we prefer to keep the distinction between g and $|g|$ clear.

Remark 4.1.8. The reader can and should object to our insistence that all of the vertices of K are in some $\mathbb{R}^N$. Why not allow an infinite set of vertices with no bound on the allowed size of the simplices? The idea is to take the topological space given by the disjoint union of the simplices of a geometric simplicial complex, ignoring their embeddings in Euclidean space, and to then form a quotient space by glueing them together along their common faces. We might instead think of sets of standard n-simplices $\Delta[n]$, and we might think of taking their disjoint union and then gluing together along prescribed faces to construct the geometric realization more abstractly. We shall allow ourselves to think of such infinite dimensional simplicial complexes, but it is best not to take them too seriously for now. We shall come back to them under the guise of simplicial sets, which are best treated later. In that context, we will make the intuition precise and show how best to define geometric realization in general.

4.2. Abstract and geometric simplicial complexes

DEFINITION 4.2.1. The abstract simplicial complex aK determined by a geometric simplicial complex K has vertex set the union of the vertex sets of the simplices of K. Its simplices are the subsets that span a simplex of K. An abstract finite simplicial complex K determines a geometric finite simplicial complex gK by choosing any bijection between the vertices of K and a geometrically independent subset of some $\mathbb{R}^N$. For specificity, we can take the standard basis elements of $\mathbb{R}^N$ where N is the number of points in the vertex set $V(K)$. The geometric simplices are spanned by the images of the simplices of K under this bijection. For an abstract simplicial complex K, agK is isomorphic to K, the isomorphism being given by the chosen bijection. Similarly, for a finite geometric simplicial complex K, gaK is isomorphic to K.

We could remove the word finite from the previous definition by defining geometric simplicial complexes more generally, without reference to a finite dimensional ambient space $\mathbb{R}^N$, as in Remark 4.1.8. We also note that we do not have to realize in such a high dimensional Euclidean space as a count of vertexes would dictate. The following result holds no matter how many vertices there are. It is rarely used, but it is conceptually attractive. A proof can be found in [**20**, 1.9.6].

THEOREM 4.2.2. *Any finite simplicial complex K of dimension n can be geometrically realized in $\mathbb{R}^{2n+1}$.*

In view of the discussion above, abstract and geometric finite simplicial complexes can be used interchangeably. In particular, the geometric realization of an abstract simplicial complex is K is understood to mean the geometric realization of any gK.

We need a criterion for when the geometric realizations of two simplicial maps are homotopic.

DEFINITION 4.2.3. Continuous maps f and g from a topological space X to the geometric realization $|K|$ of a simplicial complex are *simplicially close* if, for each $x \in X$, both $f(x)$ and $g(x)$ are in the closure of some simplex $\sigma(x)$ of K.

PROPOSITION 4.2.4. *If f and g are simplicially close continuous maps from a topological space X to some $|K| \subset \mathbb{R}^N$, then f and g are homotopic.*

PROOF. Define $h\colon X \times I \longrightarrow \mathbb{R}^N$ by

$$h(x,t) = (1-t)f(x) + tg(x).$$

Since $h(x,t)$ is in the closure of $\sigma(x)$ and therefore in $|K|$, we see that it is continuous and specifies a homotopy as required. $\square$

4.3. Cones and subdivisions of simplicial complexes

MAYBE add simple defn of subdivision of an abstract simplicial complex: introduce poset of simplices ordered under inclusion: so an n-simplex of K' is a finite sequence of simplices $\{\sigma_0 \subset \cdots \subset \sigma_n\}$. Then go to more geometric point of view. Reorganize

Let K be a finite geometric simplicial complex in $\mathbb{R}^N$.

Definition 4.3.1. Let x be a point of $\mathbb{R}^N - K$ such that each ray starting at x intersects $|K|$ in at most one point. Observe that the union of $\{x\}$ and the set of vertices of a simplex of K is a geometrically independent set. Define the cone $K * x$ on K with vertex x to be the geometric simplicial complex whose simplices are all of the faces of the simplices spanned by such unions. Then K is a subcomplex of $K * x$, x is the only vertex not in K, and $|K * x|$ is homeomorphic to $C|K|$. Define the cone $K * x$ on an abstract simplicial complex K by adding a new vertex x and taking the simplices to be all subsets of all unions of x with a simplex in K.

Example 4.3.2. A simplex is the cone of any one of its vertices with the subcomplex spanned by the remaining vertices (the opposite face).

Definition 4.3.3. A *subdivision* of K is a simplicial complex L such that each simplex of L is contained in a simplex of K and each simplex of K is the union of finitely many simplices of L.

The following observation should be clear.

Lemma 4.3.4. *If L is a subdivision of K, then $|L| = |K|$ (as spaces).*

The n-skeleton K^n of K is the union of the simplices of K of dimension at most n. It is a subcomplex. There are many ways to subdivide a simplicial complex, and in applications there can be advantages to one or another of them. However, we will focus on the standard canonical choice. We give a somewhat pedantic inductive geometric construction that should make the idea clear and then reexpress the answer combinatorially.

Construction 4.3.5. We construct the *barycentric subdivision* K' of K. We subdivide the skeleta of K inductively. Let $L_0 = K^0$. Suppose that a subdivision L_{n-1} of K^{n-1} has been constructed. Let b_σ be the barycenter of an n-simplex σ of K. The space $|\partial\sigma|$ coincides with $|L_\sigma|$ for a subcomplex L_σ of L_{n-1}, and we can define the cone $L_\sigma * b_\sigma$. Clearly $|L_\sigma * b_\sigma| = |\sigma|$ and $|L_\sigma * b_\sigma| \cap |L_{n-1}| = |L_\sigma| = |\partial\sigma|$.

If τ is another n-simplex, then $|L_\sigma * b_\sigma| \cap |L_\tau * b_\tau| = |\sigma \cap \tau|$, which is the realization of a subcomplex of L_{n-1} and therefore of both L_σ and L_τ. Define L_n to be the union of L_{n-1} and the complexes $L_\sigma * b_\sigma$, where σ runs over all n-simplices of K. Our observations about intersections show that L_n is a simplicial complex which contains L_{n-1} as a subcomplex. The union of the L_n is denoted K' and called the barycentric subdivision of K.

The second barycentric subdivision of K is the barycentric subdivision of the first barycentric subdivision, and so on inductively.

We can enumerate the simplices of K' explicitly rather than inductively.

Proposition 4.3.6. *Define $\sigma < \tau$ if σ is a proper face of τ. Then K' is the simplicial complex whose vertices are the barycenters of simplices of K and whose n-simplices σ' are the spans of the geometrically independent sets $\{b_{\sigma_0}, \cdots, b_{\sigma_n}\}$, where $\sigma_0 > \cdots > \sigma_n$. The vertex b_{σ_0} is called the* leading vertex *of the simplex σ'.*

Proof. We show this inductively for the subcomplexes L_n. Since $L_0 = K^0$, this is clear for L_0. Assume that it holds for L_{n-1}. If τ is a simplex of L_n such that

$|\tau|$ is contained in $|K^n|$ but not contained in K^{n-1}, then τ is a simplex in the cone $L_\sigma * b_\sigma$ for some n-simplex σ. By the induction hypothesis and the definition of L_σ, each simplex of L_σ is the span of a set $\{b_{\sigma_0}, \cdots, b_{\sigma_m}\}$, where $\sigma > \sigma_0 > \cdots > \sigma_m$. Therefore τ is the span of a set $\{b_\sigma, b_{\sigma_0}, \cdots, b_{\sigma_m}\}$. □

PROPOSITION 4.3.7. *There is a simplicial map $\xi = \xi_K : K' \longrightarrow K$ whose realization is simplicially close to the identity map and hence homotopic to the identity map.*

PROOF. Let ξ map each vertex b_σ of K' to any chosen vertex of σ. If σ' is a simplex of K' with leading vertex b_{σ_0}, then all other vertices of σ' are barycenters of faces of σ_0, hence are mapped under ξ to vertices of σ_0. Therefore the images under ξ of the vertices of σ' span a face of σ_0, so that ξ is a simplicial map. With these notations, if $x \in |K'|$ is an interior point of the simplex σ', then it is mapped under $|\xi|$ to a point of $\sigma_0 \supset \sigma'$, and we let $\sigma(x) = \sigma_0$. Since ξ maps every vertex of σ' to a vertex of σ_0, x and $\xi(x)$ are both in the closure of σ_0. □

DEFINITION 4.3.8. For an ordered simplicial complex K, define the *standard simplicial map* $\xi : K' \longrightarrow K$ by letting $\xi(b_\sigma)$ be the maximal vertex x_n of the simplex $\sigma = \{x_0, \cdots, x_n\}$.

REMARK 4.3.9. Observe that K' has a canonical ordering even when K does not. Explicitly, the partial ordering of the set of vertices $\{b_\sigma\}$ is given by $b_\sigma \le b_\tau$ if σ is a face of τ. This partial order clearly restricts to a total order on the vertices of each simplex.

PROPOSITION 4.3.10. *A simplicial map $g : K \longrightarrow L$ induces a subdivided simplicial map $g' : K' \longrightarrow L'$ whose realization is simplicially close to $|g|$ and hence homotopic to $|g|$. Moreover, g' is order-preserving.*

PROOF. The images under g of the vertices of a simplex σ of K span a simplex $g(\sigma)$, of possibly lower dimension than σ, and we define $g'(b_\sigma) = b_{g(\sigma)}$ on vertices. If b_{σ_0} is the leading vertex of a simplex σ' of K', then all other vertices of σ' are barycenters of faces of σ_0. Their images under g' are barycenters of faces of $g(\sigma_0)$. If x is an interior point of σ', then both $g(x)$ and $g'(x)$ are in the closure of $g(\sigma_0)$. □

REMARK 4.3.11. When K and L are ordered and g is an order-preserving simplicial map, the following "naturality" diagram commutes if we use the standard simplicial maps ξ for K and L.

$$\begin{tikzcd} K' \arrow[r, "g'"] \arrow[d, "\xi"'] & L' \arrow[d, "\xi"] \\ K \arrow[r, "g"'] & L \end{tikzcd}$$

REMARK 4.3.12. If we think of K' and K abstractly, then the barycenters of the simplices of K (other than vertices) are vertices of K' that are not vertices of K. All simplices of K' with more than one vertex have at least one vertex that is not in K. Thus the only simplices in K' that are also simplices in K are the vertices of K. However, if we think geometrically, then every simplex τ of K' is contained in a unique simplex σ of K, as should be made clear by drawing a picture of the barycentric subdivision. The simplex σ is called the *carrier* of τ.

4.4. The simplicial approximation theorem

The classical point of barycentric subdivision is its use in the simplicial approximation theorem, which in its simplest form reads as follows. Starting with $K^{(0)} = K$, let $K^{(n)} = K^{(n-1)'}$ be the nth barycentric subdivision of a simplicial complex K. By iteration of $\xi\colon K' \longrightarrow K$, we obtain a simplicial map $\xi^{(n)}\colon K^{(n)} \longrightarrow K$ whose geometric realization is homotopic to the identity map.

Theorem 4.4.1. *Let K be a finite simplicial complex and L be any simplicial complex. Let $f\colon |K| \longrightarrow |L|$ be any continuous map. Then, for some sufficiently large n, there is a simplicial map $g\colon K^{(n)} \longrightarrow L$ such that f is homotopic to $|g|$.*

This means that, for the purposes of homotopy theory, general continuous maps may be replaced by simplicial maps. Since this is proved in so many places, we shall content ourselves with a slightly sketchy proof. It relies on the classical Lebesque lemma, whose proof is not hard but just a little far afield.

Lemma 4.4.2 (Lebesque lemma). *Let (X, d) be a compact metric space with a given open cover $\mathscr{U}$. Then there exists a number $\lambda > 0$ such that every subset of X with diameter less than λ is contained in some set $U \in \mathscr{U}$. The smallest such λ is called the Lebesque number of the cover.*

Definition 4.4.3. For a vertex v of a simplicial complex K, define $star(v)$ to be the union of the interiors of all simplices of $|K|$ that contain v as a vertex. For a subcomplex L of K, define $star(L) \subset |K|$ to be the union over $v \in L$ of the open spaces $star(v)$.

Proof of the simplicial approximation theorem. We are given a map $f\colon |K| \longrightarrow |L|$. Give $|K|$ the open cover by the sets $f^{-1}(star(w))$, where w runs over the vertices of L. Since $|K|$ is a compact subspace of a metric space, the Lebesgue lemma ensures that there is a number λ such that any subset of $|K|$ of diameter less than λ is contained in one of the open sets $f^{-1}(star(w))$. The diameter of a (closed) simplex is easily seen to be the maximal length of a one-dimensional face. Each barycentric subdivision therefore has the effect of decreasing the maximal diameter of a simplex. Precisely, the maximal diameter of the subdivision of a q-simplex turns out to be $q/q+1$ times the maximal diameter of the given simplex (e.g. [**39**, p.124], [**20**, p.24], [**18**, p. 120]), but the precise estimate is not important.

What is important is that, since K is finite, for any $\delta > 0$ there is a large enough n such that every simplex of $K^{(n)}$ has diameter less than $\delta/2$. Then each $star(v)$ for a vertex v of $K^{(n)}$ has diameter less than δ, and we conclude that $f(star(v)) \subset star(w)$ for some vertex w of L. Define $g\colon V(K^{(n)}) \longrightarrow V(L)$ by letting $g(v) = w$ for some w such that $f(star(v)) \subset star(w)$. One checks that g maps simplices to simplices and so specifies a map of simplicial complexes. If u is an interior point of a simplex σ of K, then $f(x)$ is an interior point of some simplex τ of L. One can check that g maps each vertex of σ to a vertex of τ. This implies that $|g|$ is simplicially close to f and therefore homotopic to f. $\square$

4.5. Contiguity classes and homotopy classes

We are interested not just in representing maps up to homotopy as simplicial maps, but in enumerating the resulting homotopy classes of maps. For two spaces X and Y, we define the set $[X, Y]$ of homotopy classes of maps $X \longrightarrow Y$ to be the set of equivalence classes of maps $f\colon X \longrightarrow Y$, where two maps are equivalent if they are homotopic. We write $[f]$ for the homotopy class of f. This notion has a number of variants. For example, we can consider based spaces, base-point preserving maps, and homotopies that preserve the basepoints. We write $[X,Y]_*$ for the resulting set of based homotopy classes of based maps. Thus, with this notation, $\pi_n(X) = [S^n, X]_*$.

We want to understand the relationship between simplicial maps $K \longrightarrow L$ and the set $[|K|,|L|]$, where K is finite. Thus we fix K and L in the rest of this section, taking K to be finite.

We know that any homotopy class is represented by a simplicial map $f\colon K \longrightarrow L$, provided that we first subdivide K sufficiently, and we ask for a simplicial description of when two simplicial maps $f,g\colon K \longrightarrow L$ have homotopic geometric realizations. The notion of "contiguity" can be used to give an answer. If $q > n$, we agree to write $\xi\colon K^{(q)} \longrightarrow K^{(n)}$ for the map obtained by iteration of maps ξ.

Definition 4.5.1. Let $f,g\colon K \longrightarrow L$ be simplicial maps between (geometric) simplicial complexes. We say that f is *contiguous* to g if for every simplex σ of K, the union $f(\sigma)\cup g(\sigma)$ is contained in a simplex of L. More generally, let $f\colon K \longrightarrow L$ and $g\colon K^{(n)} \longrightarrow L$ be simplicial maps. We say that f is contiguous to g if for each simplex τ of $K^{(n)}$ with carrier σ in K, $f(\sigma)\cup g(\tau)$ is contained in a simplex of L.

If $q > n$, a check of definitions shows that if f and g are contiguous, then so are f and $g \circ \xi$. Similarly, if $q > 0$ and f and g are contiguous, then so are $f \circ \xi$ and g, where now $\xi\colon K^{(q)} \longrightarrow K$. The relation of contiguity is reflexive and symmetric, but it is not transitive. We let $\sim$ denote the equivalence relation generated by contiguity. Thus $f \sim g$ if there is a sequence of simplicial maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that f_i is contiguous to f_{i+1} for $i < q$.

Proposition 4.5.2. *If $f,g\colon K \longrightarrow L$ are contiguous simplicial maps, then $|f| \simeq |g|\colon |K| \longrightarrow |L|$.*

Proof. In fact, $|f|$ and $|g|$ are simplicially close by a comparison of definitions. Therefore this result is a special case of Proposition 4.2.4: the same simplex by simplex linear homotopy does the trick. □

Remember that two simplicially close maps $f,g\colon X \longrightarrow |L|$ have homotopic realizations, where X is any space, not necessarily a simplicial complex. We used that fact to show that if K is finite, then any map $f\colon |K| \longrightarrow |L|$ is homotopic to the realization of a simplicial map $g\colon K^{(n)} \longrightarrow L$ for some sufficiently large n. It is natural to ask how unique that simplicial approximation is, and the notion of contiguity gives a useful answer.

Proposition 4.5.3. *If g and g' are simplicial approximations of the same continuous map $f\colon |K| \longrightarrow |L|$, K finite, then g and g' are contiguous.*

Proof. To see this, just look back at the proof of the simplicial approximation theorem. □

Theorem 4.5.4. *If f and f' are homotopic maps $|K| \longrightarrow |L|$, K finite, and g and g' are simplicial approximations to f and f', then g is contiguous to g'. Therefore, for every pair of homotopic maps $f, f'\colon |K| \longrightarrow |L|$, there is a sufficiently large n such that f and f' are represented by contiguous simplicial maps $K^{(n)} \longrightarrow L$.*

Maybe revisit this proof to make clear L need not be finite; compare Thibault's passage to limits in his Thm 2.5.30

Sketch proof. Two slightly different detailed proofs may be found in [**20**, p. 40], [**39**, p. 132]. We follow [**20**]. Remember that $|L|$ is a subspace of some $\mathbb{R}^N$, so that we can talk about the distance between two points of $|L|$. We define the distance between two maps $f, g\colon |K| \longrightarrow |L|$ to be the maximum of the distances between $f(x)$ and $g(x)$ for $x \in |K|$. Let λ be the Lebesque number of the covering of $|L|$ by the open stars of its vertices and let $\varepsilon = (1/3)\lambda$. Then ε is small enough that if the distance between f and g is less than ε, then there is a simplicial map g that is a simplicial approximation of both f and f'. The precise estimate ε is unimportant. It is clear from the proof of the simplicial approximation theorem that some small enough ϵ will have the stated property.

Returning to the hypotheses of the theorem, let $h\colon |K| \times I \longrightarrow |L|$ be a homotopy from $f = h_0$ to $f' = h_1$, where $h_t(x) = h(x,t)$. The claim is that there is a simplicial approximation g to f, a simplicial approximation g' to f', and a sequence of simplicial maps $\{g = g_1, g_2, \cdots, g_q = g'\}$ such that g_i is contiguous to g_{i+1} for $i < q$. We use an ε, δ proof. There is a $\delta > 0$ such that $|h_s(x), h_t(x)| < \varepsilon$ for all $x \in |K|$ and all $s, t \in I$ such that $|t - s| < \delta$. Choose $q > 1/\delta$. Then, for $i < q$, the distance between $h_{(i-1)/q}$ and $h_{i/q}$ is less than ε. Therefore these two maps have a common simplicial approxmation g_i. Since g_i and g_{i+1} are both simplicial approximations of $h_{i/q}$, they are contiguous and we have chosen our maps so that $g = g_1$ is a simplicial approximation of f and $g' = g_q$ is a simplicial approximation to f'. By the previous result, they are contiguous to any other such simplicial approximations. $\square$

CHAPTER 5

The relation between A-spaces and simplicial complexes

Following McCord [**32**], we are going to relate A-spaces, and in particular F-spaces, with simplicial complexes, explaining how to go back and forth between them. Since any Alexandroff space is homotopy equivalent to a T_0-space, there is no loss of generality if we restrict attention to A-spaces. As usual, the reader may prefer to think only in terms of F-spaces.

5.1. The construction of simplicial complexes from A-spaces

Definition 5.1.1. Let X be an A-space. Define $\mathscr{K}(X)$ to be the abstract simplicial complex whose vertex set is X and whose simplices are the finite totally ordered subsets of the poset X; $\mathscr{K}(X)$ is often called the *order complex* of A. Observe that the partial order of X gives an ordering of $\mathscr{K}(X)$, since it restricts to a total order on each simplex. Observe too that if V is a subspace of X, then $\mathscr{K}(V)$ is a full subcomplex of $\mathscr{K}(X)$ since any totally ordered subset of X whose points are in V is a totally ordered subset of V. Since a map $f\colon X \longrightarrow Y$ is an order–preserving function, it may be regarded as a simplicial map $\mathscr{K}(f)\colon \mathscr{K}(X) \longrightarrow \mathscr{K}(Y)$.

Theorem 5.1.2. *For an A-space X, there is a weak homotopy equivalence*

$$\psi = \psi_X \colon |\mathscr{K}(X)| \longrightarrow X$$

such that the following diagram commutes for each map $f\colon X \longrightarrow Y$.

$$\begin{tikzcd}
|\mathscr{K}(X)| \arrow[r, "|\mathscr{K}(f)|"] \arrow[d, "\psi_X"'] & |\mathscr{K}(Y)| \arrow[d, "\psi_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof. Each point $u \in |\mathscr{K}(X)|$ is an interior point of a simplex σ spanned by some strictly increasing sequence $x_0 < x_1 < \cdots < x_n$ of points of X. We define $\psi(u) = x_0$. For $f\colon X \longrightarrow Y$, $\mathscr{K}(f)(u)$ is in the simplex spanned by the $f(x_i)$ and $f(x_0) \le f(x_1) \le \cdots \le f(x_n)$. Omitting repetitions, we see that $f(x_0)$ is the minimal vertex of this simplex, so that $\psi(f(u)) = f(x_0) = f(\psi(u))$, which proves that the diagram commutes. We must still prove that ψ is continuous and that it is a weak homotopy equivalence.

For $x \in X$, let $star(x)$ denote the union of the interiors of the simplices of $\mathscr{K}(X)$ that have x as a vertex; it is an open neighborhood of x in $|\mathscr{K}(X)|$. For an open subset V of X, define the open star, $star(V)$, to be the union over the vertices $v \in V$ of the open subspaces $star(v)$. It is the complement of $|\mathscr{K}(X - V)|$ in $|\mathscr{K}(X)|$. To see that ψ is continuous, we show that $\psi^{-1}(V) = star(V)$. If

$\psi(u) = v \in V$, then v is the initial vertex x_0 of a simplex σ. Since a vertex v is the unique interior point of the simplex $\{v\}$, $u \in star(V)$. Conversely, suppose that $u \in star(v)$, where $v \in V$. Then u is an interior point of a simplex σ determined by an increasing sequence $x_0 < x_1 < \cdots < x_n$ such that some $x_i = v \in V$. Since $x_0 \leq v$, $x_0 \in U_v$. Since V is open, $U_v \subset V$. Thus $\psi(u) = x_0$ is in V.

It remains to prove that ψ is a weak homotopy equivalence. We shall do so by applying Theorem 3.3.1 to the minimal open cover $\{U_x\}$ of X. If x is in $U_y \cap U_z$, then x is in both U_y and U_z, so that U_x is contained in both U_y and U_z. This verifies the first hypothesis of the cited theorem. For the second hypothesis, we know that each U_x is a contractible subspace of V. We also know that each $|\mathscr{K}(U_x)|$ is a contractible space. In fact, $\mathscr{K}(U_x)$ is a simplicial cone, in the sense that for every simplex σ of $\mathscr{K}(U_x)$ which does not contain x, $\sigma \cup \{x\}$ is a simplex of $\mathscr{K}(U_x)$. The realization of such a simplicial cone is contractible to the cone vertex x since $h(y,t) = (1-t)y + tx$ gives a well-defined contracting homotopy. Specializing the following general result to $L = \mathscr{K}(U_x)$, we see that $star(U_x)$ is also contractible. Therefore each restriction $\psi\colon \psi^{-1}(U_x) \longrightarrow U_x$ is a weak homotopy equivalence and Theorem 3.3.1 applies to show that ψ is a weak equivalence. □

Proposition 5.1.3. *Let L be a full subcomplex of a simplicial complex K. Then $|L|$ is a deformation retract of its open star, $star L$, in $|K|$.*

Proof. Again, $star L$, is defined to be the union of the open stars of the vertices of L. This result is a standard fact in the theory of simplicial complexes, and a more detailed proof than we shall given can be found in [**38**, 70.1 and p. 427]. Consider a simplex σ that is in the closure of $star(L)$. Then σ has vertex set the disjoint union of a set of vertices in L and a set of vertices in $K - L$. Each point u of σ that is neither in the span s of the vertices in L nor in the span t of the vertices not in L is on a unique line segment joining a point in t to a point in s. Define the required retraction r by sending u to the end point in $s \subset L$ of this line segment, letting r be the identity map on L and thus on s. Deformation along such line segments gives the required homotopy showing that $i \circ r$ is homotopic to the identity, where i is the inclusion of $|L|$ in its open star. □

Example 5.1.4. Suppose that $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere S^n. Then the dimension of $|\mathscr{K}(X)|$, which is $h(X) - 1$, must be at least n. Thus $h(X) \geq n+1$. Therefore, by Proposition 3.5.3, X has at least $2n+2$ points and, if X has exactly $2n+2$ points, then it is homeomorphic to $\mathbb{S}^n S^0$.

5.2. The construction of A-spaces from simplicial complexes

Now let K be a finite geometric simplicial complex with first barycentric subdivision K'. Remember that $|K| = |K'|$.

Definition 5.2.1. Define an A-space $\mathscr{X}(K)$ as follows. The points of $\mathscr{X}(K)$ are the barycenters b_σ of the simplices of K, that is, the vertices of K'. The required partial order $\leq$ is defined by $b_\sigma \leq b_\tau$ if $\sigma \subset \tau$. The open subspace U_{b_σ} coincides with $\mathscr{X}(\sigma)$, where σ (together with its faces) is regarded as a subcomplex of K. For a simplicial map $g\colon K \longrightarrow L$, define $\mathscr{X}(g)\colon \mathscr{X}(K) \longrightarrow \mathscr{X}(L)$ by $\mathscr{X}(g)(b_\sigma) = b_{g(\sigma)}$, and note that this function is order–preserving and therefore continuous. Using the barycenters themselves to realize the vertices geometrically, we see from the description of K' in Proposition 4.3.6 that $\mathscr{K}\mathscr{X}(K) = K'$ and $\mathscr{K}\mathscr{X}(g) = g'$.

We use Theorem 5.1.2 to obtain the following complementary result.

THEOREM 5.2.2. *For a simplicial complex K, there is a weak homotopy equivalence*

$$\phi = \phi_K \colon |K| \longrightarrow \mathscr{X}(K)$$

such that the following diagram is commutative

$$\begin{tikzcd} |K'| \arrow[r, "|g'|"] \arrow[d, "\phi_K"'] & |L'| \arrow[d, "\phi_L"] \\ \mathscr{X}(K) \arrow[r, "\mathscr{X}(g)"'] & \mathscr{X}(L) \end{tikzcd}$$

PROOF. Define

$$\phi_K = \psi_{\mathscr{X}(K)} \colon |K'| = |\mathscr{K}\mathscr{X}(K)| \longrightarrow \mathscr{X}(K).$$

Then ϕ_K is a weak homotopy equivalence and the diagram commutes by Theorem 5.1.2. Since $|K| = |K'|$ and $|L| = |L'|$, we could replace $|g'|$ by $|g|$ in the diagram. By Proposition 4.3.10, $|g'|$ is simplicially close to $|g|$ and hence homotopic to $|g|$. Therefore, after the replacement, the diagram would only be homotopy commutative, in the sense that the two composite maps in the diagram would be homotopic. □

5.3. Mapping spaces

For completeness, we record results of Stong [**40**, §6] that were obtained about the same time as the results of McCord recorded above and that give a quite different approach to the relationship between finite simplicial complexes and finite spaces. Since the proofs are fairly long and combinatorial in flavor, and since the statements do not have quite the same immediate impact as those in McCord's work, we shall not work through the details here.

Recheck: add? Expository paper topic?

Rather than constructing finite models for finite simplicial complexes, Stong studies all maps from the geometric realizations of simplicial complexes K into finite spaces X by studying the properties of the function space $X^K \equiv X^{|K|}$. More generally, he fixes a subcomplex L of K and a basepoint $* \in X$ and studies the subspace $(X,*)^{(K,L)}$ of maps $f \colon |K| \longrightarrow X$ such that $f(|L|) = *$. Homotopies relative to $|L|$ between such maps are homotopies h such that $h(p,t) = *$ for $p \in |L|$.

THEOREM 5.3.1. *Let L be a subcomplex of a finite simplicial complex K, let X be a finite space with basepoint $*$, and let $F = (X,*)^{(K,L)}$ denote the subspace of X^K consisting of those maps $f \colon |K| \longrightarrow X$ such that $f(|L|) = *$.*

(i) For any $f \in F$, there is a map $g \in F$ such that the set $V = \{h | h \leq g\} \subset F$ is a neighborhood of f in F; that is, there is an open subset U such that $f \in U \subset V$.

(ii) If $f \simeq f'$ relative to L, then there is a sequence of elements $\{g_1, \cdots, g_s\}$ in F such that $g_1 = f$, $g_s = f'$, and either $g_i \leq g_{i+1}$ or $g_{i+1} \leq g_i$ for $1 \leq i < s$.

The essential point of this analysis is the following consequence.

COROLLARY 5.3.2. *The path components and components of F coincide. That is, the homotopy classes of maps $f \colon (K,L) \longrightarrow (X,*)$ are in bijective correspondence with the components of F.*

5.4. Simplicial approximation and A-spaces

There are two papers, [**16, 17**], that start with the simplicial approximation theorem and take up where McCord and Stong leave off. In view of the explicit constructions of $\mathscr{K}(X)$ and $\mathscr{X}(K)$, the following definition is reasonable.

earlier, in section defining $\mathscr{X}$, and clarify what it looks like without barycenter terminology

DEFINITION 5.4.1. Define the *barycentric subdivision* of an A-space X to be $X' = \mathscr{X}\mathscr{K}(X)$. For a map $f\colon X \longrightarrow Y$, define $f'\colon X' \longrightarrow Y'$ to be $\mathscr{X}\mathscr{K}(f)$. Iterating the construction, define $X^{(n)} = (X^{(n-1)})'$, where $X^{(0)} = X$. Observe inductively that $\mathscr{K}(X^{(n)}) = \mathscr{K}(X)^{(n)}$ since $\mathscr{K}\mathscr{X}(K) = K'$.

PROPOSITION 5.4.2. *There is a map $\zeta = \zeta_X\colon X' \longrightarrow X$ that makes the following diagram commute, and ζ is a weak homotopy equivalence.*

$$\begin{tikzcd}
|\mathscr{K}\mathscr{X}\mathscr{K}(X)| \arrow[r, equal] \arrow[d, "\psi_{\mathscr{X}\mathscr{K}(X)}"'] & |\mathscr{K}(X)'| \arrow[r, "|\xi_{\mathscr{K}(X)}|"] & |\mathscr{K}(X)| \arrow[d, "\psi_X"] \\
X' = \mathscr{X}\mathscr{K}(X) \arrow[rr, "\zeta_X"'] & & X.
\end{tikzcd}$$

The simplicial map $\xi_{\mathscr{K}(X)}$ coincides with $\mathscr{K}(\zeta_X)\colon \mathscr{K}(X') \longrightarrow \mathscr{K}(X)$. The following diagram commutes for a map $f\colon X \longrightarrow Y$.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "\zeta_X"'] & Y' \arrow[d, "\zeta_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

PROOF. The points of $\mathscr{X}\mathscr{K}(X)$ are the barycenters of the simplices of $\mathscr{K}(X)$. These simplices σ are spanned by increasing sequences $x_0 < \cdots < x_n$ of elements of X. Let $\zeta(b_\sigma) = x_n$. Since $b_\sigma \leq b_\tau$ implies $\sigma \subset \tau$ and thus $\zeta(b_\sigma) \leq \zeta(b_\tau)$, ζ is continuous. We understand $\xi_{\mathscr{K}(X)}$ to be the standard choice specified in Definition 4.3.8. Inspection of definitions shows that $\xi_{\mathscr{K}(X)} = \mathscr{K}(\zeta_X)$. The commutativity of the first diagram follows from the "naturality" of ψ with respect to the map ζ_X. That is, this diagram is a specialization of the commutative diagram of Theorem 5.1.2, with f there taken to be ζ_X here. That ζ_X is a weak homotopy equivalence follows from the diagram, since all other maps in it are weak homotopy equivalences. The last statement is clear by inspection of definitions. $\square$

THEOREM 5.4.3. *Let X be an F-space and Y be an A-space, and let $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$ be any map. Then for some sufficiently large n there is a map $g\colon X^{(n)} \longrightarrow Y$ such that f is homotopic to $|\mathscr{K}(g)|$. We call g a* finite approximation *to f.*

PROOF. By the classical simplicial approximation theorem for simplicial complexes, for a sufficiently large n there is a simplicial approximation

$$j\colon \mathscr{K}(X^{(n-1)}) = \mathscr{K}(X)^{(n-1)} \longrightarrow \mathscr{K}(Y)$$

to f. Let g be the composite

$$X^{(n)} = \mathscr{X}\mathscr{K}(X^{(n-1)}) \xrightarrow{\mathscr{X}(j)} \mathscr{X}\mathscr{K}(Y) = Y' \xrightarrow{\zeta_Y} Y.$$

Then

$$\mathscr{K}(g) = \mathscr{K}(\zeta_Y) \circ \mathscr{K}\mathscr{X}(j) = \mathscr{K}(\zeta_Y) \circ j'.$$

We have $|j'| \simeq |j|$ by Proposition 4.3.10 and $|j| \simeq f$ by assumption. Since we also have $|\mathscr{K}(\zeta_Y)| = |\xi_{\mathscr{K}(Y)}| \simeq \mathrm{id}$, we have $|\mathscr{K}(g)| \simeq f$, as required. □

The point to emphasize here is that finite models for spaces have far too few maps between them. For example, $\pi_n(S^n, *) = \mathbb{Z}$, but there are only finitely many distinct maps from any finite model for S^n to itself. The theorem says that, after subdividing the domain sufficiently, we can realize any of these homotopy classes in terms of maps between (different) finite models for S^n.

5.5. Contiguity of maps between A-spaces

Remembering the definition of $\mathscr{K}(X)$, we may as well refer to points of an A-space X as vertices and to finite ordered subsets of X as simplices. Thus "simplex" is just a convenient abbreviation of "finite totally ordered subset". We use that language in translating the notion of contiguity from simplicial complexes to finite spaces. If $q > n$, we agree to write ζ for the composite $X^{(q)} \longrightarrow X^{(n)}$ determined by iteration of maps ζ.

Definition 5.5.1. Let $f, g\colon X \longrightarrow Y$ be continuous maps between A-spaces. We say that f is *contiguous* to g if for every simplex σ of X, there is a simplex of Y that contains both $f(\sigma)$ and $g(\sigma)$. More generally, let $f\colon X \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ be continuous maps. We say that f is contiguous to g if for each simplex σ of $X^{(n)}$, there is a simplex of Y that contains both $(f \circ \zeta)(\sigma)$ and $g(\sigma)$. If $q > n$, a check of definitions shows that if f and g are continguous, then so are f and $g \circ \zeta$. Similarly, if $q > 0$ and f and g are contiguous, then so are $f \circ \zeta$ and g, where now $\zeta\colon K^{(q)} \longrightarrow K$. The relation of contiguity is reflexive and symmetric, but it is not transitive. We let $\sim$ denote the equivalence relation generated by contiguity. Thus $f \sim g$ if there is a sequence of continuous maps $\{f = f_1, \cdots, f_q = g\}$ such that f_i is contiguous to f_{i+1} for $i < q$.

Proposition 5.5.2. *If $f\colon X \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ are contiguous maps between A-spaces, then $f \circ \zeta \simeq g\colon X^{(n)} \longrightarrow Y$.*

The analogue for simplicial maps used the notion of simplicially close maps from an arbitrary space to a simplicial complex. We have an analogous notion for maps to A-spaces. The term "approximate map" is sometimes used for either of these notions.

Definition 5.5.3. Let X be any space and let Y be an A-space. Two maps $f, g\colon X \longrightarrow Y$ are *simplicially close* if for each $x \in X$ there is a simplex $\tau = \tau_x$ of Y such that $f(x)$ and $g(x)$ are both in τ.

Clearly contiguous maps between A-spaces are simplicially close in this sense. Therefore the following result implies Proposition 5.5.2.

Proposition 5.5.4. *At least if both X and Y are A-spaces, simplicially close maps $f, g\colon X \longrightarrow Y$ are homotopic.*

Proof. Define $h\colon X \times I \longrightarrow Y$ by

$$h(x,t) = f(x) \quad \text{if} \quad 0 \le t < 1/2$$

$$h(x,1/2) = \begin{cases} g(x) & \text{if } f(x) \le g(x) \\ f(x) & \text{if } g(x) \le f(x). \end{cases}$$

$$h(x,t) = g(x) \quad \text{if} \quad 1/2 < t \le 1$$

Since $f(x)$ and $g(x)$ are both in a simplex τ_x, either $f(x) \leq g(x)$ or $g(x) \leq f(x)$. Therefore h is well-defined, and it suffices to prove that h is continuous. One way to study the problem is to introduce the three point space $J = \{0, 1/2, 1\}$ whose proper open subsets are $\{0\}$, $\{1\}$, and their union $\{0,1\}$. Define $\pi\colon I \longrightarrow J$ by

$$\pi([0,1/2)) = 0, \quad \pi(1/2) = 1/2, \quad \pi((1/2,1]) = 1.$$

Certainly π is continuous, hence so is $\mathrm{id} \times \pi\colon X \times I \longrightarrow X \times J$. There is an obvious function $j\colon X \times J \longrightarrow Y$ such that $h = j \circ (\mathrm{id} \times \pi)$, namely

$$j(x,0) = f(x), \quad j(x,1/2) = h(x,1/2), \quad j(x,1) = g(x).$$

It suffices to prove that j is continuous. When X is an A-space, this can be done by giving $X \times J$ the product order, namely $(x,i) \leq (x',i')$ if and only if both $x \leq x'$ and $i \leq i'$, and checking that j is order-preserving since f and g are order preserving. Since both $0 < 1/2$ and $1 < 1/2$ and since $x \leq x'$ implies both $f(x) \leq f(x')$ and $g(x) \leq g(x')$, the check is easy and can be left to the reader. $\square$

Comparing our two definitions of simplicially close maps, for simplicial complexes and for Alexandroff spaces, we see the following properties of the constructions $\mathscr{K}$ and $\mathscr{X}$.

PROPOSITION 5.5.5. *If $f\colon \mathscr{K}(X^{(m)}) \longrightarrow \mathscr{K}(Y)$ and $g\colon \mathscr{K}(X^{(n)}) \longrightarrow \mathscr{K}(Y)$ are contiguous maps of simplicial complexes, then $\zeta_Y \circ \mathscr{X}(f)\colon X^{(m+1)} \longrightarrow Y$ and $\zeta_Y \circ \mathscr{X}(g)\colon X^{(n+1)} \longrightarrow Y$ are contiguous maps of A-spaces. If $f\colon X^{(m)} \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ are contiguous maps of A-spaces, then $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous maps of simplicial complexes.*

Now the simplicial results Theorems 4.5.3 and 4.5.4 have the following immediate consequences.

PROPOSITION 5.5.6. *If $g\colon X^{(m)} \longrightarrow Y$ and $g'\colon X^{(n)} \longrightarrow Y$ are finite approximations of the same map $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$, then g and g' are contiguous.*

THEOREM 5.5.7. *If f and f' are homotopic maps $|\mathscr{K}X| \longrightarrow |\mathscr{K}Y|$ and g and g' are finite approximations to f and f', then g is contiguous to g'. Therefore, for every pair of homotopic maps $f, f'\colon |\mathscr{K}X| \longrightarrow |\mathscr{K}Y|$, there is a sufficiently large n such that f and f' have contiguous finite approximations $X^{(n)} \longrightarrow Y$.*

We have focused on understanding homotopy classes of maps between finite simplicial complexes in terms of contiguity classes of simplicial maps and contiguity classes of continuous maps between finite spaces, but one can also ask the relationship between homotopy classes and contiguity classes of maps between finite spaces. We have seen that contiguous maps are homotopic, but the converse is also true. To see that, we refine Proposition 2.2.12, following [**5**, 2.1.1].

DEFINITION 5.5.8. Maps $f, g\colon X \longrightarrow Y$ between Alexandroff spaces are very close if $f = g$ on all but one point $x \in X$, and either $f(x) < g(x)$ or $g(x) < f(x)$. The maps f, g are closely equivalent if there is a sequence of maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that f_i is very close to f_{i+1} for $i < q$.

LEMMA 5.5.9. *If $f, g\colon X \longrightarrow Y$ are very close, then they are contiguous.*

PROOF. Without loss of generality, we may assume that $f(x) < g(x)$ for the unique point x on which f and g differ. For a simplex σ of X that does not contain x, we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of Y. If x is in a simplex $\sigma = \{x_0 < x_1 < \cdots < x_n\}$, then $x = x_i$ for some i and $f(\sigma) \cup g(\sigma)$ is the simplex obtained by deleting repetitions from the ordered set

$$\{f(x_0) \leq f(x_1) \leq \cdots \leq f(x_i) \leq g(x_i) \leq g(x_{i+1}) \leq \cdots \leq g(x_n)\} \qquad \square$$

THEOREM 5.5.10. *If $f, g\colon X \longrightarrow Y$ are homotopic maps between finite spaces, then f and g are very closely equivalent and are therefore contiguous.*

PROOF. By Proposition 2.2.12, we may assume without loss of generality that $f \leq g$. Let $A \subset X$ be the set of points x such that $f(x) \neq g(x)$. Of course, we may assume that A is non-empty, and we let x be a maximal point in A, so that $x' > x$ implies $f(x') = g(x')$. Define f_2 by $f_2(x') = f(x')$ for $x' \neq x$ and $f_2(x) = g(x)$. Certainly f_2 is order–preserving and thus continuous. It differs from g at one less point than $f = f_1$ differs from g. Repeating the construction, we arrive at $f_q = g$ after finitely many steps since X and Y are finite. $\square$

5.6. Products of simplicial complexes

We here discuss several important constructions that we shall use later. The discussion focuses on how these concepts compare in the worlds of posets, simplicial complexes, and general spaces.

Inclusions of posets and simplicial complexes have an obvious meaning, and they are characterized as in Lemma 1.5.4. Quotients are more subtle and we shall return to them when we discuss simplicial sets.

We defined disjoint unions $X \amalg Y$ of topological spaces in Definition 1.4.3 and characterized the disjoint union by a universal property in Lemma 1.5.6. Similarly, we defined the product $X \times Y$ of topological spaces in Definition 1.4.4 and characterized the product by a universal property in Lemma 1.5.7. We can ask similarly for disjoint unions, often called "'coproducts", and products of other kinds of objects. Since posets are "the same" as A-spaces, we can translate the definitions of their coproducts and products to obtain the following definitions.

DEFINITION 5.6.1. The disjoint union of posets X and Y is the set $X \amalg Y$ with the partial order specified by requiring X and Y to be subposets, with no relations $x \leq y$ or $y \leq x$ for $x \in X$ and $y \in Y$. If $f\colon X \longrightarrow Z$ and $Y \longrightarrow Z$ are order-preserving functions to a poset Z, then there is a unique order-preserving function $X \amalg Y \longrightarrow Z$ that restricts to f and g on X and Y.

DEFINITION 5.6.2. The product of posets X and Y is the set $X \times Y$ with the partial order specified by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. The projections to X and Y are order-preserving and if $f\colon W \longrightarrow X$ and $g\colon W \longrightarrow Y$ are order-preserving maps defined on a poset W, then the unique function $W \longrightarrow X \times Y$ with coordinates f and g is order-preserving.

The specified partial orders on $X \amalg Y$ and $X \times Y$ are the only ones that satisfy the specified universal property. We shall discuss definitions like this formally when we discuss categories, but this categorical point of view can be inconsistent with properties we might like, as we illustrate by considering products of simplicial complexes. Disjoint unions behave as one would expect and require no discussion.

DEFINITION 5.6.3. The product $K \times L$ of two abstract simplicial complexes K and L has $V(K \times L) = V(K) \times V(L)$ and has simplices all subsets of products $\sigma \times \tau$ of sets σ and τ that prescribe simplices of K and L. We must take subsets here since a general subset of $\sigma \times \tau$ is not a product of subsets of σ and τ. The projections from $V(K \times L)$ to $V(K)$ and $V(L)$ prescribe simplicial maps and if $f\colon J \longrightarrow K$ and $g\colon J \longrightarrow L$ are maps of simplicial complexes then the unique function $V(J) \longrightarrow V(K) \times V(L)$ with coordinates $V(f)$ and $V(g)$ prescribes a map of simplicial complexes. The product of geometric simplicial complexes in $\mathbb{R}^M$ and $\mathbb{R}^N$ is defined similarly as a geometric simplicial complex in $\mathbb{R}^{M+N} = \mathbb{R}^M \times \mathbb{R}^N$.

It is important to distinguish beween ordered and unordered simplicial complexes here. If we construct realizations directly, without introducing orderings, it is *not* true that the realization of a product of abstract simplicial complexes is homeomorphic to the product of their realizations. The former just has too many simplices. The difference already appears when K and L each have just two vertices and their subsets. However, the difference disappears in the presence of orderings.

Proposition 5.6.4. *Let X and Y be posets. Then $\mathscr{K}(X\times Y)$ is a subdivision of $\mathscr{K}(X)\times\mathscr{K}(Y)$, hence both have the same geometric realization, and their common realization is homeomorphic to $|\mathscr{K}(X)| \times |\mathscr{K}(Y)|$.*

Proof. Clearly $\mathscr{K}(X) \times \mathscr{K}(Y)$ and $\mathscr{K}(X \times Y)$ have the same finite set of vertices. Inspection shows that every simplex of $\mathscr{K}(X\times Y)$ is contained in a product of simplices of $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ and that every simplex of $\mathscr{K}(X)\times\mathscr{K}(Y)$ is a union of finitely many simplices of $\mathscr{K}(X\times Y)$. In more detail, the n-simplices of $\mathscr{K}(X\times Y)$ are all sets of pairs $\tau = \{(x_i, y_i)|0 \le i \le n\}$ such that $(x_i, y_i) < (x_{i+1}, y_{i+1})$. This means that $x_i \le x_{i+1}$ and $y_i \le y_{i+1}$, with not both equal. If there are $p+1$ distinct x_i and $q + 1$ distinct y_j, then $\rho = \{x_i\}$ is a p-simplex of $\mathscr{K}(X)$, $\sigma = \{y_j\}$ is a q-simplex of $\mathscr{K}(Y)$, and τ is contained in $\rho \times \sigma$. There are many choices of τ that determine the same ρ and σ. Thus every simplex of $\mathscr{K}(X \times Y)$ is contained in a simplex of $\mathscr{K}(X)\times\mathscr{K}(Y)$. The projections $X\times Y \longrightarrow X$ and $X\times Y \longrightarrow Y$ induce the coordinates of a map

$$|\mathscr{K}(X \times Y)| \longrightarrow |\mathscr{K}(X)| \times |\mathscr{K}(Y)|.$$

A point on the right is a pair (u, v) where u is an interior point of some simplex σ of the geometric simplicial complexe $g\mathscr{K}(X)$ and v is an interior point of some simplex τ of $g\mathscr{K}(Y)$. Since all simplices on the left are subsimplices of some $\sigma \times \tau$, this map is a homeomorphism. $\square$

Definition 5.6.5. Let K and L be ordered simplicial complexes (abstract or geometric). Order the elements of $V(K) \times V(L)$ by $(x, y) \le (x', y')$ if $x \le x'$ and $y \le y'$. The simplices of the *ordered* simplicial complex $K \times L$ are the sets of pairs $\tau = \{(x_i, y_i)|0 \le i \le n\}$ such that $(x_i, y_i) < (x_{i+1}, y_{i+1})$, $\{x_0, \dots, x_n\}$ is a simplex of K and $\{y_0, \dots, y_n\}$ is a simplex of L.

With this definition in place, the last statement of Proposition 5.6.4 generalizes, with the same proof.

Proposition 5.6.6. *Let K and L be ordered (geometric) simplicial complexes. Then the projections induce a homeomorphism*

$$|K \times L| \longrightarrow |K| \times |L|.$$

Intuitively, the point is that the product of two geometric simplices is not a geometric simplex (a square is not a triangle) but can be subdivided into geometric simplices. In effect, the displayed homeomorphism carries out this subdivision consistently over all of the simplices of a product of simplicial complexes.

5.7. The join operation

The join operation played a very substantial role in the early decades of algebraic topology and is a very natural operation in the context of simplicial complexes. We shall only use it peripherally, when we relate simplicial complexes to finite groups, but it is best introduced here, where comparisons with disjoint unions and with products can be seen clearly.

DEFINITION 5.7.1. The *join* $X * Y$ of posets X and Y is the poset given by the disjoint union of the posets X and Y, together with the additional relations $x < y$ if $x \in X$ and $y \in Y$.

As something of a joke, consider the opposite choice available in Definition 5.7.1.

DEFINITION 5.7.2. Define the *antijoin* $(X * Y)^-$ of posets X and Y to be the poset given by the disjoint union of the posets X and Y, together with the additional relations $y < x$ if $x \in X$ and $y \in Y$.

There is no order-preserving function relating $X * Y$ and $(X * Y)^-$, but we have the following illuminating observation.

PROPOSITION 5.7.3. *The subdivisions of $X * Y$ and $(X * Y)^-$ are isomorphic.*

PROOF. Remember that $X' = \mathscr{X}\mathscr{K}X$. We define an isomorphism $f\colon (X * Y)' \longrightarrow (\mathrm{Sd}(X * Y)^-)'$ that restricts to the identity map between the subcomplexes X' and Y' of each. A typical point of $(X * Y)'$ that is in neither X' nor Y' has the form

$$(x_0 < \cdots < x_m < y_0 < \cdots < y_n)$$

where $m \geq 0$, $n \geq 0$, $x_i \in X$, and $y_j \in Y$. Define

$$f(x_0 < \cdots < x_m < y_0 < \cdots < y_n) = (y_0 < \cdots < y_n < x_0 < \cdots < x_m).$$

It is visibly clear that f is a well-defined isomorphism of posets with inverse given by

$$f^{-1}(y_0 < \cdots < y_m < x_0 < \cdots < x_n) = (x_0 < \cdots < x_n < y_0 < \cdots < y_m).$$

$\square$

If Y is a single point, then $X*Y$ is the cone CX as we defined it earlier. Quillen defines $CX = (X*Y)^-$. The choice is arbitrary and we have just seen that the two choices have isomorphic subdivisions and therefore homeomorphic realizations.

REMARK 5.7.4. It is perhaps illuminating to use both choices, and we write C^+X for the first choice and C^-X for the second. There is a canonical map i from $X * Y$ to the poset $C^+X \times C^-Y - \{(c_X, c_Y)\}$, where c_X and c_Y denote the cone points. Indeed, we set $i(x) = (x, c_Y)$ and $i(y) = (c_X, y)$. Since $x < c_X$ and $c_Y < y$, $i(x) < i(y)$ for all x and y, while $i(x) \leq i(x')$ if and only $x \leq x'$ and $i(y) \leq i(y')$ if and only if $y \leq y'$.

Just as for products, the precise definition of which is different when we consider products of posets, of simplicial complexes, and of topological spaces, we have different meanings of the notion of join, all of which are denoted by $*$. However, unlike products, which are characterized by a universal property, the different definitions of the join are primarily motivated by the comparisons among them.

DEFINITION 5.7.5. The *join* $K * L$ of abstract simplicial complexes K and L has vertex set $V(K*L)$ the disjoint union of $V(K)$ and $V(L)$ and has simplices the simplices of K, the simplices of L, and all disjoint unions of simplices of K and L.

The join of geometric simplicial complexes is defined similarly, requiring the disjoint union of $V(K)$ and $V(L)$ to be a linearly independent set.

Conceptually, it is helpful to note that, just like the product, where $X\times Y$ is not literally the same as $Y \times X$ but only isomorphic to it, we should think of disjoint union as an operation only commutative up to isomorphism. Then the evident choice of order on the join of ordered geometric simplicial complexes corresponds to the analogous choice we had when defining the join of posets in Definition 5.6.2.

DEFINITION 5.7.6. The join of topological spaces X and Y is the quotient space of $X\times I\times Y$ obtained by identifying $(x,0,y)$ with $(x',0,y)$ and $(x,1,y)$ with $(x,1,y')$ for all $x,x'\in X$ and $y,y'\in Y$. It is the space of lines connecting X to Y. If X and Y are geometrically independent subspaces of some large Euclidean space, $X*Y$ is defined geometrically as the subspace of points $tx+(1-t)y$ for $x\in X$, $y\in Y$, and $0\le t\le 1$, noting that the point is independent of x if $t=0$ and of y if $t=1$.

LEMMA 5.7.7. *For spaces X and Y, $X*Y$ is homeomorphic to the union $(CX\times Y)\cup_{X\times Y}(X\times CY)$ where the notation indicates that we identify the copies of $X\times Y$ in $CX\times Y$ and $X\times CY$.*

PROOF. We identify $X*Y$ and $(CX\times Y)\cup_{X\times Y}(X\times CY)$ as homeomorphic quotients of subspaces of $X\times Y\times I\times I$. Let J be the diagonal $\{(s,t)|s+t=1\}$ in the square. Then $X*Y$ is homeomorphic to the quotient of $X\times Y\times J$ obtained from the equivalence relation given by

$$(x,y,(1,0))\sim(x',y,(1,0)) \quad\text{and}\quad (x,y,(0,1))\sim(x,y',(0,1)).$$

Think of the cone coordinates of CX and CY as the edges $I_1=[(0,0),(1,0)]$ and $I_2=[(0,0),(0,1)]$ of $I\times I$. Let $K=I_1\cup I_2\subset I\times I$. Then the space

$$(CX\times Y)\cup_{X\times Y}(X\times CY)$$

is homeomorphic to the quotient of $X\times Y\times K$ obtained from precisely the same equivalence relation. Radial projection from the point $(1,1)$ gives a deformation

$$I\times I-\{1,1\}\longrightarrow K$$

that restricts to a homeomorphism $J\longrightarrow K$ and thus induces the claimed homeomorphism. $\square$

PROPOSITION 5.7.8. *For posets X and Y,*

$$\mathscr{K}(X*Y)\cong\mathscr{K}(X)*\mathscr{K}(Y).$$

For abstract simplicial complexes K and L,

$$g(K*L)\cong gK*gL.$$

For ordered geometric simplicial complexes K and L,

$$|K*L|\cong|K|*|L|.$$

We give another way to think about the join $|K|*|L|$ in $\mathbb{R}^N$, where K and L are geometric simplicial complexes. The notion of $X-\{x\}$, $x\in X$, is clear for a poset. For a simplicial complex K, $K-\{v\}$ for $v\in V(K)$ means the simplicial complex that is obtained from K by deleting all simplices which have v as a vertex, and

$\mathscr{K}(X - \{x\}) = \mathscr{K}(X) - \{x\}$. However, $|K - \{v\}|$ is quite different from $|K| - v$. The cone CK of a geometric simplicial complex K is obtained by by adding a vertex c_K that is geometrically independent of all vertices in K and adding a new simplex spanned by the union of c_K and the vertices of σ for each simplex σ of K. If K is ordered, then CK is ordered by requiring c_K to be greater than all other vertices.

PROPOSITION 5.7.9. *Let K and L be ordered (geometric) simplicial complexes. Then*

$$CK \times CL - \{(c_K, c_L)\} = (CK \times L) \cup_{K\times L} (K \times CL)$$

as subcomplexes of $CK \times CL$. Therefore

$$|K| * |L| \cong |CK \times CL - \{(c_K, c_L)\}|$$

.

PROOF. The simplices of $CK \times CL$ that do not have (c_K, c_L) as a vertex are the simplices in either $CK \times L$ or $K \times CL$. The gives the first conclusion. Geometric realization commutes up to homeomorphism with cones, products and unions, so that

$$|(CK \times L) \cup_{K\times L} (K \times CL)| \cong (C|K| \times |L|) \cup_{|K|\times|L|} (|K| \times C|L|).$$

Now Lemma 5.7.7 gives the second conclusion. $\square$

5.8. Remarks on an old list of problems

Revisit

We give a few problems that spring immediately to mind. To the best of my knowledge, these have not been studied, at least not thoroughly. The original 2003 list was considerably longer, but a number of people around the world have since solved many of its problems. Some of their solutions are sprinkled through the book.

PROBLEM 5.8.1. *For small n, determine all homotopy types and weak homotopy types of spaces with at most n elements.*

ADDENDUM 5.8.1. We have given the answer or left it as an exercise when $n \leq 6$. Most finite spaces with so few points are disjoint unions of (weakly) contractible spaces, but we have seen several more interesting examples. I'd like to see the answer for larger n.

PROBLEM 5.8.2. *Is there an effective algorithm for computing the homotopy groups of X in low degrees in terms of the increasing chains in X? An REU paper of Weng described in §**??** elaborated on the computation of the fundamental group by Barmak [**5**].* Not written yet

REMARK 5.8.3. The dimension of the simplicial complex $\mathscr{K}(X)$ is the maximal length of a sequence $x_0 < \cdots < x_n$ in X. A map $g\colon K \longrightarrow L$ of simplicial complexes of dimension less than n is a homotopy equivalence if and only if it induces an isomorphism of homotopy groups in dimension less than n and an epimorphism of homotopy groups in dimension n.

PROBLEM 5.8.4. *Let X be a minimal finite space. Give a descriptive interpretation of what this says about $|\mathscr{K}(X)|$.*

ADDENDUM 5.8.2. There is a nice paper of Osaki [**34**] that interprets Stong's process of passing from an F-space to its core Y. He shows that $\mathscr{K}(Y)$ is obtained from $\mathscr{K}(X)$ by a sequence of elementary simplicial collapses, so that $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ have the same "simple" homotopy type. It follows that if X and Y are homotopy equivalent F-spaces, then $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple homotopy type. If K is not collapsible, then $\mathscr{X}(K)$ is a minimal finite space. As Osaki points out and is clear from Example 3.4.8, there are non-collapsible triangulations K_1 and K_2 of S^1 such that $\mathscr{X}(K_1)$ and $\mathscr{X}(K_2)$ are not homeomorphic and therefore, being minimal, not homotopy equivalent. Barmak and Minian [**7**] went further and proved that two finite spaces X and Y are homotopy equivalent if and only if $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ have the same simple homotopy type.

reference to Barmak's book already here?

Finite spaces can be weak homotopy equivalent but not homotopy equivalent, as we have seen in Examples 3.4.5 and 3.4.8. The following problems are far more difficult than their analogues for homotopy equivalence, which we have treated in §**??**, following the REU paper of Fix and Patrias. Note that the work of Fix and Patrias implicitly addresses the problem of finding a computationally effective algorithm for enumerating the homotopy types of finite spaces.

Not written yet

PROBLEM 5.8.5. *Are there computationally effective algorithms for enumerating the weak homotopy types of finite spaces for small n? What is the asymptotic behavior of the number of weak homotopy types of spaces with at most n elements?*

ADDENDUM 5.8.3. Osaki [**34**] has given two theorems that describe when one can shrink an F-space, possibly minimal, to a smaller weakly homotopy equivalent F-space. He asks whether all weak homotopy equivalences are generated by the simple kinds that he describes. The question has since been answered in the negative, by Barmak and Minian [**6**]. Barmak's thesis, which was inspired by my 2003 REU notes and has now become the book [**5**], goes a good deal further. There is much more to be done on this problem, which is still not well understood.

PROBLEM 5.8.6. *Is there a combinatorial way of determining when a weak homotopy equivalence of finite spaces is a homotopy equivalence?*

PROBLEM 5.8.7. *Rather than restricting to finite simplicial complexes, can we model the world of finite CW complexes, or at least the world of finite regular CW complexes, in the world of finite spaces. The discussion of spheres and cones in §3.4 gives a possible starting point. This is related to the combinatorially interesting question of relating finite topological spaces to discrete Morse theory.*

CHAPTER 6

Really finite H-spaces

Later? Details! Get rid of types below? Focus on Type I, which is the standard defn of an H-space.

Expository REU paper? Research: Alexandroff H-spaces?

The circle is a *topological group*. If we regard it as the subspace of the complex plane consisting of points of norm one, then complex multiplication gives the product $S^1 \times S^1 \longrightarrow S^1$. How can we model such a basic structure in terms of a map of finite spaces?

Stong proved a rather amazing *negative* result about this problem. We will not go into the combinatorial details of his proof, contenting ourselves with the statement.

Definition 6.0.1. Let (X, e) be a finite space with a basepoint e and let $\phi\colon X \times X \longrightarrow X$ be a map We say that X is an H-space of type I if multiplication by e on either the right or the left is homotopic to the identity. That is, the maps $x \to \phi(e, x)$ and $x \to \phi(x, e)$ are each homotopic to the identity. Say that X is an H-space of type II if the *shearing maps* $X \times X \longrightarrow X \times X$ defined by sending (x, y) to either $(x, \phi(x, y))$ or $(y, \phi(x, y))$ are homotopy equivalences.

A topological group is an H-space of both types, but it is much less restrictive for a space to be an H-space than for a space to be a topological group. By definition, the notion of H-space is homotopy invariant in the sense that if one defines an H-space structure on (X, e) to be a homotopy class of products ϕ, then one has the following result.

Proposition 6.0.2. *If (X, e) and (Y, f) are homotopy equivalent, then H-space structures on (X, e) correspond bijectively to H-space structures on (Y, f).*

This motivated Stong to study H-space structures on minimal finite spaces. Here the following result is immediate from Theorem 2.4.5.

Proposition 6.0.3. *Let (X, e) be a minimal finite H-space of type I. Then the maps $X \longrightarrow X$ that send x to either $\phi(x, e)$ or $\phi(x, e)$ are homeomorphisms.*

Examining the combinatorial relationship of general points of X to the point e, Stong then arrives at the following striking conclusion.

Proposition 6.0.4. *If (X, e) is an H-space of either type, then the point e is both maximal and minimal under $\leq$.*

This means that e is a component of X. Stong shows that this implies the following conclusions for general finite H-spaces.

Theorem 6.0.5. *Let X be a finite space and let $e \in X$. Then there is a product ϕ making (X, e) an H-space of type I if and only if e is a deformation retract of its component in X. Therefore X is an H-space for some basepoint e if and only if some component of X is contractible.*

THEOREM 6.0.6. *Let X be a finite space. Then there is a product ϕ making X an H-space of type II if and only if every component of X is contractible.*

COROLLARY 6.0.7. *A connected finite space X is an H-space of either type if and only if X is contractible.*

So there is no way that we can model the product on S^1 by means of an H-space structure on some finite space X. Our standard model $\mathbb{T} = \mathbb{S}S^0$ of S^1 can be embedded in $\mathbb{C}$ as the four point subgroup $\{\pm 1, \pm i\}$, but then the complex multiplication is not continuous. However, the multiplication can be realized as a map $(\mathbb{T}\times\mathbb{T})^{(n)} \longrightarrow \mathbb{T}$ for some finite n, by the simplicial approximation theorem for finite spaces. It is natural to expect that some small n works here. The following result is proven in [**17**].

THEOREM 6.0.8. *Choosing minimal points e in $\mathbb{T}$ and $f \in \mathbb{T}'$ as basepoints, there is a map*

$$\phi \colon \mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$$

such that $\phi(f,f) = e$ and the maps $x \longrightarrow \phi(x,f)$ and $x \longrightarrow \phi(f,x)$ from $\mathbb{T}'$ to $\mathbb{T}$ are weak homotopy equivalences.

That is, we can realize a kind of H-space structure after barycentric subdivision. The proof is horribly unilluminating. The space $\mathbb{T}'$ has eight elements, the space $\mathbb{T}$ has four elements. One writes down an 8×8 matrix with values in $\mathbb{T}$, choosing it most carefully so that when the 8 point and 4 point spaces are given the appropriate partial order, and the 64 point product space the product order, the function represented by the matrix is order preserving. Then one checks the row and column corresponding to multiplication by the basepoint.

Several other interesting spaces and maps are modelled similarly in the cited paper, for example $\mathbb{R}P^2$ and $\mathbb{C}P^2$.

CHAPTER 7

Group actions and finite groups

We shall explain some of the results and questions in a beautiful 1978 paper [35] by Daniel Quillen. He relates properties of groups to homotopy properties of the simplicial complexes of certain posets constructed from the group. He does not explicitly think of these posets as finite topological spaces. He seems to have been unaware of the earlier papers of McCord [32] and Stong [40] that we have studied, and it is interesting to look at his work from their perspective. Stong himself first looked at Quillen's work this way [41], and we will include his results on the topic. We usually work with a finite group G, but the basic definitions apply more generally.

7.1. Equivariance and finite spaces

We begin with some general observations about equivariance and F-spaces, largely following Stong [41].

A topological group G is a group and a space whose product $G \times G \longrightarrow G$ and inverse map $G \longrightarrow G$ are continuous. An action of G on a topological space X is a continuous map $G \times X \longrightarrow X$, written $(g,x) \mapsto gx$, such that $g(hx) = (gh)x$ and $ex = x$, where e is the identity element of G. A map $f\colon X \longrightarrow Y$ of G-spaces is a continuous map f such that $f(gx) = gf(x)$ for $g \in G$ and $x \in X$.

For a space X, the automorphism group $\operatorname{Aut} X$ is the topological group of homeomorphisms $X \longrightarrow X$. The group operation is composition, and $\operatorname{Aut} X$ is topologized as a subspace of the space of maps $X \longrightarrow X$ with the compact open topology. Suppose a topological group G acts on X. Then the action of g on X gives a homeomorphism $g\colon X \longrightarrow X$. This gives a group homomorphism $G \longrightarrow \operatorname{Aut} X$. At least if X is first countable, this map is also continuous. That is, it is a map of topological groups.

We say that G acts trivially on X if $gx = x$ for all g and x. We let G act diagonally on products $X \times Y$, $g(x,y) = (gx, gy)$. In particular, with G acting trivially on I, we have the notion of a G-homotopy, namely a G-map $h\colon X \times I \longrightarrow Y$. There is a large subject of equivariant algebraic topology, in which one studies the algebraic invariants of G-spaces.

We begin with some basic ideas of equivalence in this context. We say that a G-map $f\colon X \longrightarrow Y$ is a G-homotopy equivalence if there is a G-map $f'\colon Y \longrightarrow X$ and there are G-homotopies $f \circ f' \simeq \mathrm{id}$ and $f' \circ f \simeq \mathrm{id}$. For a subgroup H of G, define the H-fixed point space X^H of X to be $\{x | hx = x \text{ for } h \in H\}$. Say that a G-map f is an H-equivalence if $f^H\colon X^H \longrightarrow Y^H$ is a nonequivariant homotopy equivalence. For nice G-spaces, the sort one usually encounters in classical algebraic topology, which are called G-CW complexes, a map f is a G-homotopy equivalence if and only if it is an H-equivalence for all subgroups H. Note that we have the much weaker notion of an e-equivalence, namely a G-map which is a homotopy equivalence of underlying spaces, forgetting the action of G.

We also have weak notions. A G-map f is a weak G-homotopy equivalence if each $f^H\colon X^H \longrightarrow Y^H$ is a weak homotopy equivalence in the nonequivariant sense. We also have the notion of a weak e-equivalence, meaning a G-map that is a weak homotopy equivalence of underlying spaces, forgetting the action of G.

In general, the notions of G-equivalence are very much stronger than the notions of e-equivalence. There are lots of G maps that are e-equivalences but are not G-equivalences. We show that cannot happen when G acts on a finite space. We start with some general observations.

Maybe better at start of H-space section? True with same proof for A-spaces

Lemma 7.1.1. *If an F-space G is a topological group, then it is discrete.*

Proof. If $h \le g$, then, by the continuity of the inverse map, $h^{-1} \le g^{-1}$. By the continuity of left multiplication by h, $e \le hg^{-1}$, and then, by the continuity of right multiplication by g, $g \le h$. Since G is T_0, $g = h$. Thus $U_g = g$ is open for all g and therefore every subset is open. □

We have observed that if a topological group G acts on a space X, then we can view the action as given by a map of topological groups $G \longrightarrow \mathrm{Aut}\, X$. This homomorphism has a kernel K, and the action factors through the quotient group G/K, which is a topological group with the quotient topology. When X is an F-space, $\mathrm{Aut}\, X$ is finite since there are only finitely many functions $X \longrightarrow X$. But then G/K is finite and therefore discrete. Thus we lose no generality if we restrict our attention to finite discrete groups G acting on F-spaces. Therefore G will be finite from now on.

Recall the notion of upbeat and downbeat points in an F-space X. Note that if x is upbeat, so that there is a $y > x$ such that $z > x$ implies $z \ge y$, then y is uniquely determined by x.

Theorem 7.1.2. *Let X be an F-space with an action by a group G. Then there is a core $C \subset X$ such that C is a sub G-space and equivariant deformation retract of X. We call C an equivariant core of X.*

Proof. The orbit Gx of an element x is $\{gx|g \in G\}$. If x is upbeat, then gx is also upbeat, with gy playing the role of y. The inclusion $X - Gx \subset X$ is the inclusion of a sub G-space. Define $f\colon X \longrightarrow X - Gx \subset X$ by $f(z) = z$ if $z \notin Gx$ and $f(gx) = gy$, where $y > x$ is such that $z > x$ implies $z \ge y$. Clearly $f \ge \mathrm{id}$ and thus $f \simeq \mathrm{id}$. An explicit homotopy used to show this is given by $h(z,t) = z$ if $t < 1$ and $h(z,1) = f(z)$, and this homotopy is a G-map. Removing upbeat and downbeat orbits successively until none are left, we reach an equivariant core. □

Corollary 7.1.3. *If X is a contractible F-space with an action by a group G, then X is equivariantly contractible to a G-fixed point.*

Proof. A core of X is a point, so an equivariant core must be a point with the trivial action by G. □

Corollary 7.1.4. *If X is a contractible F-space, then X has a point that is fixed by every homeomorphism of X.*

Proof. The finite group G of homeomorphisms of X acts on X, and an equivariant core is a fixed point. □

THEOREM 7.1.5. *Let X and Y be F-spaces with actions by G and $f\colon X \longrightarrow Y$ be a G-map. If f is an e-homotopy equivalence, then f is a G-homotopy equivalence.*

PROOF. Let C and D be equivariant cores of X and Y. Let $i_X\colon C \longrightarrow X$ and $r_X\colon X \longrightarrow C$ be the inclusion and retraction, and similarly for Y. Let be the composite

$$C \xrightarrow{i_X} X \xrightarrow{f} Y \xrightarrow{r_Y} D, \quad p = r_Y \circ f \circ i_X.$$

Then p is a G-map and a homotopy equivalence between minimal finite spaces. The latter property implies that p is a homeomorphism, and p^{-1} is necessarily also a G-map. Define $g\colon Y \longrightarrow X$ to be the composite

$$Y \xrightarrow{r_Y} D \xrightarrow{p^{-1}} C \xrightarrow{i_X} X, \quad g = i_X \circ p^{-1} \circ r_Y.$$

Then $g \circ f$ and $f \circ g$ are equivariantly homotopic to the respective identity maps. Indeed, we have the homotopies

$$gf = gf\, id_X \simeq gf\, i_X r_X = i_X p^{-1} r_Y f i_X r_X = i_X p^{-1} p r_X = i_X r_X \simeq id_X$$

and

$$fg = id_Y fg \simeq i_Y r_Y fg = i_Y r_Y f i_X p^{-1} r_Y = i_Y p p^{-1} r_Y = i_Y r_Y \simeq id_Y. \qquad \square$$

7.2. The basic posets and Quillen's conjecture

Fix a finite group G and a prime p. We define two posets.

Compare with Barmak's book. Anything interesting further in there?

DEFINITION 7.2.1. Let $\mathscr{S}_p(G)$ be the poset of non-trivial p-subgroups of G, ordered by inclusion. An abelian p-group is *elementary abelian* if every element has order 1 or p. This means that it is a vector space over the field of p elements. Define $\mathscr{A}_p(G)$ to be the poset of non-trivial elementary abelian p-subgroups of G, ordered by inclusion and let $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ be the inclusion.

REMARK 7.2.2. Quillen calls a non-trivial elementary abelian p-group a *p-torus*, and he defines its rank to be its dimension as a vector space.

The reason these posets are interesting is that G acts on them in such a way that their topological properties relate nicely to algebraic properties of G. The action of G is by conjugation. If H is a subgroup of G and $g \in G$, write $H^g = gHg^{-1}$. The function f_g that sends P to P^g gives an automorphism of the posets $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$. Clearly $f_e = \mathrm{id}$, where e is the identity element of G, and $f_{g'g} = f_{g'} \circ f_g$. These automorphisms are what give these posets their interest: the poset together with its group action describe how the different p-subgroups are related under subconjugation in G.

In particular, a point P in $\mathscr{A}_p(G)$ is fixed under the action of G if and only if $P^g = P$ for all $g \in G$, and this means that P is a normal subgroup of G. Thus the poset $(\mathscr{A}_p(G))^G$ of fixed points is the poset of normal p-tori of G. We can therefore relate algebraic questions about the presence of normal subgroups to topological questions about the existence of fixed points. Of course, we may regard these posets as F-spaces with G actions, and the theory of the previous section applies.

REMARK 7.2.3. Some of Quillen's language for studying these posets is similar to the language we have been using, but it can be quite confusing. For example, he says that a subset S of a poset X is *closed* if $x \in S$ and $y \le x$ implies $y \in S$. In our language, this means that $x \in S$ implies $U_x \subset S$, which says that S is *open*.

The posets $\mathscr{S}_p(G)$ and $\mathscr{A}_p(G)$ are both empty if p does not divide the order of G. At first sight, it might seem that $\mathscr{S}_p(G)$ is a lot more interesting and complicated than $\mathscr{A}_p(G)$, but that is not the case. To understand the discussion to follow, it is helpful to keep the following commutative diagram of spaces in mind, remembering that its vertical arrows are weak homotopy equivalences.

$$\begin{tikzcd}
|\mathscr{K}\mathscr{A}_p(G)| \arrow[r, "|\mathscr{K}(i)|"] \arrow[d, "\psi"'] & |\mathscr{K}\mathscr{S}_p(G)| \arrow[d, "\psi"] \\
\mathscr{A}_p(G) \arrow[r, "i"'] & \mathscr{S}_p(G)
\end{tikzcd}$$

We first consider p-groups.

Proposition 7.2.4. *If P is a non-trivial p-group, then $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are both contractible.*

Proof. There is a central subgroup B of P of order p. We will be accepting as known some basic facts in the theory of finite groups, such as this one. But the proof is just an easy counting argument. We think of P as a P-set, with P acting on itself by conjugation. As is true for any finite P-set P is isomorphic to a disjoint union of orbits, each isomorphic to some orbit P/Q. Unless the orbit consists of a single point, its number of elements is divisible by p, and the total number of elements is the order of P. Since the identity element is an orbit with a single point, there must be at least $p-1$ other orbits with a single point, and such a point is a non-identity element in the center of P.

For any subgroup A of P, we have $A \subset AB \supset B$. If A is a p-torus, then so is AB since B is central. Define three maps $\mathscr{A}_p(P) \longrightarrow \mathscr{A}_p(P)$: the identity map id, the map f that sends A to AB, and the constant map c_B that sends A to B. These are all continuous, and our inclusions say that $\mathrm{id} \leq f \geq c_B$. This implies that $\mathrm{id} \simeq f \simeq c_B$. Since the identity is homotopic to the constant map, $\mathscr{A}_p(G)$ is contractible. The proof for $\mathscr{S}_p(G)$ is the same. $\square$

Quillen calls a poset X *conically contractible* if there is an $x_0 \in X$ and a map of posets $f\colon X \longrightarrow X$ such that $x \leq f(x) \geq x_0$ for all x. He was thinking in terms of associated simplicial complexes, but we are thinking in terms of F-spaces. The previous proof says that the F-spaces $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are conically contractible. It is to be emphasized that conically contractible finite spaces are genuinely and not just weakly contractible. As we shall see, the difference is profound in the case at hand. In contrast with the previous result, we emphasize the word "weak" in the following result.

Theorem 7.2.5. *The inclusion $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is a weak homotopy equivalence. Therefore the induced map $|\mathscr{K}i|\colon |\mathscr{K}\mathscr{A}_p(G)| \longrightarrow |\mathscr{K}\mathscr{S}_p(G)|$ is a weak homotopy equivalence and hence an actual homotopy equivalence.*

Proof. We have the open cover of $\mathscr{S}_p(G)$ given by the U_P, where P is a non-trivial finite p-group. Clearly $i^{-1}U_P$ is the poset of p-tori of G that are contained in P, and this is the contractible space $\mathscr{A}_p(P)$. Our general theorem that weak homotopy equivalence is a local notion applies. $\square$

Definition 7.2.6. Define the p-rank of G, denoted $r_p(G)$, to be the maximal rank of a p-torus in G. Observe that this is one greater than the dimension of the simplicial complex $\mathscr{K}\mathscr{A}_p(G)$. (We interpret the dimension of the empty complex to be -1).

EXAMPLE 7.2.7. If the p-Sylow subgroups of G are cyclic of order p and there are q of them, then $\mathscr{A}_p(G)$ is a discrete space with q points. For example, this holds for some q if G is the symmetric group on n letters, where p is a prime and $p \le n < 2p$.

REMARK 7.2.8. Sylow's third theorem is relevant. The number of Sylow p-subgroups of G is congruent to 1 mod p and divides the order of G.

THEOREM 7.2.9. *The following statements are equivalent.*

(i) G has a non-trivial normal p-subgroup.
(ii) G has a non-trivial normal elementary abelian subgroup.
(iii) $\mathscr{S}_p(G)$ is contractible.

Moreover, they are implied by the statement

(iv) $\mathscr{A}_p(G)$ is contractible.

PROOF. Obviously (ii) implies (i). Conversely, as a matter of algebra, (i) implies (ii). To see that, let P be a non-trivial normal p-subgroup of G and let C be its center. For $g \in G$, $c \in C$, and $p \in P$,

$$gcg^{-1}p = gcg^{-1}pgg^{-1} = gg^{-1}pgcg^{-1} = pgcg^{-1}$$

since $g^{-1}pg$ is in P and therefore commutes with c. This shows that any conjugate of an element of C commutes with any element of P and is therefore in C, showing that C is normal in G. Now let B be the set of elements $b \in C$ such that $b^p = e$. Any conjugate of an element of B is in C and has pth power e, hence is in B. Therefore B is a non-trivial normal elementary abelian subgroup of G.

To see that (i) implies (iii), let P be a non-trivial normal p-subgroup of G. For any nontrivial p-subgroup Q of G, $Q \subset QP \supset P$, where QP denotes the subgroup generated by P and Q. Since P is normal in G, $QP = \{qp | q \in Q \text{ and } p \in P\}$. This implies that $\mathrm{id} \le f \ge c_P$, where $f(Q) = QP$ and $c_P(Q) = P$, hence $\mathscr{S}_p(G)$ is conically contractible, hence contractible. The same argument does not apply to show that (ii) implies (iv) since QP need not be abelian when Q and P are abelian.

Conversely, to see that (iii) implies (i) and (iv) implies (ii), we use Corollary 7.1.3, which states that contractibility implies G-contractibility to a fixed point. A fixed point of $\mathscr{S}_p(G)$ is a normal p-subgroup and a fixed point of $\mathscr{A}_p(G)$ is a normal elementary abelian p-subgroup. □

The inclusion $i: \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is not generally a homotopy equivalence. To see this, we use the following observation.

LEMMA 7.2.10. *Let $\mathscr{Q}_p(G) \subset \mathscr{S}_p(G)$ be the subposet of nontrivial intersections of Sylow p-subgroups. Then $\mathscr{Q}_p(G)$ is a G-equivariant deformation retract of $\mathscr{S}_p(G)$.*

PROOF. For $P \in \mathscr{S}_p(G)$, let $f(P)$ be the intersection of the Sylow p-subgroups that contain P. Then $f: \mathscr{S}_p(G) \longrightarrow \mathscr{Q}_p(G)$ is continuous and G-equivariant. Moreover, $f(P) = P$ if P is itself a p-Sylow subgroup. Let $j: \mathscr{Q}_p(G) \longrightarrow \mathscr{S}_p(G)$ be the inclusion. Then $fj = \mathrm{id}$. Since $P \le f(P)$, $\mathrm{id} \simeq jf$ via an equivariant homotopy. □

EXAMPLE 7.2.11. Let $G = \Sigma_5$ be the symmetric group on five letters. Then $\mathscr{A}_2(G)$ and $\mathscr{S}_2(G)$ are not homotopy equivalent. There are 6 conjugacy classes of 2-subgroups of G, as follows.

(i) Dihedral groups D_4 of order 8, the Sylow 2-subgroups.
(ii) Cyclic groups C_4 of order 4.
(iii) Elementary 2-groups $C_2 \times C_2$ generated by transpositions (ab) and (cd).
(iv) Elementary 2-groups $C_2 \times C_2$ generated by products of disjoint transpositions $(ab)(cd)$, $(ac)(bd)$, whose product in either order is $(ad)(bc)$.
(v) Cyclic groups C_2 generated by a transposition.
(vi) Cyclic groups C_2 generated by a product of two disjoint transpositions.

Of course, each $C_2 \times C_2$ contains three C_2's. Each C_2 of type (v) is contained in three $C_2 \times C_2$'s of type (iii) and each C_2 of type (vi) is contained in one $C_2 \times C_2$ of type (iii) and one $C_2 \times C_2$ of type (iv). This information shows that $\mathscr{A}_2(G)$ is minimal, hence not homotopy equivalent to any space with fewer points. The intersections of Sylow 2-subgroups of G are the dihedral groups in (i), the groups $C_2 \times C_2$ of type (iv) and the subgroups C_2 of type (v). In fact, $\mathscr{Q}_2(G)$ is a core of $\mathscr{S}_2(G)$). Counting, one sees that there are fewer points in $\mathscr{Q}_2(G)$ than there are in the minimal F-space $\mathscr{A}_2(G)$, so these two F-spaces cannot be homotopy equivalent.

Quillen conjectured the following stronger version of the implication (iii) implies (i) of Theorem 7.2.9, and he proved the conjecture for solvable groups.

CONJECTURE 7.2.12 (Quillen). *If $\mathscr{A}_p(G)$ or equivalently $\mathscr{S}_p(G)$ is weakly contractible, then G contains a non-trivial normal p-subgroup.*

The hypothesis holds if and only if $|\mathscr{K}\mathscr{A}_p(G)|$ or equivalently $|\mathscr{K}\mathscr{S}_p(G)|$ is weakly contractible and therefore contractible. We have seen that if G has a non-trivial normal p-subgroup, then $\mathscr{A}_p(G)$ is contractible and therefore weakly contractible. Quillen's conjecture is that, conversely, if $\mathscr{A}_p(G)$ is weakly contractible, then it is contractible and thus G has a non-trivial normal p-subgroup. In this form, we see that the conjecture can be thought of as a problem in the equivariant homotopy theory of F-spaces.

In particular, if G is simple and not isomorphic to C_p, then it has no non-trivial normal subgroups and the conjecture implies that $\mathscr{A}_p(G)$ cannot be weakly contractible. This consequence of the conjecture has been verified for many but not all finite simple groups, using the classification theorem and proving that the space $\mathscr{A}_p(G)$ has non-trivial homology. A conceptual proof would be a wonderful achievement!

7.3. Some exploration of the posets $\mathscr{A}_p(G)$

As an illustration of the translation of algebra to topology, we show how to compute $\mathscr{A}_p(G \times H)$ in terms of joins for finite groups G and H. We then see how the computation appears in Quillen's analysis of the poset $\mathscr{A}_p(\Sigma_{2p})$.

PROPOSITION 7.3.1. *The poset $\mathscr{A}_p(G \times H)$ is homotopy equivalent to the poset $C^-\mathscr{A}_p(G) \times C^-\mathscr{A}_p(H) - \{(c_G, c_H)\}$.*

PROOF. Let T be the subposet of $\mathscr{A}_p(G \times H)$ whose points are the p-tori in $G = G \times e$, the p-tori in $H = e \times H$, and the products $A \times B$ of p-tori A in G and B in H. (Remember that p-tori are non-trivial elementary abelian p-groups). Visibly, thinking of trivial groups as conepoints and therefore $<$ non-trivial subgroups, T

is isomorphic to $C^-\mathscr{A}_p(G) \times C^-\mathscr{A}_p(H) - \{(c_G, c_H)\}$. Let $i\colon T \longrightarrow \mathscr{A}_p(G \times H)$ be the inclusion. The projections $\pi_1\colon G \times H \longrightarrow G$ and $\pi_2\colon G \times H \longrightarrow H$ induce a map $r\colon \mathscr{A}_p(G \times H) \longrightarrow T$ such that $r \circ i = \mathrm{id}$. Explicitly, for $C \in \mathscr{A}_p(G \times H)$, $r(C) = \pi_1(C) \times \pi_2(C)$. Then $i(r(C)) \supset C$, which means that $i \circ r \geq \mathrm{id}$ and thus $i \circ r \simeq \mathrm{id}$. □

In view of Proposition 5.7.9, this has the following immediate consequence.

Corollary 7.3.2. *The space $|\mathscr{K}(\mathscr{A}_p(G \times H))|$ is homotopy equivalent to the space $|\mathscr{K}(\mathscr{A}_p(G))| * |\mathscr{K}(\mathscr{A}_p(H))|$.*

Proposition 7.3.3. *Quillen's conjecture holds if $r_p(G) \leq 2$.*

Proof. The hypothesis cannot hold if $r_p(G) = 0$, since $\mathscr{A}_p(G)$ is then empty and hence not contractible. If $r_p(G) = 1$, then the space $\mathscr{A}_p(G)$ is discrete since there are no proper inclusions. It is weakly contractible if and only if it consists of a single point, and then its single point must be fixed by the action of G. This means that there is a unique p-torus in G, and it is a normal subgroup of order p. If $r_p(G) = 2$, then $|\mathscr{K}(\mathscr{A}_p(G))|$ is one dimensional and contractible, which means that it is a tree. According to Quillen, "one knows (Serre) that a finite group acting on a tree always has a fixed point". This means that G has a normal p-torus. The trees here are of a particularly elementary sort, but the conclusion is still not altogether obvious. The following problem gives a way of thinking about it. □

Problem 7.3.4. *Consider an F-space X such that $|\mathscr{K}(X)|$ is a tree (a contractible graph). Clearly X is weakly contractible. Prove that X is contractible. (Search for upbeat or downbeat points). It follows that if a finite group G acts on X, then X is G-contractible and therefore has a G-fixed point.*

Much of Quillen's paper is devoted to proving that the conjecture holds for *solvable* groups G. This means that there is a decreasing chain of subgroups of G, each normal in the next, such that the subquotients are cyclic of prime order. We shall not repeat the proof.

However, following Quillen, we shall work out the structure of $\mathscr{A}_p(G)$ when $G = \Sigma_{2p}$ is the symmetric group on $2p$ letters for an odd prime p. This is a first interesting case since $\mathscr{A}_p(\Sigma_n)$ is empty if $n < p$ and is a discrete space with one element for each cyclic subgroup of order p if $p \leq n < 2p$. (In fact, there are $n!/(n-p)!p(p-1)$ such subgroups.) The analysis shows just how non-trivial the posets $\mathscr{A}_p(G)$ are.

Let $g \in G = \Sigma_{2p}$ have order p. The group $\langle g \rangle$ it generates has order p, and its action on the set $S = \{1, \cdots, 2p\}$ partitions S into two disjoint subsets, one given by the orbit generated by an element s such that $gs \neq s$ and the other given by its complement, on which $\langle g \rangle$ acts either freely or trivially. If $A \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is a maximal elementary abelian p-subgroup of G with generators g and g', then since g and g' commute we can see that they give the same partition of S, so that each such A gives a unique partition of the set S into two A-invariant subsets, each with p elements. The set of such partitions of S into two subsets with p elements gives a corresponding decomposition of $\mathscr{A}_p(G)$ into disjoint subposets, each consisting of those A which partition S in the prescribed way.

Under the action of G, these partitions are permuted transitively, meaning that, given two partitions, there is an element of G that permutes one into the other. Consider for definiteness the partition into the first p and last p elements of S. Let

H be the subgroup of those elements of G that fix this partition. The corresponding subposet of $\mathscr{A}_p(G)$ is $\mathscr{A}_p(H)$. Here H is the *wreath product* $\Sigma_2 \int \Sigma_p$, which is the semi-direct product of Σ_2 with $\Sigma_p \times \Sigma_p$ determined by the permutation action of Σ_2 on $\Sigma_p \times \Sigma_p$.

Since p is odd, $\mathscr{A}_p(H) = \mathscr{A}_p(\Sigma_p \times \Sigma_p)$, which, after passage to realizations of simplicial complexes, is the join $\mathscr{A}_p(\Sigma_p) * \mathscr{A}_p(\Sigma_p)$. Since Σ_p has $(p-2)!$ Sylow subgroups, each of order p, $\mathscr{A}_p(\Sigma_p)$ is the disjoint union of $(p-2)!$ points. After counting the number of partitions and inspecting the join of our two discrete spaces $\mathscr{A}_p(\Sigma_p)$, Quillen informs us, and we can work out for ourselves, that $|\mathscr{A}_p(\Sigma_{2p})|$ is a disconnected graph with $(2p)!/2(p!)^2$ components, each of which is homotopy equivalent to a one-point union of $((p-2)!-1)^2$ circles. For example, for $p = 5$, there are 25 circles. The same analysis applies to the alternating groups A_n for $n \leq 2p$ since $\mathscr{A}_p(A_n) = \mathscr{A}_p(\Sigma_n)$. Of course, these $\mathscr{A}_p(G)$ are not weakly contractible.

7.4. The components of $\mathscr{S}_p(G)$

Let p be a prime which divides the order of G. We describe the set of components $\pi_0(\mathscr{S}_p(G))$, which of course is the same as $\pi_0(\mathscr{A}_p(G))$. Recall that two elements of a poset are in the same component if they can be connected by a chain of elements, each either $\leq$ or $\geq$ the next. In the poset $\pi_0(\mathscr{S}_p(G))$, each element is a p-group and is contained in a Sylow subgroup. Therefore there is at least one Sylow subgroup in each component. Since any one Sylow subgroup P generates all the others by conjugation by elements of G, G acts transitively on $\pi_0(\mathscr{S}_p(G))$, in the sense that there is a single orbit. If $N = N_P$ denotes the subgroup of G that fixes the component $[P]$ of P, then G/N is isomorphic to the G-set $\pi_0(\mathscr{S}_p(G))$ via $gN \mapsto [P^g]$. We want to determine the subgroup N. Let $\mathrm{Syl}_p(G)$ denote the set of p-Sylow subgroups of G and let $N_G H$ denote the normalizer in G of a subgroup H. Recall that $H^g = gHg^{-1}$.

Proposition 7.4.1. *The following conditions on a subgroup M of G are equivalent.*

(i) For some $P \in Syl_p(G)$, $M \supset N_P$.
(ii) For some $P \in Syl_p(G)$, $M \supset N_G H$ for all $H \in \mathscr{S}_p(P)$.
(iii) For some $P \in Syl_p(G)$, $M \supset N_G P$ and $K \subset M$ whenever K is a p-subgroup of G that intersects M non-trivially.
(iv) p divides the order of M and $M \cap M^g$ is of order prime to p for all $g \notin M$.

Moreover, $\mathscr{S}_p(G)$ is connected if and only if there is no proper subgroup M which satisfies these equivalent conditions.

Proof. The last statement holds since G is connected if and only if $G = N_P$ for all $P \in \mathrm{Syl}_p(G)$, in which case no proper subgroup can satisfy the stated conditions.
(i) $\Longrightarrow$ (ii): If $g \in N_G H$ with $H \subset P$, then $H^g = H$ is contained in both P and P^g, so that $[P] = [P^g] = g[P]$. This means that $g \in N_P \subset M$.
(ii) $\Longrightarrow$ (iii): Obviously $M \supset N_G P$. Since P is a p-Sylow subgroup of G, it is also a p-Sylow subgroup of M. Thus if H is a non-trivial p-subgroup of M, then H is conjugate in M to a subgroup, H^m say, of P. Since $M \supset N_G(H^m)$ and $(N_G H)^m = N_G(H^m)$, $M \supset N_G H$. Let K be a p-subgroup of G such that $K \cap M$ is non-trivial. We have

$$K \cap M \subset N_K(K \cap M) = K \cap N_G(K \cap M) \subset K \cap M.$$

Since K is a p-group, the first inclusion is proper if $K \cap M$ is a proper subgroup of K. Since this is a contradiction, we must have $K \cap M = K$ and $K \subset M$.
(iii) $\Longrightarrow$ (iv): Since $M \supset P$, p divides the order of M. Assume that p divides the order of $M \cap M^g$ for some $g \in G$. Then there is a non-trivial p-subgroup $H \subset M \cap M^g$. Let $H \subset Q$ for $Q \in \mathrm{Syl}_p(G)$. Since $Q \cap M$ is non-trivial, we have $Q \subset M$. Since $H^{g^{-1}} \subset Q^{g^{-1}}$ and $H^{g^{-1}} \subset M$, we also have $Q^{g^{-1}} \subset M$. Since P, Q, and $Q^{g^{-1}}$ are p-Sylow subgroups of M, they are conjugate in M, say $Q^m = P$ and $Q^{g^{-1}} = P^n$ for $m, n \in M$. Then a quick check shows that $mgn \in N_G P \subset M$ and therefore $g \in M$, proving (iv).
(iv) $\Longrightarrow$ (i): Writing G as the disjoint union of double cosets MgM, one calculates that the index of M in G is the sum over double coset representatives g of the indices of $M \cap M^g$ in M. Since p divides the order of M and does not divide the order of $M \cap M^g$ if $g \notin M$, these indices are divisible by p except for the double coset represented by e. Thus the index of M in G is congruent to 1 mod p, hence M must contain some p-sylow subgroup P. Let $N = N_P$. For $n \in N$, P and P^n are in the same component. Considering p-Sylow subgroups containing groups in a chain connecting them, we see that there is a sequence of p-Sylow subgroups $P = P_0, P_1, \ldots, P_q = P^n$ such that $P_i \cap P_{i+1} \neq \{e\}$. There are elements g_i such that $P_{i-1}^{g_i} = P_i$, and we can choose g_q so that $g_q \cdots g_1 = n$. We have $P \subset M$, and we assume inductively that $P_{i-1} \subset M$. Then $P_{i-1} \cap P_i \subset M \cap M^{g_i}$, so this intersection contains a p-group and, by (iv), $g_i \in M$. This implies that $P_i \subset M$ and, inductively, we conclude that $n \in M$, so that $N \subset M$. $\square$

Corollary 7.4.2. *N_P is generated by the groups $N_G H$ for $H \in \mathscr{S}_p(P)$.*

Proof. N_P contains all of these $N_G H$, so it contains the subgroup they generate, and it is the smallest such subgroup by the equivalence of (i) and (ii). $\square$

By the contrapositive, G is not connected if and only if there is a proper subgroup M of G that satisfies the equivalent properties of the proposition. For example, if $r_p(G) = 1$ and G has no non-trivial normal p-subgroup, then $\mathscr{A}_p(G)$ is discrete and not contractible, and is therefore not connected. Quillen gives a condition on G under which these are the only examples.

Proposition 7.4.3. *Let H ($= O_{p'}(G)$) be the largest normal subgroup of G of order prime to p and let K ($= O_{p',p}(G)$) be specified by requiring K/H to be the largest normal p-subgroup of the quotient group G/H. If K/H is non-trivial and $\mathscr{S}_p(G)$ is not connected, then $r_p(G) = 1$.*

Proof. If Q is a p-Sylow subgroup of K, then $K = QH$ since H is a p'-group and K/H is a p-group. This implies that H acts transitively on $\pi_0(\mathscr{S}_p(K))$ since it implies that any two p-Sylow subgroups are conjugate by the action of some $h \in H$. The intersection with K of a p-Sylow subgroup P of G is a p-Sylow subgroup of K. A p-subgroup of K is a p-subgroup of G, and the induced map $\pi_0(\mathscr{S}_p(K)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is surjective since $P \cap K \subset P$ implies that $[P]$ is the image of $[P \cap K]$. Therefore H also acts transitively on $\pi_0(\mathscr{S}_p(G))$. Let A be a maximal p-torus of G. The map $\pi_0(\mathscr{S}_p(AH)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is also surjective since H acts transitively on the target and the map is H-equivariant. Therefore $\mathscr{S}_p(AH)$ is not connected. The component $[A]$ is fixed by the centralizers $C_H(B)$ for all non-trivial subgroups B of A since $B^h = B \subset A$ for $h \in C_H(B)$. By [**15**, 6.2.4], if A is not cyclic (= rank one), then H is generated by these centralizers, which contradicts the fact that $\mathscr{S}_p(AH)$ is not connected. Therefore A is cyclic. $\square$

Part 2

A categorical interlude

CHAPTER 8

A concise introduction to categories

To be written

Small versus large dichotomy

Monoids, Groups, Groupoids, Posets, Simplicial complexes, Ordered simplicial complexes, spaces, finite spaces, F-spaces, Alexandroff spaces, A-spaces

Define functor

Define Natural transformation

Define product with $\mathscr{I}$ and homotopical version of natural transformation

Define isomorphism of categories, F-space and A-space examples

Define and characterize equivalence of categories

Poset and ordered simplicial complex examples

Full and faithful functors, full embedding, essential image

Yoneda lemma

Adjoint functors

Yoneda embedding: essential image is represented functors

8.1. The adjoint relationship between S and T

now out of order due to last year reorganization

It has long been known that we can use simplicial sets pretty much interchangeably with topological spaces when studying homotopy theory. We sketch how this is seen through the categorical eyes of an adjunction. For a simplicial set K, we have defined a space $|K| = TK$, called the geometric realization of K. We write $|k,u|$ for the image of (k,u) in TK, where $k \in K_n$ and $u \in \Delta[n]$. For a space X, we have defined a simplicial set SX, called the total singular complex of X, whose n-simplices are the continuous maps $f\colon \Delta[n]^t \longrightarrow X$. The homotopical behavior is studied through an adjunction: T and S are left and right adjoint functors in the sense that we have just defined. That is, there is a bijection, natural in both variables, between morphism sets

$$\mathscr{U}(TK,X) \cong s\mathscr{S}et(K,SX).$$

It is specified by letting $f\colon TK \longrightarrow X$ correspond to $g\colon K \longrightarrow SX$ if

$$f(|k,u|) = g(k)(u).$$

There is an equivalent way of saying this. Define $\gamma : TSX \longrightarrow X$ by

$$\gamma|f,u| = f(u) \quad \text{for} \quad f : \Delta_n \longrightarrow X \quad \text{and} \quad u \in \Delta_n.$$

It is a fact that γ is a weak homotopy equivalence for every space X, although we shall not prove that here. There is also a map $\iota\colon K \longrightarrow STK$ of simplicial sets specified by $\iota(k)(u) = |k,u|$ for $k \in K_n$ and $u \in \Delta_n$. Again, as we also shall not prove, $|\iota|\colon |K| \longrightarrow |STK|$ is a homotopy equivalence. These facts are proven, for example, in [27]. The natural composite

$$SX \xrightarrow{\iota S} STSX \xrightarrow{S\gamma} SX$$

is the identity map of SX. The natural composite

$$TK \xrightarrow{T\iota} TSTK \xrightarrow{\gamma T} TK$$

is the identity map of TK. Here ιS means first apply the functor S and then the natural map γ, and similarly for γT. The natural maps ι and γ are the unit and the counit of the adjunction. This means that, in the correspondence above, $f = \gamma \circ Tg$ and $g = Sf \circ \iota$.

8.2. The fundamental category functor Π

It is also known, although this is more recent, that we can use categories pretty much interchangeably with topological spaces when studying homotopy theory. We are going to say quite a lot about this later. This comparison again starts with an adjunction. We have constructed a simplicial set $N\mathscr{C}$ called the nerve of $\mathscr{C}$. We define $B\mathscr{C} = TN\mathscr{C}$. This is called the *classifying space* of the category $\mathscr{C}$. When G is a group regarded as a category with a single object, BG is called the classsifying space of the group G. The space BG is often written as $K(G,1)$. It is called an Eilenberg-Mac Lane space. It is characterized (up to homotopy type) as a connected space with $\pi_1(K(G,1)) = G$ and with all higher homotopy groups $\pi_q(K(G,1)) = 0$. A concise summary of how that works is in [**30**, §16.5]. More generally, a detailed study of the classifying spaces of topological groups and what they classify is in [**28**]. These are fundamentally important constructions in topology and its applications.

The nerve functor N is accompanied by a functor $\Pi\colon s\mathscr{S}et \longrightarrow \mathscr{C}at$, called the "fundamental category" functor.[1] It is left adjoint to N, meaning that

$$\mathscr{C}at(\Pi K, \mathscr{C}) \cong s\mathscr{S}et(K, N\mathscr{C}).$$

This means that it is conceptually sensible, but, in contrast to such functors as S and T, it does not have good homotopical properties, as we shall see.

For a simplicial set K, the objects of the category ΠK are the vertices (that is, the 0-simplices) of K. To construct the morphisms, one starts by thinking of the 1-simplices y as maps $d_1 y \longrightarrow d_0 y$. One forms all words (formal composites) that make sense, that is, whose targets and sources match up. One then imposes the relations on morphisms determined by

$$s_0 x = \mathrm{id}_x \quad \text{for} \quad x \in K_0 \quad \text{and} \quad d_1 z = d_0 z \circ d_2 z \quad \text{for} \quad z \in K_2.$$

We use the relations $d_i d_j = d_{j-1} d_i$ for $i < j$ when (i,j) is $(0,1)$, $(1,2)$, and $(0,2)$ to see that sources and targets match up. This makes good sense since if $K = N\mathscr{C}$, then a 0-simplex is an object x of $\mathscr{C}$, a 1-simplex y is a map $d_1 y \longrightarrow d_0 y$, the 1-simplex $s_0 x$ is id_x, and a 2-simplex z is given by a pair of composable morphisms $d_2 z$ and $d_0 z$ together with their composite $d_1 z$.

Therefore there is a natural map $\varepsilon\colon \Pi N\mathscr{C} \longrightarrow \mathscr{C}$ that is the identity on objects (the zero simplices of $N\mathscr{C}$) and is induced by the identity map from the generating morphisms of $\Pi\mathscr{N}\mathscr{C}$ (the 1-simplices on $N\mathscr{C}$) to the morphisms of $\mathscr{C}$. In fact, ε is an isomorphism of categories: it is the identity on objects, and it presents the category in terms of generators given by the morphism sets modulo relations determined by the category axioms.

For the adjunction, a functor $F\colon \Pi K \longrightarrow \mathscr{C}$ is constructed from a map of simplicial sets $g\colon K \longrightarrow N\mathscr{C}$ by letting F be the unique functor that agrees with g

[1]There is no fully standard notation for this category. I've seen it denoted τ_1, π_1, π, and C.

on objects (= 0-simplices) and equivalence classes of morphisms (= 1-simplices). Applying the adjunction to the identity map of ΠK, we obtain a natural map $\eta\colon K \longrightarrow N\Pi K$, which is the unit of the adjunction, and the counit is the isomorphism ε.

8.3. The Yoneda lemma and the structure of simplicial sets

We give a construction that is a precise categorical analogue of the geometric realization of a simplicial set, and we use the Yoneda lemma to prove that it gives an amusing way of reconstructing K categorically. This kind of result is actually very useful in algebraic geometry, but we use it both to illustrate categorical ideas and to prepare for a later conceptual construction of the subdivision functor on simplicial sets.

Recall that we defined the standard simplicial n-simplex $\Delta[n]^s$ to be the simplicial set whose q-simplices are the monotonic functions $\sigma\colon [q] \longrightarrow [n]$; precomposition with monotonic functions $\xi\colon [p] \longrightarrow [q]$ gives the required contravariant functoriality on Δ. The nondegenerate q-simplices in $\Delta[n]^s$ are the monomorphisms (= strictly monotonic functions) $[q] \longrightarrow [n]$, and there is one for each subset of $[n]$ of cardinality $q+1$. We may identify the set of all non-degenerate simplices with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathscr{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathscr{K}([n])$. A monotonic function $\alpha\colon [m] \longrightarrow [n]$ gives a map $\alpha\colon \Delta[m]^s \longrightarrow \Delta[n]^s$ of simplicial sets that sends $\sigma\colon [q] \longrightarrow [m]$ to $\alpha\circ\sigma$. Thus $\Delta[-]^s$ is a covariant functor from Δ to simplicial sets. earlier

For a set C and a simplicial set L, one can form a new simplicial set $C\times L$ by letting $(C\times L)_q = C\times L_q$, and similarly letting the faces and degeneracies be induced by those of L. A simplicial set K can be reconstructed from the disjoint union over n of the simplicial sets $K_n \times \Delta[n]$ for $n\geq 0$ by taking equivalence classes under the equivalence relation generated by

$$(\alpha^*(k),\sigma) \simeq (k,\alpha_*(\sigma)) \tag{8.3.1}$$

for $k\in K_n$, $\sigma\in\Delta[m]^s_q$, and $\alpha\colon [m]\longrightarrow [n]$ in Δ. Here $\alpha^*(k)\in K_m$ is given by the fact that K is a contravariant functor from Δ to sets and $\alpha_*(\sigma)\in\Delta[n]_q$ is given by the fact that $\Delta[-]$ is a covariant functor from Δ to simplicial sets. The simplicial structure is induced from the simplicial structure on the $\Delta[n]$. The point is that an arbitrary pair (k,τ) in $K_n\times\Delta[n]_q$ is equivalent to the pair $(\tau(k),\iota_q)$ in $K_q\times\Delta[q]_q$, where $\iota_q\colon [q]\longrightarrow [q]$ is the identity map viewed as a canonical q-simplex in $\Delta[q]$, and $\tau\colon [q]\longrightarrow [n]$ is viewed as a morphism of Δ, so that $\tau = \tau_*(\iota_q)$. Identifying equivalence classes of q-simplices with elements of K_q in this faction, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p]\longrightarrow [q]$, $\xi\circ\iota_p = \iota_q\circ\xi$ and

$$(k,\xi^*(\iota_q)) = (k,\xi_*(\iota_p)) \simeq (\xi^*(k),\iota_p).$$

8.4. Tensor products of functors?

Give the idea, relate to geometric realization of simplicial spaces and $K\cong K\otimes_\Delta \Delta^s$. Motivate by coming analogy with subdivision.

MORE TO COME: $K\cong K\otimes_\Delta\Delta^s$

QUESTION: Does the canonical map $\Delta'\longrightarrow\Delta$ define a map of cosimplicial simplicial sets. For each n, it is a map of simplicial sets, and it is natural, so surely yes! See Definition 4.3.8, Proposition 4.3.10, Remark 4.3.11.

Add in: see §9 examples

Part 3

Topological spaces, Simplicial sets, and categories

CHAPTER 9

Simplicial sets

9.1. Motivation for the introduction of simplicial sets

Simplicial sets, and more generally simplicial objects in a given category, are central to modern mathematics. While I am not a mathematical historian, I thought I would describe in conceptual outline how naturally simplicial sets arise from the classical study of simplicial complexes. I suspect that something like this recapitulates the historical development.

We have described simplicial complexes in several different forms: abstract simplicial complexes, ordered simplicial complexes, geometric simplicial complexes, ordered geometric simplicial complexes and realizations of geometric simplicial complexes. It is possible to go directly from abstract simplicial complexes to realizations without passing through geometric simplicial complexes, but the construction is perhaps not as intuitive and will not be included.

An abstract simplicial complex is equivalent to a geometric simplicial complex, and neither of these notions involves anything about ordering the vertices. If one has a simplicial complex of either type, one can choose a partial ordering of the vertices that restricts to a linear ordering of the vertices of each simplex, and this gives the notion of an ordered simplicial complex. This can be done most simply, but not most generally, just by choosing a total ordering of the set of all vertices and restricting that ordering to simplices. However, there is no canonical choice.

We have seen in studying products of simplicial complexes that geometric realization behaves especially nicely only in the ordered setting. Both the category $\mathscr{SC}$ of simplicial complexes and the category $\mathscr{OSC}$ of ordered simplicial complexes have categorical products. Geometric realization preserves products when defined on $\mathscr{OSC}$, but it does not preserve products when defined on $\mathscr{SC}$. The functor $\mathscr{K}$ is best viewed as a functor from the category $\mathscr{P}$ of partially ordered sets to the category $\mathscr{OSC}$ rather than just to the category $\mathscr{SC}$. Observe that there are generally many different ordered simplicial complexes with the same poset of vertices. The functor $\mathscr{K}$ picks out the largest choice, the one in which every finite totally ordered subset of the set of vertices is a simplex.

The functor $\mathscr{X}$, on the other hand, starts in $\mathscr{SC}$ and lands in $\mathscr{P}$, which can be identified with the category of A-spaces. The composite $\mathscr{K}\mathscr{X}$ is the barycentric subdivision functor $Sd\colon \mathscr{SC} \longrightarrow \mathscr{OSC}$. It can be viewed as the construction of a canonical ordered simplicial complex SdK starting from a given unordered simplicial complex K, at the price of subdividing. Since the geometric realization functor gives a space $|SdK|$ that can be identified with $|K|$ there is no loss of topological generality working in $\mathscr{OSC}$ instead of $\mathscr{SC}$.

The most important motivation for working with ordered rather than unordered simplicial complexes is that the ordering leads to the definition of an associated chain complex and thus to a quick definition of homology. I'll explain that in the talks and add it to the notes if I have time.

promise

As noted earlier, a topological space X is called a polytope if it is homeomorphic to $|K|$ for a (given) simplicial complex K. Such a homeomorphism $|K| \longrightarrow X$ is called a triangulation of X, and X is said to be triangulable if it admits a triangulation. Then we can define the homology of X to be the homology of K. This is a quick definition, and useful where it applies, but it raises many questions and is quite unsatisfactory conceptually. Not every space is triangulable, and triangulable spaces can admit many different triangulations. It is far from obvious that the homology is independent of the choice of triangulation.

Simplicial sets abstract the notion of ordered simplicial complexes, retaining enough of the combinatorial structure that homology can be defined with equal ease. The generalization allow myriads of examples that do not come from simplicial complexes. The original motivating example gives a functor from topological spaces to simplicial sets. Composing with the functor from simplicial sets to homology groups gives the quickest way of defining the homology groups of a space and leads to the proof that these groups depend only on the weak homotopy type of the space, not on any triangulation, and to the proofs that different triangulations, when they exist, give canonically isomorphic homology groups.

Perhaps the quickest and most intuitive way to motivate the definition of simplicial sets is to start from structure clearly visible in the case of ordered simplicial complexes. Let X denote the partially ordered set $V(K)$ of vertices of an ordered simplicial complex K. The reader might prefer to start with an ordered simplicial complex of the form $\mathscr{K}(X)$, where X is a poset. The reader may also want to insist that X is finite, but that is not necessary to the construction, and we later want to allow infinite sets.

Then an n-simplex σ of K is a totally ordered $n+1$-tuple of elements of X. Write such a tuple as $(x_0, \cdots, x_n)$. When studying products, we saw that it can become essential to consider tuples $(x_0, \cdots, x_n)$, where $x_0 \leq x_1 \leq \cdots \leq x_n$. Of course, $(x_0, \cdots, x_n)$ is no longer a simplex, but one can obtain a simplex from it by deleting repeated entries. When there are repeated entries, we think of $(x_0, \cdots, x_n)$ as a "degenerate" n-simplex. Let K_n denote the set of such generalized n-simplices, degenerate or not. For $0 \leq i \leq n$, define functions

$$d_i \colon K_n \longrightarrow K_{n-1} \quad \text{and} \quad s_i \colon K_n \longrightarrow K_{n+1},$$

called face and degeneracy operators, by

$$d_i(x_0, \cdots, x_n) = (x_0, \cdots x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, \cdots, x_n) = (x_0, \cdots x_i, x_i, \cdots, x_n).$$

Of course, the d_i and s_i just defined also depend on n, but it is standard not to indicate that in the notation. In words, d_i deletes the i^{th} entry and s_i repeats the i^{th} entry. If $i < j$ and we first delete the j^{th} entry and then the i^{th} entry, we get the same thing as if we first delete the i^{th} entry and then delete the (new) $(j-1)^{\text{st}}$ entry. Similarly, elementary inspections give commutation relations between the d_i and s_j and between the s_i. Here is a list of all such relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \quad i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } \ i < j \\ \mathrm{id} & \text{if } \ i = j \ \text{ or } \ i = j+1 \\ s_j \circ d_{i-1} & \text{if } \ i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \ \text{ if } \ i \leq j$$

The reader can easily check that these identities really do follow immediately from the definition of the K_n, d_i, and s_i above.

The K_n are defined in terms of the partially ordered vertex set $V(K)$ of K, but there are many examples of precisely similar structure that arise differently.

9.2. The definition of simplicial sets

We obtain our first definition of simplicial sets by formalizing structure that, as we have just seen, is implicit in the definition of an ordered simplicial complex.

DEFINITION 9.2.1. A simplicial set K is a sequence of sets K_n, $n \geq 0$, and functions $d_i \colon K_n \longrightarrow K_{n-1}$ and $s_i \colon K_n \longrightarrow K_{n+1}$ for $0 \leq i \leq n$ that satisfy the identities just displayed. The elements of the set K_n are called n-simplices, following the historic precedent of simplicial complexes. Just as if K were a simplicial complex, a map $f \colon K \longrightarrow L$ of simplicial sets is a sequence of functions $f_n \colon K_n \longrightarrow L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$. With these objects and morphisms, we have the category $s\mathscr{S}et$ of simplicial sets.

Now our motivating example can be recapitulated in the following statement.

PROPOSITION 9.2.2. *There is a canonical functor $i \colon \mathscr{O}\mathscr{S}\mathscr{C} \longrightarrow s\mathscr{S}et$ from the category of ordered simplicial complexes to the category of simplicial sets. It assigns to an ordered simplicial complex K the simplicial set K^s given by the sequence of sets K^s_n and the functions d_i and s_i defined above. It assigns to a map $f \colon K \longrightarrow L$ of ordered simplicial complexes the map $f^s \colon K^s \longrightarrow L^s$ induced by its map of vertex sets:*

$$f^s_n(x_0, \cdots, x_n) = (f(x_0), \cdots, f(x_n)).$$

It is a full embedding, meaning that the maps $K \longrightarrow L$ of ordered simplicial complexes map bijectively to the maps $K^s \longrightarrow L^s$ of simplicial sets.

The identities listed above are hard to remember and do not appear to be very conceptual. The definition admits a conceptual reformulation that may or may not make things clearer, depending on personal taste, but definitely allows many arguments and constructions to be described more clearly and conceptually than would be possible without it. We define the category Δ of finite ordered sets.

DEFINITION 9.2.3. The objects of Δ are the finite ordered sets $[n]$ with $n+1$ elements $0 < 1 < \cdots < n$. Its morphisms are the monotonic functions $\mu \colon [m] \leq [n]$. This means that $i < j$ implies $\mu(i) \leq \mu(j)$. Define particular monotonic functions

$$\delta_i \colon [n-1] \longrightarrow [n] \quad \text{and} \quad \sigma_i \colon [n+1] \longrightarrow [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = j \ \text{ if } \ j < i \quad \text{and} \quad \delta_i(j) = j+1 \ \text{ if } \ j \geq i$$

and

$$\sigma_i(j) = j \ \text{ if } \ j \leq i \quad \text{and} \quad \sigma_i(j) = j-1 \ \text{ if } \ j > i.$$

In words, δ_i skips i and σ_i repeats i.

There are identities for composing the δ_i and σ_i that are "dual" to those for composing the d_i and s_i that appear in the definition of a simplicial set. Precisely, the duality amounts to reversing the direction of arrows. The following pair of commutative diagrams should make clear how to interpret this, where $i < j$.

$$\begin{tikzcd} K_n \arrow[r, "d_j"] \arrow[d, "d_i"'] & K_{n-1} \arrow[d, "d_i"] \\ K_{n-1} \arrow[r, "d_{j-1}"'] & K_{n-2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[n]} & {[n-1]} \arrow[l, "\delta_j"'] \\ {[n-1]} \arrow[u, "\delta_i"] & {[n-2]} \arrow[l, "\delta_{j-1}"] \arrow[u, "\delta_i"'] \end{tikzcd}$$

A moment's reflection should convince the reader that every monotonic function $\mu\colon [m] \longrightarrow [n]$ can be written as a composite of monotonic functions δ_i and σ_j for varying i and j. That is, μ can be obtained by omitting some of the i's and repeating some of the j's. Just as a group can be defined by specifying a set of generators and relations, so a category can often be specified by a set of generating morphisms and relations between their composites. The category Δ is generated by the δ_i and σ_i subject to our "dual" relations. This leads to the proof of the following reformulation of the notion of a simplicial set. Recall that a contravariant functor F assigns a morphism $FY \longrightarrow FX$ of the target category to each morphism $X \longrightarrow Y$ of the source category.

Proposition 9.2.4. *The category of simplicial sets can be identified with the category of contravariant functors $K\colon \Delta \longrightarrow \mathscr{S}et$ and natural transformations between them.*

Proof. The correspondence is given by viewing the functions d_i and s_i that define a simplicial set as the morphisms of sets induced by the morphisms δ_i and σ_i of the corresponding functor $\Delta \longrightarrow \mathscr{S}et$. It is convenient to write $\mu^*\colon K_n \longrightarrow K_m$ for the function induced by contravariance from a morphism $\mu\colon [m] \longrightarrow [n]$, and then $d_i = \delta_i^*$ and $s_i = \sigma_i^*$. For a map f, the corresponding natural transformation is given on the object $[n]$ by the function f_n. $\square$

While we do not want to emphasize abstraction in the first instance, we nevertheless cannot resist the temptation to generalize the definition of simplicial sets to simplicial objects in a perfectly arbitrary category. The generalization has a huge number of applications throughout mathematics, and we shall use it when defining homology.

Definition 9.2.5. A simplicial object in a category $\mathscr{C}$ is a contravariant functor $K\colon \Delta \longrightarrow \mathscr{C}$. A map $f\colon K \longrightarrow L$ of simplicial objects in $\mathscr{C}$ is a natural tranformation $K \longrightarrow L$; it is given by morphisms $f_n\colon K_n \longrightarrow L_n$ in $\mathscr{C}$. We have the category $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. By composition of functors and natural transformations, any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a functor $sF\colon s\mathscr{C} \longrightarrow s\mathscr{D}$. By duality, a *covariant* functor $\Delta \longrightarrow \mathscr{C}$ is called a cosimplicial object in $\mathscr{C}$.

9.3. Standard simplices and their role

We explain a general conceptual way to relate simplicial sets to "standard simplices". Standard simplices exist in many categories. We have standard simplices in topological spaces, simplicial sets, and even posets and categories. In general, fixing a category $\mathscr{V}$, we often have a standard cosimplicial object in $\mathscr{V}$, that is a certain covariant functor $\Delta[\bullet]^v\colon \Delta \longrightarrow \mathscr{V}$. The superscript v is meant as a reminder that the functor is assigning objects in $\mathscr{V}$ to objects in Δ; it should also help to distinguish the functor $\Delta[\bullet]^v$ from the category Δ. On objects, we write the functor $\Delta[\bullet]^v$ as $[n] \mapsto \Delta[n]^v$, but we agree to write μ_* rather than $\Delta[\mu]^v$ for the map $\Delta[m]^v \longrightarrow \Delta[n]^v$ in $\mathscr{V}$ obtained by applying our functor to a morphism μ in Δ. For each object V of $\mathscr{V}$ we obtain a *contravariant* functor, denoted $SV\colon \Delta \longrightarrow \mathscr{S}et$, by letting the set S_nV of n-simplices be the set $\mathscr{V}(\Delta[n]^v, V)$ of morphisms $\Delta[n]^v \longrightarrow V$ in the category $\mathscr{V}$. The faces and degeneracies are induced by precomposition with the maps

$$\delta_i\colon \Delta[n-1]^v \longrightarrow \Delta[n]^v \quad \text{and} \quad \sigma_i\colon \Delta[n+1]^v \longrightarrow \Delta[n]^v$$

obtained by applying the functor $\Delta[\bullet]^v$ to the generating morphisms δ_i and σ_i of Δ. That is, for a morphism $\nu\colon \Delta[n]^v \longrightarrow V$ in $\mathscr{V}$,

$$d_i(\nu) = \nu \circ \delta_i \quad \text{and} \quad s_i(\nu) = \nu \circ \sigma_i.$$

Before turning to the motivating examples, in which $\mathscr{V}$ is the category $\mathscr{U}$ of topological spaces or the category $\mathscr{C}at$ of small categories, we apply this construction to the case $\mathscr{V} = s\mathscr{S}et$.

DEFINITION 9.3.1. Define the standard simplicial n-simplex $\Delta[n]^s$ to be the contravariant functor $\Delta \longrightarrow s\mathscr{S}et$ represented by $[n]$. This means that the set $\Delta[n]^s_q$ of q-simplices is the set of all morphisms $\phi\colon [q] \longrightarrow [n]$ in Δ. For a morphism $\nu\colon [p] \longrightarrow [q]$ in Δ, the function $\nu^*\colon \Delta[n]^s_q \longrightarrow \Delta[n]^s_p$ is given by composition, $\nu^*(\phi) = \phi \circ \nu\colon [p] \longrightarrow [q]$.

DEFINITION 9.3.2. We define a covariant functor $\Delta[\bullet]^s$ from Δ to the category $s\mathscr{S}et$ of simplicial sets. On objects, the functor sends $[n]$ to the standard simplicial n-simplex $\Delta[n]^s$. On morphisms $\mu\colon [m] \longrightarrow [n]$ in Δ, define $\mu_*\colon \Delta[m]^s_q \longrightarrow \Delta[n]^s_q$ by $\mu_*(\psi) = \mu \circ \psi\colon [q] \longrightarrow [m] \longrightarrow [n]$. Thus the simplicial set $\Delta[n]^s$ is defined using pre-composition with morphisms of Δ, and then the covariant functoriality of $\Delta[\bullet]^s$ is defined using post-composition with morphisms of Δ. The object $\Delta[\bullet]^v$ is a cosimplicial simplicial set, that is, a cosimplicial object in the category of simplicial sets.

We may identify the set of all non-degenerate simplices of $\Delta[n]^s$ with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathscr{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathscr{K}([n])$.

Although we shall give a direct proof, the following result is an application of the Yoneda lemma. Let $\iota_n \in \Delta[n]^s_n$ be the identity map $\mathrm{id}\colon [n] \longrightarrow [n]$.

PROPOSITION 9.3.3. *Let K be a simplicial set. For $x \in K_n$, there is a unique map of simplicial sets $Y(x)\colon \Delta[n]^s \longrightarrow K$ such that $Y(x)(\iota_n) = x$. Therefore K is naturally isomorphic to the simplicial set whose n-simplices are the maps of simplicial sets $\Delta[n]^s \longrightarrow K$.*

PROOF. The map $Y(x)$ is a natural transformation from the contravariant functor $\Delta[n]^s$ to the contravariant functor K from Δ to $\mathscr{S}et$. Since a q-simplex $\phi\colon [q] \longrightarrow [n]$ is $\phi^*(\iota_n)$, we can and must specify $Y(x)$ at the object $[q] \in \Delta$ by the function $\Delta[n]^s_q \longrightarrow K_q$ that sends ϕ to the q-simplex $\phi^*(x)$. $\square$

We can vary the construction in a way that may look unnatural but that will lend itself to generalization to other examples. We show how to reconstruct K directly from the $\Delta[n]^s$.

Construction 9.3.4. For a set J and a simplicial set L, one can form a new simplicial set $J \times L$ by setting $(J \times L)_q = J \times L_q$ and letting the faces and degeneracies be induced by those of L. Said another way, we think of J as a "discrete" simplicial set with each $J_q = J$ and all faces and degeneracies the identity map of J, and we then take the product $J \times L$ of simplicial sets. We apply this with $J = K_n$ and $L = \Delta[n]^s$ as n varies to obtain a simplicial set

$$\overline{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^s.$$

We define an equivalence relation $\simeq$ on $\overline{K}$ by requiring

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \tag{9.3.5}$$

for $k \in K_n$, $\sigma \in \Delta[m]^s_q$, and $\alpha\colon [m] \longrightarrow [n]$ in Δ. Here $\alpha^*(k) \in K_m$ is given by the fact that K is a contravariant functor from Δ to sets and $\alpha_*(\sigma) \in \Delta[n]^s_q$ is given by the fact that $\Delta[-]^s$ is a covariant functor from Δ to simplicial sets. With the simplicial structure induced from the simplicial structure on the $\Delta[n]^s$, passage to equivalence classes gives us a new simplicial set that we shall denote by T^sK for the moment. Then T^s is a functor from simplicial sets to simplicial sets.

Proposition 9.3.6. *The simplicial set T^sK is naturally isomorphic to K.*

Proof. We claim that an arbitrary pair (k, τ) in $K_n \times \Delta[n]^s_q$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]^s_q$ where, as above, $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical q-simplex in $\Delta[q]^s$. Viewing $\tau\colon [q] \longrightarrow [n]$ as a morphism of Δ, we have $\tau = \tau_*(\iota_q)$, and the claim follows. Identifying equivalence classes of q-simplices with elements of K_q in this fashion, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p). \qquad \square$$

9.4. The total singular complex SX and the nerve $N\mathscr{C}$

We turn to the historical motivating example $\mathscr{V} = \mathscr{U}$ by constructing the total singular complex SX of a topological space X. We need a covariant functor $\Delta[\bullet]^t\colon \Delta \longrightarrow \mathscr{U}$, and that is given by the standard topological simplices $\Delta[n]^t$.

Definition 9.4.1. Recall that the standard topological n-simplex $\Delta[n]^t$ is the subspace

$$\{(t_0, \cdots, t_n) \,|\, 0 \leq t_i \leq 1 \ \text{ and } \ \Sigma_i t_i = 1\}$$

of $\mathbb{R}^{n+1}$. Define

$$\delta_i\colon \Delta[n-1]^t \longrightarrow \Delta[n]^t \ \text{ and } \ \sigma_i\colon \Delta[n+1]^t \longrightarrow \Delta[n]^t$$

by

$$\delta_i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_n)$$

and

$$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}).$$

Then the δ_i and σ_i satisfy the commutation relations required to specify a covariant functor $\Delta[\bullet]^t$ from Δ to the category $\mathscr{U}$ of topological spaces, that is, a cosimplicial object in the category of topological spaces.

Definition 9.4.2. The *total singular complex* SX of a space X is the simplicial set whose set S_nX of n-simplices is the set of continuous maps $\Delta[n]^t \longrightarrow X$ and whose faces d_i and degeneracies s_i induced by precomposition with δ_i and σ_i. By composition of continuous maps, a map $f\colon X \longrightarrow Y$ induces the map $f_* = Sf\colon SX \longrightarrow SY$ of simplicial sets that sends an n-simplex $s\colon \Delta[n]^t \longrightarrow X$ to the n-simplex $f \circ s$. This defines the total singular complex functor S from topological spaces to simplicial sets.

We shall return to this example after giving an analogue that may seem astonishing at first sight. Although it has become a standard and commonplace construction, its importance and utility were only gradually recognized. Recall that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \leq y$, and then there is a unique arrow $x \longrightarrow y$, or $x \nleq y$, and then there is no arrow $x \longrightarrow y$. Composition is defined in the only possible way. By definition $[n]$ is a totally ordered set, hence of course it is a partially ordered set. We can view it as a category and then the monotonic functions $\mu\colon [m] \longrightarrow [n]$ are precisely the functors $[m] \longrightarrow [n]$: monotonicity says that if there is an arrow $i \to j$, then there is an arrow $i \leq j$, which must be the value of the functor μ on that arrow.

Definition 9.4.3. Let $\mathscr{C}at$ denote the category whose objects are small categories and whose morphisms are the functors between them. Define a covariant functor $\Delta[\bullet]^c\colon \Delta \longrightarrow \mathscr{C}at$ by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu\colon [m] \longrightarrow [n]$ to the corresponding functor $\mu_*\colon [m] \longrightarrow [n]$. Thus $\Delta[\bullet]^c$ is a cosimplicial category. When necessary for clarity, we write $[n]^c$ for the ordered set $[n]$ regarded as a category.

It is consistent with our previous notations to write $\Delta[n]^c$ for the poset $[n]$ regarded as a category. With that notation, the analogy with the definition of the total singular complex becomes especially obvious.

Definition 9.4.4. Let $\mathscr{C}$ be a small category. We define a simplicial set $N\mathscr{C}$, called the nerve of $\mathscr{C}$. Its set $N_n\mathscr{C}$ of n-simplices is the set of covariant functors $\phi\colon [n]^c \longrightarrow \mathscr{C}$. The function $\mu^*\colon N_n\mathscr{C} \longrightarrow N_m\mathscr{C}$ induced by $\mu\colon [m] \longrightarrow [n]$ is given by $\mu^*(\phi) = \phi \circ \mu$, where μ is viewed as a functor $[m]^c \longrightarrow [n]^c$. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a function $F_n = N_nF\colon N_n \longrightarrow N_n\mathscr{D}$ by composition of functors, $F_n(\phi) = F \circ \phi$. These functions specify a map $F_* = NF\colon N\mathscr{C} \longrightarrow N\mathscr{D}$ of simplicial sets. Thus we the nerve functor N from $\mathscr{C}at$ to the category of simplicial sets.

The definition can easily be unravelled. The category $[0]^c$ has one object and its identity morphism, hence a functor $\phi\colon [0]^c \longrightarrow \mathscr{C}$ is just a choice of an object of $\mathscr{C}$. That is, if we write $\mathscr{O}\mathscr{C}$ for the set of objects of $\mathscr{C}$, then $N_0\mathscr{C} = \mathscr{O}\mathscr{C}$. For $n \geq 1$, a functor $\phi\colon [n]^c \longrightarrow \mathscr{C}$ is a choice of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n.$$

Denoting such a string by $(f_1, \cdots, f_n)$, the faces and degeneracies are given by

$$d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n) & \text{if } i = 0 \\ (f_1, \cdots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \cdots, f_n) & \text{if } 0 < i < n \\ (f_1, \cdots, f_{n-1}) & \text{if } i = n \end{cases} \tag{9.4.5}$$

$$s_i(f_1, \cdots, f_n) = (f_1, \cdots, f_{i-1}, \mathrm{id}, f_i, \cdots, f_n)$$

In words, the 0^{th} and n^{th} faces send $(f_1, \cdots, f_n)$ to the $(n-1)$-simplex obtained by deleting f_1 or f_n; when $n = 1$ this is to be interpreted as giving the object c_1 or c_0. For $0 < i < n$, the i^{th} face composes f_{i+1} with f_i. The i^{th} degeneracy operation inserts the identity morphism of c_i. The ordering may look unnatural, since $f_{i+1} \circ f_i$ means first f_i and then f_{i+1}, and many authors prefer to reverse the ordering in a composable sequence so that for $n \geq 1$, a functor $\phi \colon [n]^c \longrightarrow \mathscr{C}$ is a choice of n composable morphisms

$$c_0 \xleftarrow{f_1} c_1 \longleftarrow \qquad \cdots \longleftarrow c_{n-1} \xleftarrow{f_n} c_n.$$

This amounts to replacing the categories $\Delta[n]^c$ by their opposite categories. It is the choice taken in the following hugely important example.

Example 9.4.6. Let G be a group regarded as a category with a single object $*$; the elements of the group are the morphisms $* \longrightarrow *$, and every pair of morphisms is composable. The nerve NG is often written B_*G and called the bar construction. It is the simplicial set with $B_nG = G^n$, with n-tuples of elements written $[g_1|\cdots|g_n]$ (hence the name "bar") and with faces and degeneracies specified for $0 \leq i \leq n$ by

$$d_i[g_1|\cdots|g_n] = \begin{cases} [g_2|\cdots|g_n] & \text{if } i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] & \text{if } 0 < i < q \\ [g_1|\cdots|g_{n-1}] & \text{if } i = q. \end{cases}$$

$$s_i[g_1|\cdots|g_n] = [g_1|\cdots|g_{i-1}|e|g_i|\cdots|g_n]$$

However $N\mathscr{A}$ is written, in general it looks nothing like our original example of the simplicial set associated to an ordered simplicial complex! In one important case, which we will find is far more common than one might reasonably expect, it does look like that.

Example 9.4.7. Let X be a poset. We can obtain a simplicial set by regarding X as a category and taking its nerve. Alternatively, we can take the ordered simplicial complex $\mathscr{K}X$ and then take the simplicial set associated to that. It is an instructive exercise to check that we get the same simplicial set via either route. That is, NX is naturally isomorphic to $(\mathscr{K}X)^s$.

9.5. The geometric realization of simplicial sets

We have observed that the category Δ is generated by the injections δ_i and surjections σ_i. Decomposing a morphism $\mu \colon [m] \longrightarrow [n]$ as a composite of δ_i's and σ_j's records which elements of the target $[n]$ are not in the image of μ and which elements of the source $[m]$ have the same image under μ. It is helpful to be more precise about this. Let $i_1, \cdots, i_q$ in reverse order $0 \leq i_q < \cdots < i_1 \leq n$ be the elements of $[n]$ that are not in the image $\mu([m])$. Let $j_1, \ldots, j_p$ in order $0 \leq j_1 < \cdots < j_p < m$ be the elements $j \in [m]$ such that $\mu(j) = \mu(j+1)$. With these notations, $m - p + q = n$ and

$$\mu = \delta_{i_1} \cdots \delta_{i_q} \sigma_{j_1} \cdots \sigma_{j_p}. \tag{9.5.1}$$

That is, we record duplications in such a manner that the indices record the repeated and skipped elements in a sensible canonical order. The sequences of i's and j's in this description of μ are uniquely determined.

Using this canonical decomposition implicitly, we can be precise about the definition and description of the geometric realization of a simplicial set K. The construction is precisely analogous to Construction 9.3.4 and might well be denoted by T^tK.

Construction 9.5.2. For a set J and a space L, we regard J as a discrete topological space and obtain the space $J \times L$. Applying this with $J = K_n$ and $L = \Delta[n]^t$ for $n \geq 0$, we obtain the space

$$\bar{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^t$$

with the topology of the union. That is, we take the union of one topological simplex for each n-simplex $k \in K_n$. Say that an n-simplex k is degenerate if $k = s_i \ell$ for some $(n-1)$-simplex ℓ and some i and nondegenerate otherwise. We shall glue the simplices together in such a way that we obtain a space with one "n-cell" for each nondegenerate n-simplex of K. That means in particular that in the resulting space every point will be the interior point of the image of exactly one simplex $\{k\} \times \Delta[n]^t$, where k is nondegenerate. Note that the unique point of $\Delta[0]$ is an interior point. We say that a point (k, u) of $\bar{K}$ is nondegenerate if k is nondegenerate and u is interior.

Define an equivalence relation $\approx$ on $\bar{K}$ by letting

$$(\mu^* k, u) \approx (k, \mu_* u)$$

for each $k \in K_n$, $u \in \Delta[m]$, and $\mu \colon [m] \longrightarrow [n]$. This equivalence relation is generated by the relations obtained by specializing to $\mu = \delta_i$ or $\mu = \sigma_i$. These can be rewritten as

$$(d_i k, u) \approx (k, \delta_i u) \quad \text{and} \quad (s_i k, u) \approx (k, \sigma_i u).$$

Each n-simplex k_n can be written uniquely in the form $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$, where k_{n-p} is nondegenerate and $0 \leq j_1 < \cdots < j_p < n$. Define a function $\lambda \colon \bar{K} \longrightarrow \bar{K}$ by

$$\lambda(k_n, u_n) = (k_{n-p}, \sigma_{j_1} \cdots \sigma_{j_p} u_n)$$

where $u_n \in \Delta[n]^t$. Similarly, every $u_n \in \Delta[n]^t$ can be written uniquely in the form $u_n = \delta_{i_q} \cdots \delta_{i_1} u_{n-q}$, where u_{n-q} is interior and $0 \leq i_q < \cdots < i_1 \leq n$. Define a function $\rho \colon \bar{K} \longrightarrow \bar{K}$ by

$$\rho(k_n, u_n) = (d_{i_q} \cdots d_{i_1} k_n, u_{n-q}).$$

Lemma 9.5.3. *The composite $\lambda \circ \rho$ carries each point of $\bar{K}$ into the unique nondegenerate point that is equivalent to it.*

Define the geometric realization of K, which is usually denoted $|K|$ but which we shall usually denote by TK, to be the set of equivalence classes $\bar{K}/(\approx)$. Define F_pTK to be the image of $\coprod_{0 \leq n \leq p} K_n \times \Delta[n]$ in TK and give it the quotient space topology. Then topologize TK by giving it the topology of the union of the F_pTK. This means that a subset C is closed if and only if it intersects each F_pTK in a closed subset. We shall shortly give an equivalent description of this topology.

9.6. CW complexes

We explain the nature of the space TK by introducing two equivalent definitions of a CW complex. We start with the original 1949 definition of J.H.C. Whitehead [42], which explains the name. We then observe that TK satisfies the specifications of that definition. Finally, we give the more modern and now standard definition of a CW complex. Let D^n be the disc $\{x||x| \leq 1\} \subset \mathbb{R}^n$.

DEFINITION 9.6.1. A *cell complex* is a Hausdorff space X such that X is a disjoint union of subspaces e^n, called "open cells", each of which is homeomorphic to an open disc $\mathring{D}^n$. The closure of e^n in X is denoted $\bar{e}^n$, and it is *not* required to be homeomorphic to the closed disc D^n. Rather, for each open cell e^n, there must be a map $\bar{j}\colon \Delta[n] \longrightarrow \bar{e}^n$ such that

(i) The restriction of $\bar{j}$ maps $\mathring{\Delta}[n]$ homeomorphically onto e^n.
(ii) The restriction of $\bar{j}$ maps the boundary $\partial\Delta[n]$ into the union of the cells of dimension less than n.

A subcomplex A of X is a union of some of the cells of X such that if $e^n \subset A$, then $\bar{e}^n \subset A$. A cell complex is a CW complex if

(i) X is Closure finite, meaning that each $\bar{e}^n$ is contained in a finite subcomplex.
(ii) X has the Weak topology, meaning that a subset is closed if and only if its intersection with each $\bar{e}^n$ is a closed subspace.

The capitalized C and W are the source of the name "CW complex", but this form of the definition is so rarely used nowadays that younger experts often have no idea where the name came from. However, it is convenient for describing TK.

THEOREM 9.6.2. *The space TK is a CW complex with one n-cell for each nondegenerate n-simplex $k_n \in K_n$.*

PROOF. The n-cells e^n of TK are the images of the subspaces $\{k_n\} \times \mathring{\Delta}[n]$, and the map $j\colon \Delta[n] \longrightarrow \bar{e}^n$ is the restriction of the map $\bar{K} \longrightarrow TK$ to $\{k_n\} \times \mathring{\Delta}[n]$. The topology of the union we prescribed before is in fact the "weak topology". It is "weak" in the sense that in general it has more open sets than the quotient space topology, but the novice may not want to worry about the verification, preferring to simply accept that our original definition of the topology gives what once upon a time was called the weak topology. $\square$

Here is the modern redefinition of a CW complex.

DEFINITION 9.6.3. A CW complex is a space X that is the union of an expanding sequence of subspaces X^n, where X^n is called the n-skeleton of X. It is required inductively that

(1) X^0 is a set with the discrete topology.
(2) X^{n+1} is constructed from X^n as a "pushout"

$$\begin{tikzcd} \coprod S^n \arrow[r, "j"] \arrow[d, "\cap"'] & X^n \arrow[d] \\ \coprod D^{n+1} \arrow[r, "\bar{j}"'] & X^{n+1}. \end{tikzcd}$$

This means that X^{n+1} is the quotient space

$$X^n \cup_{\amalg S^n} (\amalg D^{n+1}) \equiv X^n \amalg (\amalg D^{n+1})/(\approx)$$

specified by the equivalence relation $s \approx j(x)$ for $s \in S^n \subset D^{n+1}$.

The space X is given the topology of the union; equivalently, a subset is closed if its intersection with each closed cell $\bar{j}(D^n)$ is closed.

We leave it as an exercise for the reader to see that the two definitions of a CW complex give exactly the same spaces. The compactness of the spheres that are the domains of attaching maps ensures that a CW complex with the second definition is closure finite, as required in the first definition.

The intuition is that we glue discs D^{n+1} to X^n as dictated by attaching maps defined on their boundaries S^n. The attaching maps can be quite badly behaved. For an ordered simplicial complex K, the classical geometric realization $|K|$ is homeomorphic to the geometric realization $T(K^s)$ of its associated simplicial set K^s. This is visually apparent since each has an n-cell for each n-simplex of K. Remember that the n-simplices of K itself are of the form $\{x_0 < \cdots < x_n\}$ whereas the elements of K_n are of the form $\{x_0 \leq \cdots \leq x_n\}$. The degeneracy identifications in the construction of TK^s serve to eliminate the degenerate elements in which some of the vertices are repeated.

In $T(K^s)$ the closed cells are homeomorphic to $\Delta[n]$ and the attaching maps are homeomorphisms on boundaries. Spaces can be "triangulated" as CW complexes using many fewer cells than are required for polyhedral triangulations. For example, we can triangulate the n-sphere S^n as a CW complex with just two cells. Clearly S^0 is a CW complex with two 0-cells, or vertices. For $n > 0$, we start with a single 0-cell $*$, take $(S^n)^{n-1} = *$ and attach a single n-cell with attaching map the trivial map $S^{n-1} \longrightarrow *$. Then the n-skeleton is $* \cup_{S^{n-1}} D^n = D^n/S^{n-1}$, which is already homeomorphic to S^n.

There is a natural half-way house between simplicial complexes and CW complexes that will later play a role in our study.

Definition 9.6.4. A CW complex is *regular* if each of its attaching maps $S^n \longrightarrow X^n$ is a homeomorphism onto its image.

Remark 9.6.5. Earlier we neglected to give a precise definition of $|K|$ for a geometric simplicial complex with a possibly infinite number of vertices and thus with possibly infinite dimension: while every simplex has a finite dimension, simplices of all finite dimensions can occur. When K is ordered, we now have such a definition. We just take the geometric realization of the associated simplicial set; the result is a functor from the category of ordered simplicial sets to the category of spaces. When K is finite, TK^s is homeomorphic to $|K|$ as we defined it originally. We can also start with A-spaces, alias posets X. Then $T\mathscr{K}(X)^s$ gives a composite functor from the category of posets to the category of spaces.

Remember that the product $K \times L$ of ordered simplicial complexes K and L has simplices all subsets of products $\sigma \times \tau$ of simplices, where the ordering on vertices is given by $(x,y) \leq (x',y')$ if $x \leq x'$ and $y \leq y'$.

Definition 9.6.6. Define the product $K \times L$ of simplicial sets K and L by letting $(K \times L)_n = K_n \times L_n$, with $d_i = (d_i, d_i)$ and $s_i = (s_i, s_i)$, which implies that $\mu^* = (\mu^*, \mu^*)$ for all morphisms μ in Δ.

This definition is forced by two considerations. First, it ensures the consistency statement $(K\times L)^s \cong K^s\times L^s$. That is, if we start with ordered simplicial complexes K and L, then the simplicial set $(K \times L)^s$ is naturally isomorphic to the product simplicial set $K^s \times L^s$. Second, the definition is dictated by the universal property that we require of products in any category. Recall that the n-simplices of $K \times L$ involve repeated vertices of K and L. These correspond to the use of degeneracy operators in the factors K^s and L^s of the associated simplicial set. It clarifies matters to be precise about this. We state the following lemma for general simplicial sets K and L, but the reader should think about what it is saying when we apply it to K^s and L^s for ordered simplicial complexes K and L.

LEMMA 9.6.7. *Let K and L be simplicial sets. The nondegenerate n-simplices of $K \times L$ can be written uniquely in the form*

$$(s_{i_p}\cdots s_{i_1}k, s_{j_q}\cdots s_{j_1}\ell),$$

where k is a nondegenerate $(n-p)$-simplex of K, ℓ is a nondegenerate $(n-q)$-simplex of L, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the sets $\{i_a\}$ and $\{j_b\}$ are disjoint.

The set $\{i_a\}\cup\{j_b\}$ has $p+q$ elements and corresponds to a (p, q) shuffle permutation of a set with $p+q$ elements. The term "shuffle" comes from thinking of a permutation of a deck of $p+q$ cards that starts with a cut into p cards and q cards, which are kept in order by the permutation. The reader will easily see that when we started with posets X and Y and showed that $\mathscr{K}(X \times Y)$ is a subdivision of $\mathscr{K}(X) \times \mathscr{K}(L)$, we were actually verifying an instance of essentially this lemma. From here, the reader will have no trouble believing the following result, the proof of which amounts to appropriately subdividing topological simplices $\Delta[n]^t$.

THEOREM 9.6.8. *For simplicial sets K and L, the map*

$$T(K\times L) \longrightarrow TK\times TL$$

whose coordinates are the maps $T\pi_1$ and $T\pi_2$ induced by the projections of $K \times L$ on K and L is a homeomorphism.

We shall not repeat the proof, which adds precision and decreases intuition, referring the reader, for example, to [**27**, 14.3] or [**14**, 4.3.15] for details. The latter book is especially recommended as a very good and relatively recent treatment of CW complexes, simplicial complexes, and simplicial sets.

CHAPTER 10

The big picture: a schematic diagram and the role of subdivision

The n-skeleton K^n of a simplicial set K is the subsimplicial set generated by the q-simplices for all $q \leq n$. Visibly, ΠK depends only on the 2-skeleton K^2. Therefore the inclusion $K^2 \longrightarrow K$ of simplicial sets induces an isomorphism of categories $\Pi K^2 \longrightarrow \Pi K$ for any K. In particular, Π takes the inclusion $\iota\colon \partial\Delta[n]^s \longrightarrow \Delta[n]^s$ of the boundary of the n-simplex to the identity functor when $n > 2$. Thus Π loses homotopical information: upon realization, $|\iota|$ is equivalent to the inclusion $S^{n-1} \longrightarrow D^n$. What is amazing is that this extreme loss of information disappears after subdividing twice. This is something I have been trying to better understand for quite some time.

The reader will find it easy to believe that there is a subdivision functor on simplicial sets that generalizes the subdivision functor Sd on simplicial complexes in the sense that $(SdK)^s \cong Sd(K^s)$ for a simplicial complex K. This allows one to define a subdivision functor on categories by setting $Sd\mathscr{C} = \Pi SdN\mathscr{C}$. One can iterate subdivision, forming functors Sd^2 on both simplicial sets and categories. What is mind blowing at first is that the iterated subdivision $Sd^2\mathscr{C}$ is actually a poset whose classifying space $BSd^2\mathscr{C}$ is homotopy equivalent to $B\mathscr{C}$. I will start from a more combinatorial definition of $Sd\mathscr{C}$, and I will use it to give what I hope the reader will find an easy combinatorial proof that $Sd^2\mathscr{C}$ is indeed a poset.

However, before heading for that, let us summarize a schematic and technically oversimplified global picture of all of the big categories that we are constructing and comparing by functors. This is the same diagram as in the introduction, and it gives an interesting picture of lots of kinds of mathematics that come together with a focus on simplicial sets.

Add left adjoint to i, from Cat to Poset?

$$\begin{tikzcd}
\mathscr{G}rp \arrow[rr, shift left, "K(-{,}1)"] \arrow[d, "i"'] & & \mathscr{U} \arrow[ll, shift left, "\pi_1"] \arrow[d, shift left, "S"] & & \\
\mathscr{C}at \arrow[rr, shift left, "N"] \arrow[urr, "B"] \arrow[d, shift left, "Sd^2"] & & s\mathscr{S}et \arrow[ll, shift left, "\Pi"] \arrow[u, shift left, "T"] \arrow[rr, dashed, shift left, "Sd^2"] & & \mathscr{O}\mathscr{S}\mathscr{C} \arrow[ll, shift left, "i"] \arrow[d, "U"] \\
\mathscr{P}oset \arrow[u, shift left, "i"] \arrow[rr, shift left, "\cong"] & & A\text{-}\mathscr{S}pace \arrow[ll, shift left, "\cong"] \arrow[urr, "\mathscr{K}"] & & \mathscr{S}\mathscr{C} \arrow[ll, "\mathscr{X}"]
\end{tikzcd}$$

Our earlier work focused on finite spaces, but the basic theory generalizes with the finiteness removed, provided we understand simplicial complexes to mean abstract simplicial complexes. As noted above, we didn't define geometric realization in general earlier, but we have done so now. The equivalence of posets with A-spaces and the constructions $\mathscr{K}$ and $\mathscr{X}$ that we worked out in detail for finite spaces work in exactly the same way when we no longer restrict ourselves to the finite case. The functors i in the diagram are thought of as inclusions of categories. Remember that we write $i(K) = K^s$ for the simplicial set associated to an ordered simplicial complex. We have defined all of the categories and functors exhibited in the diagram except for Sd^2, which is second subdivision.

Describe features of the diagram: posets vs ordered simplicial complexes (latter: some but not all totally ordered subsets of the poset of vertices. [Said earlier]) Remember no canonical ordering, u cannot be a right adjoint, etc.

CHAPTER 11

Subdivision and Properties A, B, and C in $s\mathscr{S}et$

We shall define three properties of a simplicial set, called Properties A, B, and C. We say that a category satisfies property A, B, or C if its nerve satisfies that property. Remember that the nerve functor N is a right adjoint whose left adjoint is the fundamental category functor Π. We shall define the subdivision of a simplicial set in such a way as to generalize the subdivision of simplicial complexes that plays such a fundamental role in our study of finite spaces. We shall define the companion notion of the subdivision of a category in the next chapter. We write Sd^s for the subdivision functor on simplicial sets and Sd^c for the subdivision functor on categories when necessary for clarity. These are the main characters in our story. We want to understand the relationships between these functors and the rest of the categories and functors in our big picture. There are a number of surprising and interesting implications.

11.1. Properties A, B, and C of simplicial sets

Definition 11.1.1. We define and name three properties that a simplicial set might have.

(A) Property A, the *nondegenerate simplex property*: K has property A if every face of a nondegenerate simplex x of K is nondegenerate.
(B) Property B, the *distinct vertex property*: K has property B if the $n+1$ vertices of any nondegenerate n-simplex x of K are distinct.
(C) Property C, the *unique simplex property*: K has property C if for any set of $n+1$ distinct vertices of K, there is at most one nondegenerate n-simplex of K whose vertices are the elements of that set.

Remark 11.1.2. In Property A, we mean that all faces $d_i x$ are nondegenerate. But then all faces of all $d_i x$ are also nondegenerate. Iterating, all of the face q-simplices of x for $q < n$ are nondegenerate.

In line with this remark, there is a less succinct but useful characterization of Property B. We express it with a notation that we shall use frequently later.

Notation 11.1.3. For a simplex $x \in K_n$ and a (nonempty) subset S of the set $[n] = \{0, 1, \cdots, n\}$, let S^*x denote the simplex $\mu^*x \in K_m$, where $\mu\colon [m] \longrightarrow [n]$ is the unique injection in Δ with image S. Then the cardinality of S, which we write as $|S|$, is $m+1$.

Proposition 11.1.4. *A simplicial set K has Property B if and only if for every n and every nondegenerate simplex $x \in K_n$, μ^*x and ν^*x are distinct simplices of K for every pair μ and ν of distinct injections with target $[n]$ in Δ; equivalently, $S^*x \neq T^*x$ for every pair of distinct subsets S and T of $[n]$.*

PROOF. Property B is the case when μ and ν have source $[0]$, so it is clear that the new property implies Property B. For the converse, suppose that K satisfies Property B and that $S^*x = T^*x$ for a nondegenerate simplex $x \in K_n$ and nonempty subsets S and T of $[n]$. This clearly implies that $|S| = |T| = m + 1$, say, where $0 \leq m \leq n$. Write $S = \{s_0, \cdots, s_m\}$ and $T = \{t_0, \cdots, t_m\}$, each in strictly increasing order. Consider the singleton subsets $\{i\} \subset [m]$, $\{s_i\} \subset [n]$, and $\{t_i\} \subset [n]$, where $0 \leq i \leq m$. Using the language of Notation 11.1.3, we have

$$\{s_i\}^*x = \{i\}^*S^*x = \{i\}^*T^*x = \{t_i\}^*x.$$

Since these are vertices of x, they are equal by Property B. This implies that $s_i = t_i$ and thus $S = T$. □

It is natural to ask if there are implications among Properties A, B, and C.

THEOREM 11.1.5. *Property B implies Property A, but there are no other implications between these properties.*

PROOF. Suppose that K does not have Property A. There is an $n \geq 1$ and a nondegenerate n-simplex with a degenerate face. Using the commutation relations between faces and degeneracies, we see that any degenerate simplex has a degenerate 1-simplex as one of its 1-faces. Since both vertices of a degenerate 1-simplex s_0x are x, our original nondegenerate n-simplex cannot have distinct vertices. The non-implications are proven by exhibiting counterexamples. We choose nerves of categories, so that these non-implications will also be clear for categories. □

EXAMPLE 11.1.6. Here are some examples which exhibit various non-implications.
(i) Let $K = N\mathscr{C}$ where $\mathscr{C}$ is the category with one object x and one non-identity morphism p, with $p \circ p = p$. Then K satisfies Property A but not Property B.
(ii) Let $K = N\mathscr{C}$, where $\mathscr{C}$ is the category with two vertices x and y, two non-identity morphisms $x \longrightarrow y$, and no morphisms $y \longrightarrow x$. Then K satisfies Properties A and B but not Property C.
(iii) Let $K = NC_2$, where C_2 is the cyclic group of order 2 regarded as a category with one object. Then K satisfies Property C but not Properties A or B. For each q, K has a unique nondegenerate q-simplex $(g, \cdots, g)$, where g is the generator of C_2. Since $g^2 = e$, that simplex has a degenerate face when $q \geq 2$.
(iv) More generally, if $K = NC_n$, where C_n is the cyclic group of order $n > 2$ with generator g, the simplices $x = (g, \cdots, g) \in K_q$ have all faces d_ix nondegenerate, but iterated face operations reach degenerate simplices when $q \geq n$.

Here is a thought exercise. Consider the simplicial set K^s associated to an ordered simplicial complex K. Clearly it has all three properties. What about a converse? Recall that there is a natural order on the set of vertices of the standard n-simplex $\Delta[n]^s$. After all, they are the i with $0 \leq i \leq n$. Since the set K_n can be identified with the set of simplicial maps $\Delta[n]^s \longrightarrow K_n$, each simplex has an induced ordering of its vertices. It need not be consistent as the simplices vary. We can try to give the set of vertices a partial order that restricts to a total order on each simplex by setting $v \leq w$ if and only if v and w are vertices of some simplex x in some K_n and $v \leq w$ in the ordering of the vertices of that simplex.

EXERCISE 11.1.7. Suppose that a simplicial set K satisfies Properties B and C. Then $\leq$ is a well-defined partial order on the set $V = K_0$ that restricts to a total order on the vertices of each non-degenerate simplex of K. With simplices those finite sets of vertices that are the vertices of some nondegenerate $x \in K_n$, we obtain a simplicial complex L, and K is isomorphic to L^s. Conversely, if K does not satisfy either Property B or Property C, then it cannot be isomorphic to L^s for any simplicial complex L.

By abuse of language, we say that a simplicial set is a simplicial complex if it is isomorphic to L^s for some ordered simplicial complex L. In fact, L is canonically determined by K in the manner that we have described. The exercise proves the following result.

THEOREM 11.1.8. *A simplicial set is a simplicial complex if and only if it satisfies Properties B and C.*

11.2. The definition of the subdivision of a simplicial set

For both simplicial sets and categories, there is both a conceptual definition and an equivalent combinatorial definition. For simplicial sets, we begin with the perhaps ugly looking and hard to grasp combinatorial definition and then show that it is equivalent to a conceptual definition that is closely analogous to the definition of geometric realization.

DEFINITION 11.2.1. We define the subdivision $\mathrm{Sd}K = \mathrm{Sd}^s K$ of a simplicial set K. The q-simplices of $\mathrm{Sd}K_q$ are the equivalence classes of tuples

$$(x; S_0, \cdots, S_q),$$

where, for some $n \geq 0$, $x \in K_n$, each S_i is a subset of $[n]$, and $S_i \subset S_{i+1}$ for $0 \leq i < q$. The equivalence relation is specified by

$$(\mu^* x; S_0, \cdots, S_q) \ \sim \ (x; \mu_*(S_0, \cdots, S_q))$$

for a morphism $\mu\colon [m] \longrightarrow [n]$ in Δ, where $x \in K_n$, hence $\mu^* x \in K_m$; here $\{S_i\}$ is an increasing sequence of subsets of $[m]$ and

$$\mu_*(S_0, \cdots, S_q) = (\mu(S_0), \cdots, \mu(S_q)).$$

The simplicial operations are induced by

$$\nu^*(x; S_0, \cdots, S_q) = (x; S_{\nu(0)}, \cdots, S_{\nu(p)})$$

for a map $\nu\colon [p] \longrightarrow [q]$ in Δ, where $x \in K_n$ and $\{S_i\}$ is an increasing sequence of subsets of $[n]$ for some n. Subdivision is functorial. For a map $f\colon K \longrightarrow L$ of simplicial sets, $f_* = \mathrm{Sd}f\colon \mathrm{Sd}K \longrightarrow \mathrm{Sd}L$ is induced by

$$f_*(x; S_0, \cdots, S_q) = (f(x); S_0, \cdots, S_q).$$

This definition is convenient for doing combinatorics and is directly motivated by the following comparison, which we will prove in §11.3.

THEOREM 11.2.2. *If K is an ordered simplicial complex, then the simplicial sets $\mathrm{Sd}(K^s)$ and $(\mathrm{Sd}K)^s$ are naturally isomorphic.*

However, it obscures the idea behind the definition, which we now elucidate. The conceptual definition parallels Constructions 9.3.4 and 9.5.2. The parallel with the geometric realization functor is particularly useful, but the parallel with the reconstruction functor $T^s K$ is especially illuminating.

Recall that $\Delta[n]^s$ is the represented simplicial set with q-simplices the maps $\alpha\colon [q] \longrightarrow [n]$ in Δ. Its nondegenerate simplices are the injections. It is a simplicial complex. That is, it can be viewed as $(\mathscr{K}[n])^s$. As a simplicial complex it has the subdivision studied earlier, which we now regard as a simplicial set and denote by $Sd\Delta[n]^s$. Then the nondegenerate q-simplices of $Sd\Delta[n]^s$ are the ordered q-tuples $\underline{\alpha} = \{\alpha_0, \cdots, \alpha_q\}$ of $\Delta[n]^s$, where α_i is a face of α_{i+1}, so that α_i is obtained from α_{i+1} by precomposition with an injection in Δ. For a map $\nu\colon [p] \longrightarrow [q]$ in Δ, the simplicial operation ν^* on $Sd\Delta[n]$ is given by

$$\nu^*(\underline{\alpha}) = (\alpha_{\nu(0)}, \cdots, \alpha_{\nu(p)}).$$

As n varies, the subdivisions $Sd\Delta[n]$ define a covariant functor

$$Sd\Delta[\bullet]^s\colon \Delta \longrightarrow s\mathscr{S}et,$$

that is, a cosimplicial simplicial set. For $\mu\colon [m] \longrightarrow [n]$, $\mu_*\colon Sd\Delta[m]^s \longrightarrow Sd\Delta[n]^s$ is given by

$$\mu_*\underline{\alpha} = (\mu \circ \alpha_0, \cdots, \mu \circ \alpha_q).$$

Strictly speaking, to write simplices in terms of injections only, we must interpret $\mu\circ\alpha_i$ as the injective part δ of the canonical decomposition of $\mu\circ\alpha_i$ as the composite $\delta\sigma$ of a surjection σ and an injection δ. Here is the conceptual definition of $\mathrm{Sd}K$.

Construction 11.2.3. As in the construction of T^sK given in Construction 9.3.4, regard each set K_n as just a set, or as a discrete simplicial set with each $(K_n)_q = K$ and all faces and degeneracies the identity map. Then form the product simplicial sets $K_n \times Sd\Delta[n]^s$ and take their disjoint union to obtain the simplicial set

$$\overline{SdK} = \coprod_{n\geq 0} K_n \times Sd\Delta[n].$$

Again as in the construction of T^sK, define an equivalence relation on $\overline{SdK}$. For $\mu\colon [m] \longrightarrow [n]$ in Δ, we let

$$(\mu^*x, \underline{\alpha}) \sim (x, \mu_*\underline{\alpha}).$$

where $x \in K_n$ and $\underline{\alpha} \in Sd\Delta[m]^s$. We suppress from the notation that this defines an equivalence relation on q-simplices for each q. Now $(\mathrm{Sd}K)_q$ is the set of equivalence classses of q-simplices. The simplicial operations on the simplicial sets $K_n \times Sd\Delta[n]^s$ are of the form $\mathrm{id}\times\nu^*$. They induce the simplicial operations on $\mathrm{Sd}K$.

Remark 11.2.4 (Categorical remark). The definitions of T^sK, $\mathrm{Sd}K$ and TK are all examples of "tensor products of functors", often written $K \otimes_\Delta L$ for a contravariant functor K and a covariant functor L defined on Δ (which could be replaced by any other small category) but we shall not go into the general categorical framework. However, as a specialization of a general categorical result about such categorical tensor products, there is an associativity isomorphism of simplicial sets

$$(K \otimes_\Delta L) \otimes_\Delta M \cong K \otimes_\Delta (L \otimes_\Delta M)$$

where K is a simplicial set and L and M are cosimplicial simplicial sets. Inductively, this implies that

$$Sd^nK \cong K \otimes_\Delta Sd^n\Delta[-] = \coprod_n K_n \times Sd^n\Delta[n]/(\sim),$$

where the equivalence relation is defined exactly as in Construction 11.2.3. This gives a good hold on these functors, since $Sd^n\Delta[-] = (\mathscr{K}^{(n)}\Delta[-])^s$ is just the classical iterated barycentric subdivision, regarded as a simplicial set.

To reconcile the combinatorial and conceptual definitions of $\mathrm{Sd}K$, observe that injective maps α in Δ are uniquely determined by their images. The q-tuples $(\alpha_0, \cdots, \alpha_q)$ of injections above can just as well be viewed as the q-tuples $(S_0, \cdots, S_q)$ of the images of the α_i, which are increasing sequences of subsets of $[n]$ for some n. After this replacement, the two definitions coincide. Observe that the degenerate simplices of $Sd\Delta[n]^s$ are those for which $S_i = S_{i+1}$ for some i.

The conceptual definition is the one best suited for the proof of the following basic result.

THEOREM 11.2.5. *The geometric realization of a simplicial set K is homeomorphic to the geometric realization of* $\mathrm{Sd}K$*, but there is no natural simplicial map between the two that realizes the homeomorphism. There is a natural map of simplicial sets* $\mathrm{Sd}K \longrightarrow K$ *that induces a homotopy equivalence* $T\mathrm{Sd}K \longrightarrow TK$*.*

PROOF. We compare SdK with the simplicial set isomorphic to K given by Proposition 9.3.6. That simplicial set is constructed from K and the $\Delta[n]$ rather than from K and the $Sd\Delta[n]$. The standard homeomorphisms between the $|\Delta[n]|$ and the $|Sd\Delta[n]|$ induce the claimed homeomorphism between $|K|$ and $|\mathrm{Sd}K|$.

The standard maps of simplicial sets $\xi\colon Sd\Delta[n]^s \longrightarrow \Delta[n]^s$ given by Definition 4.3.8 together specify a map $\xi\colon Sd\Delta[\bullet]^s \longrightarrow \Delta[\bullet]^s$ of cosimplicial simplicial sets since they are natural, as observed in Remark 4.3.11. Using the conceptual definition of $\mathrm{Sd}K$ and the description of K as T^sK in Proposition 9.3.6, we see that ξ induces a natural map of simplicial sets $\xi\colon SdK \longrightarrow K$. The geometric realization of the maps $\xi\colon Sd\Delta[n]^s \longrightarrow \Delta[n]^s$ are homotopy equivalences by Proposition 4.3.7. It follows that the induced map $T\xi\colon T\mathrm{Sd}K \longrightarrow TK$ is a homotopy equivalence. The proof of the implication is just a bit beyond the scope of this book; an old reference is [**?**, A.4(ii)]. The idea is that application of the maps ξ gives a map that by inspection of the filtrations of $T\mathrm{Sd}K$ and TK can be proven to be a local weak homotopy equivalence, so that Theorem 3.3.1 gives that $T\xi$ is a weak homotopy equivalence. Since it is a map between CW complexes, it is a homotopy equivalence. $\square$

11.3. Combinatorial properties of subdivision

We use the combinatorial definition to derive some basic combinatorial properties of subdivision.

DEFINITION 11.3.1. A q-simplex $(x; S_0, \cdots, S_q)$ of SdK is in *minimal form* if $x \in K_n$ is nondegenerate and $S_q = [n]$.

PROPOSITION 11.3.2. *Every simplex of* $\mathrm{Sd}K$ *is equivalent to a unique simplex in minimal form. When so written, a simplex is degenerate if and only if $S_i = S_{i+1}$ for some i.*

PROOF. Conceptually, this is analogous to the description of the points of the geometric realization TK in nondegenerate form. We think of q-simplices of $Sd\Delta[n]^s$ as "interior" if $S_q = [n]$, and we then use the same canonical form for morphisms of Δ as composites of σ's and δ's that we used to prove the analogue for realization. If we start with an element $(y; T_1, \cdots, T_q)$ with $y \in K_p$, $T_i \subset [p]$ and $|T_q| = m+1$, we have a unique injection $\delta\colon [m] \longrightarrow [p]$ such that $\delta([m]) = T_q$. There are unique subsets R_i of $[m]$ such that $\delta(R_i) = T_i$, and $(y; T_1, \cdots, T_q)$ is equivalent to $(\delta^* y; R_1, \cdots, R_q)$, where $R_q = [m]$. Now there is a surjection $\sigma\colon [m] \longrightarrow [n]$ and a nondegenerate simplex x of K_n such that $\sigma^* x = \delta^* y$. Then $(\delta^* y; R_1, \cdots, R_q)$ is equivalent to $(x; S_1, \cdots, S_q)$ where $S_i = \sigma^*(R_i)$. By the surjectivity of σ, $S_q = [n]$. It is left as a thought exercise to see that this process reaches the unique minimal element equivalent to the element we started with.

Now suppose that $z = (x; S_1, \cdots, S_q)$ is in minimal form. If $S_i = S_{i+1}$, then z is certainly degenerate. We must show that if z is degenerate, then some $S_i = S_{i+1}$. The assumption means that z is equivalent to $z' = (y; T_0, \cdots, T_q)$, where $T_j = T_{j+1}$ for some j. However, unlike z, z' might not be in minimal form. Just as above, let $y \in K_p$, so that the T_i are contained in $[p]$. Let $|T_q| = m+1$ and choose an injection $\delta \colon [m] \longrightarrow [p]$ such that $\delta([m]) = T_q$. Define $R_i = \delta^{-1}(T_i)$ for all i and note that $R_q = [m]$. Then z' is equivalent to $z'' = (\delta^* y; R_0, \cdots, R_q)$. Now let $\delta^* y = \sigma^* w$ where σ is a surjection and $w \in K_n$ is nondegenerate. Then z'' is equivalent to $(w; \sigma(R_0), \cdots, \sigma(R_q))$. This simplex is in minimal form since $\sigma([m]) = [n]$, so it must be z. Thus $x = w$ and $S_i = \sigma(R_i) = \sigma_i \delta^{-1}(T_i)$. Since $T_j = T_{j+1}$, $S_j = S_{j+1}$. This proves the result. □

Corollary 11.3.3. *Let $x \in K_n$ be nondegenerate. Then there is a nondegenerate q-simplex y_q in* SdK *with qth vertex $(x; [n])$ if and only if $q \le n$.*

Proof. If $q \le n$, set $y_q = (x; [n-q], [n-q+1], \cdots, [n])$. Then y_q is in minimal form and nondegenerate, and its qth vertex is $(x; [n])$. Conversely, if we have a nondegenerate y_q with qth vertex $(x; [n])$, then, in minimal form, we must have $y_q = (x; S_0, \cdots, S_{q-1}, S_q)$ with S_i strictly contained in S_{i+1} for $0 \le i < n$ and $S_q = [n]$. Clearly that implies $q \le n$. □

Proof of Theorem 11.2.2. The nondegenerate q simplices of the barycentric subdivision SdK are the strictly increasing chains $\sigma_0 \subset \cdots \subset \sigma_q$ of faces of a simplex. If σ_q has cardinality $n+1$, its elements specify a nondegenerate n-simplex x of K^s. Viewing x as a map $\Delta[n] \longrightarrow K^s$ via Proposition 9.3.3, the inverse images of the σ_i specify an increasing sequence of subsets S_i of $[n]$ with $S_q = [n]$. The rest is left as a thought exercise about elements of $Sd^s(K^s)$ of minimal form. □

11.4. Subdivision and Properties A, B, and C of simplicial sets

Here is how subdivision relates to Properties A, B, and C.

Theorem 11.4.1. *Subdivision of simplicial sets has the following properties.*

(i) K has Property A if and only if SdK *has Property A.*
(ii) K has Property A if and only if SdK *has Property B.*
(iii) K has Property B if and only if SdK *has Property C.*

The following two corollaries are immediate.

Corollary 11.4.2. *If K does not have Property A, then* $\mathrm{Sd}^n K$ *does not have any of the three properties for any $n \ge 1$. If K does have property A, then* $\mathrm{Sd}^n K$ *has all three properties for all $n \ge 2$.*

Corollary 11.4.3. *K has Property A if and only if* $\mathrm{Sd}^2 K$ *has Property C, and then* $\mathrm{Sd}^2 K$ *also has Property B.*

Now the following very satisfactory theorem follows directly from Theorem 11.1.8.

Theorem 11.4.4. *A simplicial set K satisfies Property A if and only* $\mathrm{Sd}^2 K$ *is a simplicial complex.*

We might also ask whether our properties shed light on the question of whether or not a simplicial complex is the nerve of a category. We have the following complement to the previous result. It is an analogue of the fact that the subdivision of a simplicial complex is a poset. We will prove it later, in §12.6.

THEOREM 11.4.5. *A simplicial set satisfies Property A if and only if* SdK *is the nerve of a category, namely the category* ΠSdK.

The last clause is a consequence of the following general observation.

PROPOSITION 11.4.6. *If a simplicial set K is isomorphic to $N\mathscr{C}$ for some category $\mathscr{C}$, then the category $\mathscr{C}$ is isomorphic to ΠK.*

PROOF. If $K \cong N\mathscr{C}$, then $\Pi K \cong \Pi N\mathscr{C} \cong \mathscr{C}$. □

Since ordered simplicial complexes satisfy Property A when regarded as simplicial sets, Theorem 11.4.5 has the following result as a special case. It says that the subdivision of a simplicial complex is the nerve of a category. Remarkably, this appears to be a new result.

THEOREM 11.4.7. *If K is an ordered simplicial complex, then* $\mathrm{Sd}(K^s)$ *is isomorphic to* $N\Pi Sd(K^s)$.

11.5. The proof of Theorem 11.4.1

Since Property B implies Property A, by Theorem 11.1.5, the following two implications prove both (i) and (ii) of Theorem 11.4.1.

PROOF THAT IF SdK HAS PROPERTY A, THEN SO DOES K. Suppose for a contradiction that we have a nondegenerate $x \in K_n$ with a degenerate face $d_i x = s_j z$, where $z \in K_{n-2}$. Recall that $d_j s_j = \mathrm{id}$. In SdK, we have the 2-simplex[1]

$$(x; \delta_i\delta_j[n-2], \delta_i[n-1], [n]).$$

It is written in minimal form and is nondegenerate. Its last face is the 1-simplex

$$(x; \delta_i\delta_j[n-2], \delta_i[n-1]) \sim (d_i x; \delta_j[n-2], [n-1]) = (s_j z; \delta_j[n-2], [n-1]) \sim (z; [n-2], [n-2])$$

since $\sigma_j\delta_j = \mathrm{id}$ and $\sigma_j \colon [n-2] \longrightarrow [n-2]$ is a surjection. This simplex is in minimal form and degenerate, which contradicts the assumption that SdK has Property A. □

PROOF THAT IF K HAS PROPERTY A, THEN SdK HAS PROPERTY B. Consider a nondegenerate q-simplex $y = (x; S_0, \cdots, S_q)$ written in minimal form. For some n, $x \in K_n$ is nondegenerate and the S_i give a strictly increasing sequence of subsets of $[n]$, with $S_q = [n]$. The vertices of y are the $(x; S_i)$. Suppose that $(x; S_i) \sim (x; S_j)$ where $0 \le i < j \le q$. Let $\mu \colon [m_i] \longrightarrow [n]$ and $\nu \colon [m_j] \longrightarrow [n]$ be injective maps in Δ with images S_i and S_j, respectively. Then

$$(\mu^* x; [m_i]) \sim (x; S_i) \sim (x; S_j) \sim (\nu^* x; [m_j]).$$

Since K has Property A, the faces $\mu^* x$ and $\nu^* x$ are nondegenerate. Therefore, by the uniqueness of the minimal form, we must have $m_i = m_j$. Since $S_i \subset S_j$, this implies that $S_i = S_j$. The contradiction proves that SdK has Property B. □

[1] Here and below, we write $\alpha[n]$ to denote the set $\alpha([n])$.

Finally, the following two implications prove (iii) of Theorem 11.4.1.

Proof that if K has Property B, then SdK has Property C. Let

$$z_1 = (x; S_0, \cdots, S_q) \text{ and } z_2 = (y; T_0, \cdots, T_q)$$

be nondegenerate q-simplices of SdK that have the same set of $q+1$ distinct vertices. We must show that $z_1 = z_2$. We may assume without loss of generality that z_1 and z_2 are in minimal form, with $x \in K_m$, $S_q = [m]$, $y \in K_n$, and $T_q = [n]$ for some m and n. Let $m_i + 1 = |S_i|$ and $n_i + 1 = |T_i|$ and note that $m_0 < \cdots < m_q = m$ and $n_0 < \cdots < n_q = n$. Using Proposition 11.1.4, we see that the vertices of z_1 and z_2, in minimal form, are the $(S_i^* x; [m_i])$ and the $(T_i^* x; [n_i])$, respectively.

We are assuming that these two sets of vertices are the same. We claim that they are the same as *ordered* sets. That is, $(S_i^* x; [m_i]) = (T_i^* y; [n_i])$ for $0 \leq i \leq q$. Suppose not. Then $(S_i^* x; [m_i]) = (T_j^* y; [n_j])$ for some $i \neq j$, and we may assume $i < j$. Since these are both in minimal form, $m_i = n_j$. By the pigeonhole principle, we must have some $j' < j$ and $i' > i$ such that $m_{i'} = n_{j'}$. But then we have $m_i < m_{i'} = n_{j'} < n_j = m_i$, which is a contradiction.

Thus $m_i = n_i$ and $S_i^* x = T_i^* y$ for all i. Since $S_q = [m] = [n] = T_q$, we have $x = S_q^* x = T_q^* y = y$. Then, by Proposition 11.1.4 again, S_i and T_i must be defined by the same injection and so must be equal. Therefore $z_1 = z_2$ and SdK has Property C. □

Proof that if SdK has Property C, then K has Property B. Suppose that K does not have Property B. Let $x \in K_n$, $n > 0$, be nondegenerate with repeated vertices $\alpha^* x$ and $\beta^* x$ for injections $\alpha, \beta \colon [0] \longrightarrow [n]$. By the uniqueness of the minimal form, $(x; \alpha[0], [n])$ and $(x; \beta[0], [n])$ are distinct 1-simplices of SdK. However, these 1-simplices have the same vertex sets since one of the vertices of each is $(x; [n])$ and the other is

$$(x; \alpha[0]) \sim (\alpha^* x; [0]) = (\beta^* x; [0]) \sim (x; \beta[0]).$$

Thus SdK does not have Property C. □

Not worth a section?

11.6. Isomorphisms of subdivisions

We saw in **??** that if X and Y are posets, then the subdivisions of $X * Y$ and $(X * Y)^-$ are isomorphic, hence so are their associated simplicial sets. However, the posets $X * Y$ and $(X * Y)^-$ are not isomorphic, and neither are their associated simplicial sets. We round out the picture with the following rather strange looking result, which puts this example in a more general context.

Proposition 11.6.1. *If K and L are simplicial sets such that* SdK *and* SdL *are isomorphic, then although K and L need not be isomorphic, for each n there is a bijection of sets $f_n \colon K_n \cong L_n$ such that the faces of a simplex $x \in K_n$ correspond bijectively under f_{n-1} to the faces of $f(x)$.*

Proof. Let $g \colon \mathrm{Sd}K \longrightarrow \mathrm{Sd}L$ be an isomorphism of simplicial sets. For a nondegenerate n-simplex $x \in K_n$, we have the vertex $(x; [n])$ in SdK. Write $g(x; [n]) = (y; [m])$ in minimal form. Using Corollary 11.3.3, we see that $m = n$, and we define $f_n(x) = y$. If $x \in K_n$ is degenerate, there is a unique surjection σ and nondegenerate simplex z such that $x = \sigma^* z$. Define $f_n(x) = \sigma^* f(z)$. If we apply the same construction starting from $g^{-1} \colon \mathrm{Sd}L \longrightarrow \mathrm{Sd}K$, we obtain an inverse function f_n^{-1} to f_n. The $(n+1)$ faces $d_i x$ of a nondegenerate $x \in K_n$ correspond to the $(n+1)$ 1-simplices $y_i = (x; \delta_i[n-1], [n])$ of $\mathrm{Sd}K$, counted with multiplicities in case of repetitions. The vertices of y_i are $d_0 y_i = (x; \delta_i[n-1]) \sim (d_i x; [n-1])$ and $d_1 y_i = (x; [n])$ in minimal form. Since the nondegenerate faces of L admit a similar description, we see that these faces correspond under f_{n-1} to the faces of $f_n(x)$. The following example shows that K and L need not be isomorphic. $\square$

11.7. Regular simplicial sets and regular CW complexes

Property A of a simplicial set is an analogue of the classical notion of regularity for a CW complex X. The results of this section are peripheral to our main interests here, but they help contrast simplicial sets with CW complexes.

Incomplete section, see Piccinini? Or expository REU paper project

Definition 11.7.1. A CW complex X is *regular* if its closed cells are homeomorphisms onto their images so that each cell map $(D^n, S^{n-1}) \longrightarrow (e^n, \partial e^n)$ is a homeomorphism.

Definition 11.7.2. A nondegenerate simplex $x \in K_n$ is *regular* if the following diagram is a pushout, where $[x]$ denotes the subsimplicial set generated by x.

$$\begin{tikzcd}
\Delta[n-1] \arrow[r, "d_n x"] \arrow[d, "\delta^n"'] & {[d_n x]} \arrow[d] \\
\Delta[n] \arrow[r, "x"'] & {[x];}
\end{tikzcd}$$

K is *regular* if all of its nondegenerate simplices are regular.

Theorem 11.7.3. *For any K, SdK is regular.*

Theorem 11.7.4. *If K is a regular simplicial set, then $|K|$ is a regular CW complex.*

Theorem 11.7.5. *If X is a regular CW complex, then X is triangulable; that is X is homeomorphic to $|K^s|$ for some simplicial complex K.*

CHAPTER 12

Subdivision and Properties A, B, and C in $\mathscr{C}at$

12.1. Properties A, B, and C of categories

Categories are implicitly small unless they are obviously large, like the categories of spaces, simplicial sets, or (small) categories. We may interpret properties A, B, and C of the simplicial set $N\mathscr{C}$ as properties of a category $\mathscr{C}$.

Definition 12.1.1. A (small) category $\mathscr{C}$ has Property A, B, or C if the simplicial set $N\mathscr{C}$ has Property A, B, or C.

Theorem 12.1.2. *Let $\mathscr{C}$ be a category. The following statements hold.*

(i) $N\mathscr{C}$ has property A if and only if $\mathscr{C}$ has the no retracts property*, meaning that retractions are identity maps: if we have morphisms $i\colon a \longrightarrow b$ and $r\colon b \longrightarrow a$ in $\mathscr{C}$ such that $r \circ i = \mathrm{id}_a$, then $a = b$ and $i = r = \mathrm{id}$.*

(ii) $N\mathscr{C}$ has property B if and only if $\mathscr{C}$ has the no loops property*, meaning that loops are identity maps: if we have morphisms $f\colon a \longrightarrow b$ and $g\colon b \longrightarrow a$ in $\mathscr{C}$, then $a = b$ and $f = g = id$.*

(iii) $N\mathscr{C}$ has property C if and only if $\mathscr{C}$ has the one way property*: there is at most one sequence of nonidentity morphisms $f_i\colon C_i \longrightarrow C_{i+1}$ connecting any finite ordered set of objects $\{C_i\}$.*

(iv) $\mathscr{C}$ is a poset if and only if $N\mathscr{C}$ has properties B and C.

Proof. A nondegenerate n-simplex of $N\mathscr{C}$ is a composable sequence

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n$$

of nonidentity morphisms. It has a degenerate face if and only if one of the composites $f_{i+1} \circ f_i$ is an identity map. This proves (i).

For (ii), Property B says that the objects c_i of a nondegenerate n-simplex are distinct, which clearly implies the no loops property. Conversely, if $c_i = c_j$ for some $i < j$, the composite of f's from c_i to c_j is a loop $c_i \longrightarrow c_i$. We can write the composite as $g \circ f_i$, The no loops property implies that f_i and g are identity maps, so that our simplex is degenerate. This proves (ii)

Statement (iii) is immediate from the definition of Property C.

For (iv), it is immediate from (ii) and (iii) that $\mathscr{C}$ satisfies Properties A and B if and only if there is at most one morphism between any pair of objects of $\mathscr{C}$. That is precisely the characterization of posets regarded as categories. □

12.2. The definition of the subdivision of a category

Let $\mathscr{C}$ be a category. We start with a combinatorical definition of $\mathrm{Sd}\mathscr{C} = \mathrm{Sd}^c\mathscr{C}$. It may be hard to assimilate, but it is the right definition to start with. We will eventually see that Sd is actually nothing but the composite functor $\Pi\mathrm{Sd}^s N$, but that will require a fair amount of proof.

The intuition is that $\mathrm{Sd}\mathscr{C}$ has objects all chains of non-identity maps, and the set of morphisms from (f_i, n) to (g_i, m) is the set of all ways that (f_i, n) can be mapped injectively to a subchain of (g_i, m). These ways are to be distinct after accounting for degeneracies, which motivates the definition of the equivalence relation in the following definition.

To define $\mathrm{Sd}\mathscr{C}$ rigorously, we first define a category $\mathscr{D}\mathscr{C}$. The objects of $\mathscr{D}\mathscr{C}$ are the chains of composable arrows in $\mathscr{C}$. To abbreviate notation, we sometimes write $A = (f_i, m)$ as shorthand for a chain

$$a_0 \xrightarrow{f_1} a_1 \longrightarrow \cdots \longrightarrow a_{m-1} \xrightarrow{f_m} a_m.$$

We may think of such a chain as an m-simplex of $N\mathscr{C}$.

The morphisms from (f_i, m) to (g_i, n) are the equivalence classes of maps $\mu\colon [m] \longrightarrow [n]$ in Δ such that $\mu^*(g_i, n) = (f_i, m)$ in $N\mathscr{C}$. The equivalence relation is generated under composition by the following basic equivalences. For a surjective map $\sigma\colon [q] \longrightarrow [p]$ in Δ and for right inverses $\alpha, \beta\colon [p] \longrightarrow [q]$ to σ, so that $\sigma\alpha$ and $\sigma\beta$ are both the identity morphism of $[p]$, set $\alpha \sim \beta\colon (h_i, p) \longrightarrow \sigma^*(h_i, p)$ for any object (h_i, p). Thisz makes sense since $\alpha^*\sigma^* = \mathrm{id} = \beta^*\sigma^*$. Composition in $\mathscr{D}\mathscr{C}$ is induced by composition in Δ. Then define $\mathrm{Sd}\mathscr{C}$ to be the full subcategory of $\mathscr{D}\mathscr{C}$ whose objects are the non-degenerate chains. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{C}'$ induces a functor $NF\colon N\mathscr{C} \longrightarrow N\mathscr{C}'$, which in turn induces a functor $\mathrm{Sd}F\colon \mathrm{Sd}\mathscr{C} \longrightarrow \mathrm{Sd}\mathscr{C}'$. With these definitions, Sd is a functor $\mathscr{C}at \longrightarrow \mathscr{C}at$.

There is another way to view the definition, which may be easier to grasp. The letter $\mathscr{D}$ above is meant to indicate that we allow degenerate chains as objects of the category $\mathscr{D}\mathscr{C}$. We can instead start with the smaller category $\mathscr{C}\mathscr{C}$ whose objects (f_i, m) are the nondegenerate chains, so that no f_i is an identity map. The maps from (f_i, m) to (g_i, n) in $\mathscr{C}\mathscr{C}$ are the maps $\nu\colon [m] \longrightarrow [n]$ in Δ such that $\nu^*(g_i, n) = (f_i, m)$. Notice that such a map ν must be an injection since (f_i, m) is nondegenerate. Now define $\mathrm{Sd}\mathscr{C}$ to be the quotient category of $\mathscr{C}\mathscr{C}$ with the same objects but with equivalence classes of morphisms under the equivalence relation generated by setting $\nu\alpha \sim \nu\beta$ when

$$\nu^*(g_i, n) = (f_i, m) = \sigma^*(h_i, q)$$

for some surjection $\sigma\colon [m] \longrightarrow [q]$ with right inverses $\alpha, \beta\colon [q] \longrightarrow [m]$.

The difference is whether we choose to first restrict to nondegenerate simplices and then impose an equivalence relation or to first impose an equivalence relation and then restrict to nondegenerate simplices. We get the same category either way.

REMARK 12.2.1. It is useful to observe that if $\mathscr{C}$ has Property A, then no $\nu^*(g_i, n)$ can be degenerate and therefore $\mathscr{C}\mathscr{C} = \mathrm{Sd}\mathscr{C}$.

12.3. Subdivision and Properties A, B, and C of categories

Despite the analogy with simplicial sets, the conclusions here read rather differently.

THEOREM 12.3.1. *Subdivision of categories has the following properties.*

(i) For any category $\mathscr{C}$, $\mathrm{Sd}\mathscr{C}$ has Property B.
(ii) A category $\mathscr{C}$ has Property B if and only if $\mathrm{Sd}\mathscr{C}$ is a poset.

Again, the following remarkable theorem follows directly. Since this result applies to any category $\mathscr{C}$, it does not make sense to ask for a converse.

THEOREM 12.3.2. *For any category $\mathscr{C}$,* $\mathrm{Sd}^2\mathscr{C}$ *is a poset.*

EXAMPLE 12.3.3. The nerve of a poset need not be the subdivision of a simplicial set. The poset $\mathbb{Z}$ of integers with its usual ordering provides a counterexample. If $N\mathbb{Z} \cong SdK$ and 0 corresponds to $(x;[n])$ in minimal form, then for any nondegenerate q-simplex $(y;S_0,\cdots,S_q)$ in minimal form that has qth vertex $(x;[n])$, we have $S_q = [n]$ and thus $q \le n$. However, in $N\mathscr{C}$ there are nondegenerate simplices $(-r,-r+1,\cdots,0)$ for arbitrarily large r.

Since we have subdivision functors on both categories and simplicial sets, it is natural to ask how these functors relate to the adjoint pair (Π, N). The following result is either a theorem or a definition, depending on whether one chooses to start with the combinatorial or the conceptual definition of the subdivision of a category. We shall take it as a theorem and prove it in §12.5.

THEOREM 12.3.4. *For any category $\mathscr{C}$,* $\mathrm{Sd}^c\mathscr{C}$ *is isomorphic to* $\Pi\mathrm{Sd}^sN\mathscr{C}$.

This implies another characterization of categories having Property A.

COROLLARY 12.3.5. *A category $\mathscr{C}$ has Property A if and only if* $\mathrm{Sd}^sN\mathscr{C}$ *is isomorphic to* $N\mathrm{Sd}^c\mathscr{C}$.

PROOF. If $\mathscr{C}$ has Property A, then Theorem 11.4.5 implies that $Sd^sN\mathscr{C}$ is isomorphic to $N\Pi\mathrm{Sd}^sN\mathscr{C}$. By Theorem 12.3.4, the latter is isomorphic to $N\mathrm{Sd}^c\mathscr{C}$. For the converse, $N\mathrm{Sd}^c\mathscr{C}$ has Property B and therefore Property A by Theorems 12.3.1(i) and 11.4.1(ii). If $\mathrm{Sd}^sN\mathscr{C} \cong N\mathrm{Sd}^c\mathscr{C}$, then $\mathscr{C}$ has Property A by Theorem 11.4.1(i). $\square$

REMARK 12.3.6. For posets X, we obtain naturally isomorphic simplicial sets if we regard X as a category and take its nerve or if we regard X as the simplicial complex $\mathscr{K}X$ and take the associated simplicial set $(\mathscr{K}X)^s$. It is natural to ask whether NSd^cX is isomorphic to $\mathrm{Sd}^s(\mathscr{K}X)^s$. Since X satisfies Property A (and B and C), the previous result gives that

$$N\mathrm{Sd}^cX \cong \mathrm{Sd}^sNX \cong Sd^s(\mathscr{K}X)^s.$$

Remarkably, Theorem 12.3.4 also implies that the categorical analogue of Theorem 11.2.5 is a direct implication of that result.

THEOREM 12.3.7. *There is a natural functor* $\mathrm{Sd}^c\mathscr{C} \longrightarrow \mathscr{C}$ *that induces a homotopy equivalence on passage to classifying spaces.*

PROOF. We apply the natural map of simplicial sets of Theorem 11.2.5 and the fact that the composite ΠN is isomorphic to the identity functor to obtain ξ as the composite

$$\mathrm{Sd}^c\mathscr{C} \cong \Pi\mathrm{Sd}^sN\mathscr{C} \longrightarrow \Pi N\mathscr{C} \cong \mathscr{C}.$$

$\square$

12.4. The proof of Theorem 12.3.1

We have three implications to prove.

PROOF THAT $\mathrm{Sd}\mathscr{C}$ HAS PROPERTY B. We first prove that $\mathscr{C}\mathscr{C}$ has Property B. Let $A = (f_i, m)$ and $B = (g_i, n)$ be objects of $\mathscr{C}\mathscr{C}$ and suppose that we have morphisms $\mu\colon A \longrightarrow B$ and $\nu\colon B \longrightarrow A$. Since these morphisms are given by injections in Δ, $m = n$. Since the only injection $[n] \longrightarrow [n]$ is the identity map, we have $A = B$ and $\mu = \mathrm{id} = \nu$. Thus $\mathscr{C}\mathscr{C}$ has the no loops property, which is equivalent to Property B. This property is inherited by the quotient category $\mathrm{Sd}\mathscr{C}$. If we have maps $\overline{\mu}\colon A \longrightarrow B$ and $\overline{\nu}\colon B \longrightarrow A$ in $\mathrm{Sd}\mathscr{C}$, they must be represented by maps μ and ν in $\mathscr{C}\mathscr{C}$, but these maps are identity maps by what we have just shown, hence $\overline{\mu}$ and $\overline{\nu}$ are identity maps. □

Proof that if $\mathscr{C}$ has Property B, then $\mathrm{Sd}\mathscr{C}$ is a poset. Since Property B implies Property A, $\mathscr{C}\mathscr{C} = \mathrm{Sd}\mathscr{C}$ by Remark 12.2.1. We must show that $\mathscr{C}\mathscr{C}$ is a poset. Let A and B be objects of $\mathscr{C}\mathscr{C}$. We must show that there is at most one morphism between A and B. Suppose there is a morphism $\mu\colon A$ and B. Since we have just shown that $\mathscr{C}\mathscr{C}$ has the no loops property, there is no morphism $B \longrightarrow A$ unless $A = B$ and $\mu = \mathrm{id}$. Suppose there is another morphism $\nu\colon A$ and B. We must show that $\mu = \nu$. Since $A = \mu^* B = \nu^* B$, we have $a_i = b_{\mu(i)} = b_{\nu(i)}$ for all i, where the a_i and b_j are the objects appearing in the chains A and B. Since B must be nondegenerate when thought of as an element of $N\mathscr{C}$ and $\mathscr{C}$ has the no loops property, we have $b_i \neq b_j$ for $i \neq j$. Therefore $\mu(i) = \nu(i)$ for all i and $\mu = \nu$. □

Proof that if $\mathrm{Sd}\mathscr{C}$ is a poset, then $\mathscr{C}$ has Property B. Suppose that $\mathscr{C}$ does not have Property B. Then there are objects A and B (possibly the same) and non-identity maps $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$. Consider the objects $A \xrightarrow{f} B \xrightarrow{g} A$ and A in $\mathrm{Sd}\mathscr{C}$. Let $\alpha, \gamma\colon [0] \longrightarrow [2]$ be the maps with images $\{0\}$ and $\{2\}$, respectively. Then

$$\alpha^*(A \xrightarrow{f} B \xrightarrow{g} A) = A = \gamma^*(A \xrightarrow{f} B \xrightarrow{g} A).$$

Since no degeneracy operator on A is a face of $A \xrightarrow{f} B \xrightarrow{g} A$, we cannot have $\alpha \sim \gamma$; that is, they represent distinct morphisms of $\mathrm{Sd}\mathscr{C}$. But that contradicts the assumption that $\mathrm{Sd}\mathscr{C}$ is a poset. □

12.5. Relations among Sd^s, Sd^c, N, and Π

We are heading towards the proof of Theorem 12.3.4. We recall that ΠK has objects the vertices $x \in K$, morphisms generated by the 1-simplices $y \in K$, and relations dictated by the 2-simplices z. For a vertex x, $s_0 x$ is the identity map of x. For a 1-simplex y, $d_1 y$ is the source of y and $d_0 y$ is the target of y. For a 2-simplex z, $d_1 z = d_0 z \circ d_2 z$. The functor Π is left adjoint to N, and the counit of the adjunction is a natural isomorphism $\Pi N\mathscr{C} \cong \mathscr{C}$. We start work with the following understanding of the category $\Pi\mathrm{Sd}^s K$ for simplicial sets K.

Proposition 12.5.1. *Every morphism of the category $\Pi\mathrm{Sd}^s K$ can be represented by a 1-simplex in $\mathrm{Sd}^s K$, and the category $\Pi\mathrm{Sd}^s K$ has Property B.*

Proof. By definition, every morphism is a formal composite of 1-simplices, say $y_q \circ \cdots \circ y_1$. Since $y_{i+1} \circ y_i$ is defined, the target $d_0 y_i$ is equal to the source $d_1 y_{i+1}$. We will show that such a formal composite of length q is equivalent to a formal composite of length $q - 1$. By induction, it must be equivalent to a formal composite of length 1, which is just a 1-simplex.

Write y_i in minimal form $(x_i; S_i, [n_i])$, where $x_i \in K_{n_i}$ is nondegenerate. Let $|S_i| = m_i \leq n_i$ and let $\alpha_i\colon [m_i] \longrightarrow [n_i]$ be the injection with image S_i. Since

$$(x_q; S_q) = d_1(x_q; S_q, [n_q]) = d_0(x_{q-1}; S_{q-1}, [n_{q-1}]) = (x_{q-1}; [n_{q-1}]),$$

there must be some surjection $\sigma\colon [m_q] \longrightarrow [n_{q-1}]$ in Δ such that $\alpha_q^* x_q = \sigma^*[x_{q-1}]$. Let $\beta\colon [n_{q-1}] \longrightarrow [m_q]$ be a right inverse to σ. Then

$$(x_q; \alpha_q\beta[n_{q-1}], S_q) \sim (\sigma^* x_{q-1}; \beta[n_{q-1}], [m_q]) \sim (x_{q-1}; [n_{q-1}], [n_{q-1}]),$$

which is degenerate and thus an identity morphism in $\Pi\mathrm{Sd}K$. Consider the 2-simplex $z = (x_q; \alpha_q\beta[n_{q-1}], S_q, [n_q])$. The relation $d_1 z = d_0 z \circ d_2 z$ gives that

$$(x_q; \alpha_q\beta[n_{q-1}], [n_q]) = (x_q; S_q, [n_q]) = y_q$$

as morphisms in $\Pi\mathrm{Sd}K$. Now use that $\beta^*\sigma^* = \mathrm{id}$ on $[n_{q-1}]$ to see that

$$y_{q-1} = (x_{q-1}; S_{q-1}, [n_{q-1}]) \sim (x_q; \alpha_q\beta S_{q-1}, \alpha_q\beta[n_{q-1}]).$$

Finally, consider the 2-simplex $w = (x_q; \alpha_q\beta S_{q-1}, \alpha_q\beta[n_{q-1}], [n_q])$. The relation $d_1 w = d_0 w \circ d_2 w$ gives that $(x_q; \alpha_q\beta S_{q-1}, [n_q]) = y_q \circ y_{q-1}$ in $\Pi\mathrm{Sd}K$. This gives the claimed reduction from word length q to word length $q-1$.

To prove that $\Pi\mathrm{Sd}^s K$ has Property B, we must verify the no loop condition. Thus suppose that $f\colon (x; [m]) \longrightarrow (y; [n])$ and $g\colon (y; [n]) \longrightarrow (x; [m])$ are morphisms in $\Pi\mathrm{Sd}^s K$, where $x \in K_m$ and $y \in K_n$ are nondegenerate simplexes. We have just shown that f and g can be represented by 1-simplices. It suffices to show that both are degenerate, so that they are identity morphisms in $\Pi\mathrm{Sd}^s K$. We have

$$d_0 f = d_1 g = (y; [n]) \quad\text{and}\quad d_0 g = d_1 f = (x; [m]).$$

By the conditions on d_0, we can write $f = (y; T, [n])$ and $g = (x; S, [m])$ in minimal form. By the conditions on d_1, we then have $(y; T) \sim (x; [m])$ and $(x; S) \sim (y; [n])$. Choose injections $\alpha\colon [p] \longrightarrow [m]$ and $\beta\colon [q] \longrightarrow [n]$ with images S and T. We then have

$$(x; [m]) \sim (y; T) \sim (\beta^* y; [p]) \quad\text{and}\quad (y; [n]) \sim (x; S) \sim (\alpha^* x; [q]).$$

Write $\alpha^* x = \sigma^* u$ where $u \in K_j$ is nondegenerate and $\sigma\colon [q] \longrightarrow [j]$ is a surjection. Then

$$(y; [n]) \sim (\alpha^* x; [q]) = (\sigma^* u; [q]) \sim (u; [j]).$$

Since these are both in minimal form, $n = j \le q$. Similarly $m \le p$. Since α and β are injections, $n = q$, $m = p$, and α and β are identity maps. Thus $S = [m]$ and $T = [n]$, showing that f and g are degenerate. $\square$

Proof of Theorem 12.3.4. We shall prove that the categories $Sd^c\mathscr{C}$ and $\Pi Sd^s N\mathscr{C}$ are isomorphic by exhibiting inverse functors between these categories. Moreover, these inverse isomorphisms of categories will be natural in $\mathscr{C}$.

We first define $F\colon \mathrm{Sd}\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$ and its inverse G on objects. The objects $A = (f_i, m)$ of $\mathrm{Sd}\mathscr{C}$ are the nondegenerate simplices of $N\mathscr{C}$. The objects of $\Pi Sd^s N\mathscr{C}$ are the vertices of $Sd^s N\mathscr{C}$. We may write these in minimal form as $(A; [m])$, where A is an object of $Sd^c\mathscr{C}$. We define F and G on objects by

$$F(A) = (A; [m]) \quad\text{and}\quad G(A; [m]) = A.$$

Visibly, $FG = \mathrm{Id}$ and $GF = \mathrm{Id}$ on objects.

We next define F on morphisms and we first define it on the morphisms of $\mathscr{C}\mathscr{C}$, which has the same objects as $\mathrm{Sd}\mathscr{C}$. For objects $A = (f_i, m)$ and $B = (g_i, n)$, a morphism $\nu\colon A \longrightarrow B$ is an injection $\nu\colon [m] \longrightarrow [n]$ such that $\nu^* B = A$. We let $F(\nu)$ be the morphism of $\Pi Sd^s N\mathscr{C}$ represented by the 1-simplex $\overline{\nu} = (B; \nu[m], [n])$ of $Sd^s N\mathscr{C}$. It is straightforward and left to the reader to check that F is indeed a functor, respecting composition and identities.

To see that F induces a functor $Sd^c\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$, we must show that F respects the equivalence relation used to define morphisms in $Sd^c\mathscr{C}$ from morphisms in $\mathscr{C}\mathscr{C}$. Thus suppose that we have an injection $\nu\colon [m] \longrightarrow [n]$ and a surjection $\sigma\colon [m] \longrightarrow [q]$ such that $\nu^*B = A = \sigma^*C$ for some object C. Let $\alpha, \beta\colon [q] \longrightarrow [m]$ be right inverses to σ. Then $\nu\alpha \sim \nu\beta$ and we must show that $\overline{\nu\alpha} = \overline{\nu\beta}$ in $\Pi Sd^s N\mathscr{C}$. Observe first that

$$(B; \nu\alpha[q], \nu[q]) \sim (\sigma^*A; \alpha[q], [m]) \sim (A; [q], [q]) \sim (\sigma^*A; \beta[q], [m]) \sim (B; \nu\beta[q], \nu[q])$$

are degenerate 1-simplices of $SdN\mathscr{C}$. Therefore they are identity morphisms of $\Pi SdN\mathscr{C}$. We now use the definition of Π to see that

$$\overline{\nu\alpha} = (B; \nu\alpha[q], [n]) = (B; \nu\beta[q], [n]) = \overline{\nu\beta}$$

$\Pi \mathrm{Sd}^s N\mathscr{C}$. In fact, both are equivalent to $(B; \nu[m], [n])$, as we see by considering the relations of the form $d_1 z = d_0 z d_2 z$ induced by the 2-simplices

$$(B; \nu\alpha[q], \nu[m], [n]) \quad \text{and} \quad (B; \nu\beta[q], \nu[m], [n])$$

of $NSd^s\mathscr{C}$. Therefore F induces a well-defined functor $Sd^c\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$.

We next define $G\colon \Pi Sd^s N\mathscr{C} \longrightarrow Sd^c\mathscr{C}$ on morphisms. We claim that every morphism $(A; [m]) \longrightarrow (B; [n]$ in $\Pi Sd^s N\mathscr{C}$ is of the form $\overline{\nu}$, and we define $G(\overline{\nu}) = \nu$. Visibly this will ensure that $FG = \mathrm{Id}$ and $GF = \mathrm{Id}$ on morphisms. By Proposition 12.5.1, a morphism $(A; [m]) \longrightarrow (B; [n])$ in $\Pi Sd^s N\mathscr{C}$ can be represented by some 1-simplex $(D; S, [r])$ in $Sd^s N\mathscr{C}$. Inspection of source and target shows that we must have

$$d_1(D; S, [r]) = (D; S) \sim (A; [m]) \quad \text{and} \quad d_0(D; S, [r]) = (D; [r]) \sim (B; [n]).$$

By the uniqueness in minimal form $r = n$ and $D = B$. Then $(B; S) \sim (A; [m])$. Let S be the image of an injection $\nu\colon [p] \longrightarrow [n]$, and note that ν is uniquely determined by S. Then $(B; S) \sim (\nu^*B; [p])$. By the uniqueness in minimal form, $[p] = [m]$ and $\nu^*B = A$. Thus our morphism is given in minimal form by the 1-simplex $\overline{\nu} = (B; \nu[m], [n])$, where $\nu^*B = A$. We have effectively used the defining relations for $\Pi Sd^s N\mathscr{C}$ in the reduction to 1-simplices of Proposition 12.5.1, and G is well-defined.

We have not checked that G is actually a functor, but fortunately we don't have to. It is a familiar observation that a homomorphism of groups that is a bijection of sets is an isomorphism of groups. In our situation, the same argument works to prove that G preserves identity morphisms and respects composition. Indeed

$$G(\mathrm{id}_{(A;[m])}) = GF(\mathrm{id}_A) = \mathrm{id}_A$$

and, for composable morphisms $\overline{\mu}$ and $\overline{\nu}$ of $\Pi Sd^s N\mathscr{C}$,

$$G(\overline{\nu} \circ \overline{\mu}) = G(F(\nu) \circ F(\mu)) = GF(\nu \circ \mu) = \nu \circ \mu$$

and

$$G(\overline{\nu}) \circ G(\overline{\mu}) = GF(\nu) \circ GF(\mu) = \nu \circ \mu. \qquad \square$$

12.6. Horn-filling conditions and nerves of categories

There are special kinds of simplicial sets that appear ubiquitously and are central to the applications of simplicial sets to other areas of mathematics. They are closely related to our focus on the relationship between simplicial sets and categories, and understanding them leads to several equivalent characterizations of those simplicial sets which are the nerves of categories.

Define Λ^k_n to be the subsimplicial set of $\Delta[n]^s$ generated by the faces $d_i\iota_n$ for all $i \neq k$. The name horn comes from the picture that one sees after passage to geometric realization. The realization of $\Delta[n]^s$ is $\Delta[n]^t$, and the realization of Λ^k_n is the "horn" that one sees after removing one of the faces of the boundary $\partial\Delta[n]^t$. If one has a map f from the realization $T\Lambda^k_n$ to a space X, then one can extend the map to $T\Delta[n]^s = \Delta[n]^t$. In fact, the topological n-simplex retracts onto any of its horns, as one sees by pushing in along the missing face. Composing f with such a retraction extends f over the simplex. This leads to the following definition and example.[1]

Definition 12.6.1. A simplicial set K is a *Kan complex* if every map of simplicial sets $\Lambda^k_n \longrightarrow K$ extends to a map $\Delta[n]^s \longrightarrow K$. There is a concrete combinatorial way to rephrase the condition. For every set of n-simplices $x_i \in K_{n-1}$, $0 \leq i \leq n$ and $i \neq k$ that satisfy the necessary compatibility condition $d_i x_j = d_{j-1} x_i$ for $i < j$ with neither $i = k$ nor $j = k$, there must exist an n-simplex $x \in K_n$ such that $d_i x = x_i$ for $i \neq k$.

The equivalence of the two formulations is immediate from Proposition 9.3.3.

Proposition 12.6.2. *For every space X, the simplicial set SX is a Kan complex.*

One might ask whether the extensions in Definition 12.6.1 are unique. If they are, we say that K has the *unique horn filling property.* Looking at the definition of the faces of the nerve of a category, (9.4.5), we see that not all horns are created equal. We say that Λ^k_n is an *inner horn* if $0 < k < n$; the outer horns are those with $k = 0$ or $k = n$.

Looking at $N\mathscr{C}$ or at ΠK, one sees that the inner horns play a special role. If we have faces $d_0 z$ and $d_2 z$, their composite is $d_1 z$. In a category, if we are given morphisms f_0 and f_2 such that the source of f_2 is the target of f_0, they define a map $\Lambda^1_2 \longrightarrow N\mathscr{C}$, and the composable pair (f_0, f_2) gives a 2-simplex that extends the horn. This doesn't work if we are given f_0 and f_1 or f_1 and f_2, since we cannot compose those. We can use inverses to fill these outer horns when $\mathscr{C}$ is a groupoid. This leads to the following result whose meaning should I hope be clear. We leave some details of proof to the reader. For $1 \leq i \leq n$, let $\nu_i \colon [1] \longrightarrow [n]$ denote the injection with image $\{i-1, i\}$.

Theorem 12.6.3. *Let K be a simplicial set. The following conditions are equivalent.*

(i) K is isomorphic to the nerve of a category.
(ii) Every inner horn of K has a unique filler.
(iii) For any $n \geq 2$ and any n-tuple of simplices $x_i \in K_1$, $1 \leq i \leq n$, such that $d_0 x_{i-1} = d_1 x_i$ for $2 \leq i \leq n$, there is a unique $y \in K_n$ such that $\nu_i^* y = x_i$.

K is isomorphic to the nerve of a groupoid if and only if every horn of K, inner or outer, has a unique filler.

Sketch Proof. First suppose that $K \cong N\mathscr{C}$. We deduce (ii) and (iii). It helps to recall the formulas for the faces and degeneracies of $N\mathscr{C}$ as given in (9.4.5).

If we have an inner horn $\Lambda^k_n \longrightarrow K$ given by compatible $(n-1)$-simplices x_i for $i \neq k$, then we can reconstruct from these simplices a unique string $(f_1, \cdots, f_n)$

[1]These are so basic that they appear on pages 2 and 3 of my book [27].

of composable arrows, and they give a filler for the given inner horn. One way of seeing this is to look at the ordered string of $n-1$ 1-simplices obtained from x_0 and x_n by applying all iterated face operations. Applied to x_0, we obtain 1-simplices in order that we denote by f_i, $2 \le i \le n$. Applied to x_n, we obtain 1-simplices that we also denote by f_i, but now for $1 \le i \le n-1$. The duplicate f_i for $2 \le i \le n-1$ are equal by the assumed compatibility condition, and the required y is the n-simplex $(f_1, \cdots, f_n)$. If we have simplices $x_i \in K_1$ as in (iii), they are a string of composable morphisms $(f_1, \cdots, f_n)$, and that string is the required simplex y.

If $\mathscr{C}$ is a groupoid, we can use inverses to modify the proof of (ii) so that it applies to outer as well as inner horns.

Conversely, assume (ii) or (iii). We claim that either suffices to prove that the unit $\eta\colon K \longrightarrow N\Pi K$ of the (N,Π)-adjunction is an isomorphism. The meaning is that the formal words of length n in the 1-simplices that appear in the definition of ΠK are all realized uniquely by simplices in K_n. We show that η is an isomorphism on n-simplices for all n by induction on n. The induction starts with $n = 0$ and $n = 1$, where there is nothing to prove. Assume that η is an isomorphism on $(n-1)$-simplices. Let y be an n-simplex of $N\Pi K$. Its faces give inner horns Λ^k_n in K, and they also give the data of (iii). With either hypothesis, a filler gives an n-simplex x of K such that y and $\eta(x)$ have the same faces. This means $\eta(x)$ is the same composite of 1-simplices as y, so that $\eta(x) = y$. If also $\eta(x') = y$, then x and x' have the same faces and so are equal by the uniqueness assumed in (ii) or (iii).

If we have fillers for all horns, then $K \cong N\Pi K$ and the fillers for the outer horns defined on Λ^0_2 and Λ^2_2 give left and right inverses for all morphisms. Just as for groups, the left and right inverses must be equal, and $N\Pi K$ must be a groupoid. $\square$

We use this characterization to prove Theorem 11.4.5.

Proof of Theorem 11.4.5. Suppose that K has Property A. We show that SdK satisfies condition (iii) of Theorem 12.6.3. Thus let $(x_i; S_i, [q_i])$, $1 \le i \le n$, be 1-simplices of SdK in minimal form such that

$$d_0(x_{i-1}; S_{i-1}, [q_{i-1}]) = d_1(x_i; S_i, [q_i])$$

for $2 \le i \le n$. Choose an injection $\alpha_i\colon [p_i] \longrightarrow [q_i]$ with image S_i for $0 \le i \le n$. Note that $p_1 = q_0$, where $q_0 = |S_0|$. The compatibility condition is equivalent to

$$(x_{i-1}, [q_{i-1}]) \sim (x_i; S_i) \sim (\alpha_i^* x_i; [p_i])$$

for $2 \le i \le n$. Since K has Property A, the faces $\alpha_i^* x_i$ are nondegenerate. By the uniqueness in minimal form, $q_{i-1} = p_i$ and $x_{i-1} = \alpha_i^* x_i$ for $2 \le i \le n$. Letting $x_0 = \alpha_1^* x_1$, this still holds for $i = 1$. The composite $\alpha_n \cdots \alpha_1\colon [p_1] \longrightarrow [q_n]$ is defined. Let

$$y = (x_n; \alpha_n \cdots \alpha_1[p_1], \alpha_n \cdots \alpha_2[p_2], \cdots, \alpha_n[p_n], [q_n]).$$

Then $\nu_n y = (x_n; S_n, [q_n])$ and, for $1 \le i < n$,

$$\nu_i^* y = (x_n; \alpha_n \cdots \alpha_i[p_i], \alpha_n \cdots \alpha_i[p_{i+1}]) \sim (x_i; S_i, [q_i])$$

For the uniqueness, suppose that we have another extension $z = (w; T_0, \cdots, T_n)$ in minimal form such that $\nu_i z = (x_i; S_i, [q_i])$ for $1 \le i \le n$. The nth vertex $(w; T_n)$ of z must be $(x_n; [q_n])$, so that $(w; T_n) \sim (x_n; [q_n])$. Since K satisfies Property A and w is nondegenerate, it follows from the uniqueness in minimal form that

$w = x_n$ and $T_n = [q_n]$. Similarly, for $0 \leq i < n$, the ith vertex of z must be the ith vertex of y, hence

$$(x_n; T_i) \sim (x_n; \alpha_n \cdots \alpha_{i+1}[p_{i+1}]).$$

Therefore T_i must be $\alpha_n \cdots \alpha_{i+1}[p_{i+1}]$ and $z = y$.

We shall prove a strengthened form of the converse statement in Proposition 12.7.3 below. $\square$

Remark 12.6.4 (Categorical remark). The functor Sd is a left adjoint. Its right adjoint is denoted Ex. Iterating it leads to an endofunctor Ex^∞ on $s\mathscr{S}et$ that assigns a Kan complex $Ex^\infty K$ to a simplicial set K. The composite ST is another such functor. They fit into a more sophisticated context of Quillen model category theory. One recent reference is [**31**, 17.5].

12.7. Quasicategories, subdivision, and posets

Looking at the definition of Kan complexes and the characterization of nerves of categories, one sees that they have a natural common generalization.

Definition 12.7.1. A simplicial set is a *quasicategory* if and only if every inner horn has a filler, not necessarily unique.

The idea is that compositions are defined, but they need not be unique. This is a very fashionable notion, and in much current literature the rather grandiose terms "∞-category" or "$(\infty, 1)$-category" are used for quasicategories. To go with this, the term "∞-groupoid" is then often used for Kan complexes. There is even some motivation for the terminology. In view of their importance, it seems reasonable to ask how these concepts behave with respect to subdivision and our Properties A, B, and C.

Proposition 12.7.2. *If* $\mathrm{Sd}K$ *is a Kan complex, then* K *is discrete, meaning that it has no nondegenerate simplices other than vertices.*

Proof. Suppose that K has a nondegenerate n-simplex, where $n > 0$. Let v be a vertex of x and let $\alpha \colon [0] \longrightarrow [n]$ be an injection such that $\alpha^* x = v$. Define an outer horn $\Lambda^2_2 \longrightarrow SdK$ by sending the vertices $0, 1, 2$ to the vertices $(x; [n])$, $(v; [0])$, $(x; [n])$ of SdK and sending the 1-simplices $(1, 2)$ and $(0, 2)$ to $(x; \alpha[0], [n])$ and $(x; [n], [n])$. Since $v \in K_0$, there is clearly no 1-simplex $(y; S, [m])$ with vertices $(x; [n])$ and $(v; [0])$, so $\mathrm{Sd}K$ cannot be a Kan complex. $\square$

Proposition 12.7.3. *If* SdK *is a quasicategory, then* K *satisfies Property* A.

Proof. Assume that K does not satisfy Property A. We construct an inner horn $f \colon \Lambda^2_3 \longrightarrow SdK$ that cannot be extended to a map $\Delta[3] \longrightarrow K$, thus showing that SdK cannot be a quasicategory. Since Property A fails for K, we can choose a nondegenerate simplex $x \in K_n$, an injection $\alpha \colon [m] \longrightarrow [n]$, and a surjection $\sigma \colon [m] \longrightarrow [p]$, $m > p$, such that $\alpha^* x = \sigma^* y$ in K_m for some nondegenerate simplex $y \in K_p$. Choose a right inverse $\beta \colon [p] \longrightarrow [m]$ to σ. The three 2-faces of $\Lambda^2_3 \subset \Delta[3]$ are $d_0\iota_3$, $d_1\iota_3$, $d_3\iota_3$, where ι_3 is the identity simplex that generates $\Delta[3]$. We specify f on these three 2-simplices by sending them to

$$(x; \alpha\beta[k], \alpha[m], [n]), \quad (x; \alpha[m], \alpha[m], [n]), \quad (y; [p], [p], [p])$$

respectively. It is a straightforward to check that they satisfy the required consistency on 1-faces of the horn. However, f cannot be extended to the last 2-face $d_2\iota_3$.

Any possible image would have a minimal form $(x; S, T, [n])$. For consistency with the prescribed faces, we would have

$$(x; S, [n]) \sim (x; \alpha[m], [n]) \text{ and } (x; T, [n]) \sim (x; \alpha\beta[p], [n]).$$

By the uniqueness of the minimal form, $S = \alpha[m]$ and $T = \alpha\beta[p]$. Thus, since $p < m$, T is a proper subset of S. Since $S \subset T$ by definition, $S = T$. This contradicts the choice of β as a non-identity injection. □

REMARK 12.7.4. There is a curious analogue for quasicategories of the result that a simplicial set is a simplicial complex if and only it satisfies Properties B and C. If K is the nerve of a poset, then it satisfies Properties B and C by Theorem 12.3.1, and of course it is a category and thus a quasicategory. It is reasonable to ask whether a quasicategory K that satisfies Properties B and C is a poset. By Theorem 11.1.8, K is the simplicial set associated to a simplicial complex, and we now write K for the latter. The set of vertices of K is a poset, and its order restricts to a total order on each simplex, so that we can write simplices in the form $\{x_0 < \cdots < x_n\}$ for vertices x_i. Then K is isomorphic to the nerve of the poset K_0 if and only if every finite totally ordered set $\{x_0 < \cdots < x_n\}$ is a simplex.

The example of $\partial\Delta[1]^s$ shows that for two vertices $x_0 < x_1$, $\{x_0 < x_1\}$ need not be a simplex of K. However, suppose that all such sets $\{x_0 < x_1\}$ are 1-simplices. Then K is a poset. To see this assume by induction that all totally ordered subsets of K_0 with at most n elements are simplices. Suppose for a contradiction that $\{x_0 < \cdots < x_n\}$ is totally ordered but not a simplex. Since all faces of this missing simplex are simplices, it is easy to construct an inner horn $f\colon \Lambda^k_n \longrightarrow K$, in fact one for each $0 < k < n$, from all but one of the faces. A filler is an n-simplex of K, hence a totally ordered set $\{y_0, \ldots, y_n\}$; it must be totally ordered since otherwise it would have degenerate faces, which it clearly does not have; that its vertices must be the x_i follows from the fact that the map $\Delta[n] \longrightarrow K$ determined by $\{y_0, \ldots, y_n\}$ extends f, and f maps onto the vertices.

We also remark that Properties B and C clearly fail to imply that K is a quasicategory. The inner horn Λ^1_2 is a simplicial complex, and its identity map does not extend to a simplex $\Delta[2] \longrightarrow \Lambda^1_2$.

CHAPTER 13

An outline summary of point set topology

We have implicitly given a quick outline of a bare bones introduction to point set topology in Chapter 1. The focus was on basic concepts and definitions rather than on the usual examples that give substance to the subject. We thought the reader might like to see a brief summary of some of the most basic parts of point-set topology that were not discussed in Chapter I, including but not limited to those results we that we have used in our exposition.

13.1. Metric spaces

The intuition for and the most important examples in point-set topology come from metric spaces, where the topology is defined in terms of a distance function.

Definition 13.1.1. A metric d on a set X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

The basis $\mathscr{B}$ determined by a metric d consists of the sets $B(x, r) = \{y | d(x, y) < r\}$. The topology generated by $\mathscr{B}$ is called the metric topology on X determined by d. A topological space X is *metrizable* if its topology is determined by a metric.

A subset A of a metric space X has an induced metric, and the metric and subspace topologies coincide. Any metric space is Hausdorff.

Of course, $\mathbb{R}^n$ has the standard metric

$$d(x, y) = (\sum (y_i - x_i)^2)^{1/2}.$$

The metric topology that it determines coincides with the product topology. The product of countably many copies of $\mathbb{R}$ is metrizable, but the product of uncountably many copies of $\mathbb{R}$ is not. There is a metric topology on any product of copies of $\mathbb{R}$, called the uniform topology, but it is finer than yjr product topology when the product is infinite.

For metric spaces, Lemma 1.5.8 leads to the familiar ε, δ formulation of continuity.

Lemma 13.1.2. *A function $f : X \longrightarrow Y$ between metric spaces is continuous if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(B(x, \delta)) \subset B(f(x), \varepsilon);$$

that is, if the distance from x to y is less than δ, then the distance from $f(x)$ to $f(y)$ is less than ε.

Moreover, we can characterize continuity in terms of convergent sequences.

Definition 13.1.3. A sequence $\{x_n\}$ of points in a space X *converges* to a point x if every neighborhood of x contains all but finitely many of the x_n. We then write $\{x_n\} \to x$. If X is Hausdorff, then the limit of $\{x_n\}$ is unique if it exists.

Observe that if $\{x_n\} \subset A$ and $\{x_n\} \to x$, then $x \in \bar{A}$. The converse does not hold for general topological spaces, but it does hold for metric spaces. Actually, what is relevant is not the metric but something it implies.

defined and used earlier

Definition 13.1.4. A space X is *first countable* if for each $x \in X$, there is a countable set of neighborhoods U_n of x such that any neighborhood of x contains at least one of the U_n; X is *second countable* if its topology has a countable basis.

Using the neighborhoods $B(x, 1/n)$, we see that a metric space is first countable.

Lemma 13.1.5. *Let X be first countable. Then $x \in \bar{A}$ if and only if there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$.*

Using Lemma 1.5.2 this leads to the promised characterization of continuity.

Proposition 13.1.6. *Let $f : X \longrightarrow Y$ be a function, where X is first countable and Y is any space. Then f is continuous if and only for every convergent sequence $\{x_n\} \to x$ in X, $\{f(x_n)\} \to f(x)$ in Y.*

13.2. Compact and locally compact spaces

Definition 13.2.1. A space X is *compact* if every open cover contains a finite subcover. That is, if X is the union of open sets U_i, then there are finitely many indices i_j, such that X is the union of the U_{i_j}.

Using standard facts about complements, one can reformulate the notion of compactness as follows. Say that a set of subsets of X has the finite intersection property if any finite subset has nonempty intersection.

defined and used earlier

Proposition 13.2.2. *A space X is compact if and only if any set of closed subsets of X with the finite intersection property has nonempty intersection. In particular, if X is compact and if $\{C_n\}$ is a nested sequence of closed subsets of X, $C_n \supset C_{n+1}$, then $\cap C_n$ is nonempty.*

A metric space X is *bounded* if $d(x, y) \leq D$ for some fixed D and all $x, y \in X$; the least such D is called the *diameter* of X. Boundedness is not a "topological" property, since it depends on the choice of metric: different metrics can define the same topology but have very different bounded subsets. With the standard Euclidean metric, we have the following result.

Theorem 13.2.3 (Heine-Borel). *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

In general, we have the following observations about the compactness of subspaces. For a subset A of a space X, a cover of A in X is a set of subsets of X whose union contains A.

Proposition 13.2.4. *Let A be a subspace of a space X. Then A is compact if and only if every cover of A in X has a finite subcover. If X is compact, then every closed subspace of X is compact.*

For compact Hausdorff spaces, the second statement has a converse.

PROPOSITION 13.2.5. *Every compact subspace of a compact Hausdorff space is closed.*

PROPOSITION 13.2.6. *If $f : X \longrightarrow Y$ is a continuous function and X is compact, then the image of f is a compact subspace of Y. In particular, any quotient space of a compact space is compact.*

THEOREM 13.2.7. *Let X be compact and Y be Hausdorff. Then a continuous bijection $f : X \longrightarrow Y$ is a homeomorphism (hence X is Hausdorff and Y is compact).*

PROOF. If C is closed in X, then C is compact, hence $f(C)$ is compact, hence $f(C)$ is closed in Y. This proves that f^{-1} is continuous. □

The results above give the behavior of compactness with respect to subspaces and quotient spaces. The behavior with respect to products is deeper than anything that we have stated so far.

THEOREM 13.2.8 (Tychonoff). *Any product of compact spaces is compact.*

The case of finite products is not difficult, but the general case is.

For metric spaces, compactness can be characterized in terms of limit points and convergent sequences.

THEOREM 13.2.9. *Consider the following conditions on a space X.*

(i) X is compact.
(ii) Every infinite subset of X has a limit point.
(iii) Every sequence in X has a convergent subsequence.

In general, $(i) \Rightarrow (ii) \Rightarrow (iii)$. If X is a metric space, the three conditions are equivalent.

We say that X is *sequentially compact* if it satisfies (iii). The following important fact is used in proving that $(iii) \Rightarrow (i)$ when X is a metric space.

LEMMA 13.2.10 (Lebesque Lemma). *Let $\mathscr{O}$ be an open cover of a sequentially compact metric space X. Then there is a $\delta > 0$ such that if $A \subset X$ is bounded with diameter less than δ, then A is contained in some $U \in \mathscr{O}$.*

PROOF. If not, then for each n we can choose a subset A_n of diameter less than $1/n$ which is not contained in any $U \in \mathscr{O}$. Choose a point $x_n \in A_n$ for each n. Suppose that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ that converges to some x. Certainly $x \in O$ for some $U \in \mathscr{O}$. For small enough ε and large enough n_i, $B(x, 2\varepsilon) \subset U$, $d(x, x_{n_i}) < \varepsilon$ and $1/n_i < \varepsilon$. It follows easily that $A_{n_i} \subset U$, which is a contradiction. □

DEFINITION 13.2.11. A space X is locally compact if each point of X has a neighborhood that is contained in a compact subspace of X.

Clearly $\mathbb{R}^n$ is locally compact but not compact.

LEMMA 13.2.12. *Let X be a Hausdorff space. Then X is locally compact if and only if for any point x and any neighborhood U of x, there is a smaller neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.*

This criterion is needed to prove the second part of the following result.

LEMMA 13.2.13. *Let A be a subspace of a locally compact subspace X. If A is closed or if A is open and X is Hausdorff, then A is locally compact.*

Locally compact Hausdorff spaces admit a canonical compactification, as we now make precise.

DEFINITION 13.2.14. A *compactification* of a space X is an inclusion of X as a dense subspace in a compact Hausdorff space Y. Observe that a compactification of a compact Hausdorff space must be a homeomorphism. Two compactifications Y and Y' are equivalent if there is a homeomorphism $Y \longrightarrow Y'$ which restricts to the identity map on X.

Compactifications are of fundamental importance in topology and algebraic geometry. The most naive example is the one-point compactification. The construction applies to any Hausdorff space, but it only gives a Hausdorff space when X is locally compact.

CONSTRUCTION 13.2.15. Let X be a Hausdorff space and let Y be the union of X and a disjoint point denoted ∞. Then Y is a topological space whose open sets are the open sets in X together with the complements of the compact sets in X. The space Y is called the *one point compactification of* X.

If X is itself compact, then $\{\infty\}$ is open and closed in Y and Y is the union of its components X and $\{\infty\}$.

PROPOSITION 13.2.16. *If X is a locally compact Hausdorff space that is not compact, then the one point compactification Y of X is in fact a compactification: Y is compact Hausdorff and X is a dense subspace.*

Since X is itself one of the open sets in Y, Lemma 13.2.13 gives the following implication.

COROLLARY 13.2.17. *A space X is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.*

13.3. Further separation properties

We have defined T_0, T_1 spaces and T_2, or Hausdorff spaces. We give three analogous definitions, and we describe various implications relating these separation properties to each other and to local compactness.

DEFINITION 13.3.1. Let X be a T_1 space (points are closed), let $x \in X$, and let A and B be closed subsets of X.

(i) X is *regular* if whenever $x \notin A$, there are open subsets U and V such that $x \in U$ and $A \subset V$.
(ii) X is *completely regular* if whenever $x \notin A$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for $a \in A$.
(iii) X is *normal* if whenever $A \cap B = \emptyset$, there are open subsets U and V such that $A \subset U$ and $B \subset V$.

Together with Lemma 13.2.12, the following result makes clear that these separation properties are closely related to local compactness.

LEMMA 13.3.2. *Let X be a T_1 space.*

(i) X is regular if and only if for any point x and any neighborhood U of x, there is a smaller neighborhood V of x such that $\bar{V} \subset U$.
(ii) X is normal if and only if for any closed set A contained in an open set U, there is an open set V such that $A \subset V$ and $\bar{V} \subset U$.

Language varies. The terms regular, completely regular, and normal are often defined without assuming that X is T_1. Then what we call regular and normal spaces are called T_3 and T_4 spaces and what we call completely regular spaces are called Tychonoff spaces. (As already noted, the T_i notation goes back to a 1935 paper of Alexandroff and Hopf [**2**], but some later references confuse things further by forgetting history and using T_i differently).

LEMMA 13.3.3. *The following implications hold: A normal space is completely regular. A completely regular space is regular. A regular space is Hausdorff.*

$$\textit{normal} \Rightarrow \textit{completely regular} \Rightarrow \textit{regular} \Rightarrow \textit{Hausdorff}$$

The implications normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff are obvious. The implication normal $\Rightarrow$ completely regular is a consequence of the following important result.

THEOREM 13.3.4 (Urysson's lemma). *If X is normal and A and B are disjoint closed subsets of X, then there is a continuous function $f : X \longrightarrow I$ such that $f(a) = 0$ if $a \in A$ and $f(b) = 1$ if $b \in B$.*

The proof is non-trivial, and the closely analogous assertion that regular implies completely regular is false. Urysson's lemma can be used to prove the following equally important result.

THEOREM 13.3.5 (Tietze extension theorem). *If A is a closed subspace of a normal space X and $f : A \longrightarrow I$ is a continuous function, then f can be extended to a continuous function $X \longrightarrow I$.*

Normality is the most desirable separation property, but it is much less nicely behaved than our other separation properties.

PROPOSITION 13.3.6. *A subspace of a Hausdorff, regular, or completely regular space is again Hausdorff, regular, or completely regular. A product of Hausdorff, regular, or completely regular spaces is again Hausdorff, regular, or completely regular. Neither of these assertions is true in general for normal spaces.*

For example, the product of uncountably many copies of $\mathbb{R}$ is not normal. Since $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$ and Tychonoff's theorem implies that the product of uncountably many copies of I is compact Hausdorff, this example also shows that a subspace of a normal space need not be normal. Nevertheless, most spaces of interest are normal.

THEOREM 13.3.7. *If X is metrizable or compact Hausdorff, then X is normal.*

Some indication of the importance of complete regularity is given by the following sequence of results, the second of which should be compared with Corollary 13.2.17.

THEOREM 13.3.8. *If X is completely regular, then it can be embedded as a subspace of a product of copies of the unit interval.*

COROLLARY 13.3.9. *The following conditions on a space X are equivalent.*

(i) X is completely regular.
(ii) X is homeomorphic to a subspace of a compact Hausdorff space.
(iii) X is homeomorphic to a subspace of a normal space.

COROLLARY 13.3.10. *A space X admits a compactification if and only if it is completely regular.*

PROOF. If Y is a compactification of X, then X is a subspace of the compact Hausdorff space Y and is thus completely regular. Conversely, if X is completely regular and thus homeomorphic to a subspace of some compact Hausdorff space Z, then the closure of the image of X in Z is a compactification of X, called the compactification induced by the inclusion of X in Z. □

The very definition of complete regularity leads to a canonical compactification.

CONSTRUCTION 13.3.11. Let X be completely regular. Let $F = F(X)$ be the set of all continuous functions $f : X \longrightarrow I$, let $Z = Z(X)$ be the product of copies of I indexed on the set F, and let $i : X \longrightarrow Z$ be the map whose fth coordinate is the map f. Then i is an inclusion. The induced compactification is denoted βX and called the *Stone-Čech compactification* of X.

The Stone-Čech compactification is characterized as the unique compactification (up to equivalence) that satisfies the following "universal property".

PROPOSITION 13.3.12. *Let X be a completely regular space. A map $f : X \longrightarrow Y$, where Y is a compact Hausdorff space, extends uniquely to a map $\tilde{f} : \beta X \longrightarrow Y$.*

PROOF. Uniqueness holds by Lemma 1.5.3. When $Y = I$, the existence is immediate from the construction: f is one of the coordinate maps, and the projection from $Z(X)$ to this coordinate restricts to $\tilde{f} : \beta X \longrightarrow I$. In general, Y is homeomorphic to $\beta Y \subset Z(Y)$. The map $f_g : X \xrightarrow{f} Y \cong \beta Y \subset Z(Y) \xrightarrow{\pi_g} I$ obtained from the gth coordinate projection π_g, $g \in Z(Y)$, extends to a map $\tilde{f}_g : \beta X \longrightarrow I$, and $\tilde{f}_g$ is the gth coordinate of a map $\beta X \longrightarrow Z(Y)$. This map sends X into the closed set βY, hence it sends the closure βX into $\beta Y \cong Y$, giving $\tilde{f}$. □

13.4. Metrization theorems and paracompactness

Since we are much more comfortable with metric spaces than with general spaces, it is important to be able to recognize when the topology on a given space is that induced by some metric. The simplest criterion is the following. Metrization theorems are proven by embedding a given space as a subspace of a space that is known to be metrizable. Let I^ω denote the product of countably many copies of I. It is a metric space, which would be false for an uncountable product.

THEOREM 13.4.1 (Uryssohn metrization theorem). *The following conditions on a T_1 space X are equivalent.*

(1) X is regular and second countable.
(2) X is homeomorphic to a subspace of I^ω.
(3) X is metrizable and has a countable dense subset.

Remember that second countable means that there is a countable basis for the topology. This ensures the following analogue of compactness.

Lemma 13.4.2. *If X is second countable, then any open cover of X has a countable subcover and X has a countable dense subset.*

Second countability is a strong condition, and a weaker countability condition, plus regularity, is necessary and sufficient for metrizability.

Definition 13.4.3. A set $\mathscr{V}$ of subsets of X is *locally finite* if each $x \in X$ has a neighborhood that intersects at most finitely many subsets of $\mathscr{V}$. A cover $\mathscr{O}$ of X is *σ-locally finite* if it is the union of countably many locally finite subsets.

Theorem 13.4.4 (Nagata-Smirnov metrization theorem). *A space is metrizable if and only if it is regular and has a σ-locally finite basis.*

The "σ" here is essential: if a Hausdorff space has a locally finite cover, then it is discrete.

There is another characterization of metrizability that is perhaps more intuitive.

Definition 13.4.5. A space X is locally metrizable if every point $x \in X$ has a neighborhood U such that U (with its subspace topology) is metrizable.

Clearly any metric space is locally metrizable. There is a property, called paracompactness, that is very often used to patch local conditions to obtain a global condition, and Stone proved that any metric space is paracompact.

Theorem 13.4.6 (Smirnov metrization theorem). *A space is metrizable if and only if it is paracompact and locally metrizable.*

We explain paracompactness. A *refinement* of a cover $\mathscr{O}$ of X is a collection of subspaces each of which is contained in at least one of the spaces in $\mathscr{O}$.

Definition 13.4.7. A space X is paracompact if every open cover of X has a locally finite refinement that is again an open cover of X.

Clearly a compact Hausdorff space is paracompact. The following sharpening of part of Theorem 13.3.7 holds.

Theorem 13.4.8. *A paracompact space X is normal.*

Like normality, paracompactness is not preserved by standard constructions. For this reason, Stone's theorem that metrizable $\Rightarrow$ paracompact seems more useful than the converse implication of Smirnov's metrization theorem.

Proposition 13.4.9. *A closed subspace of a paracompact space is paracompact. In general, subspaces of paracompact spaces and products of paracompact spaces need not be paracompact.*

The point of paracompactness is that it ensures the existence of particularly convenient open covers. This is very important in the theory of fiber bundles in algebraic topology.

Definition 13.4.10. An open cover $\mathscr{O}$ of X is *numerable* if it is locally finite and for each $U \in \mathscr{O}$ there is a continuous function $\phi_U : X \longrightarrow I$ such that $\phi_U(x) > 0$ only if $x \in U$. A numerable cover $\mathscr{U}$ is a *partition of unity* if $\sum_U \phi_U(x) = 1$ for each $x \in X$.

Given a numerable cover $\mathscr{O}$, we can define $\phi(x) = \sum_U \phi_U(x)$ and $\psi_U(x) = \phi_U(x)/\phi(x)$, thereby obtaining a partition of unity.

PROPOSITION 13.4.11. *If X is paracompact, then any open cover of X has a numerable refinement.*

DEFINITION 13.4.12. An *n-manifold M* is a second countable Hausdorff space each point of which has a neighborhood homeomorphic to $\mathbb{R}^n$.

By the Uryssohn metrization theorem, an n-manifold is metrizable. By Stone's theorem, it is therefore paracompact. The following theorem can be proven by use of a numerable cover of M.

THEOREM 13.4.13. *Any n-manifold M can be embedded as a subspace of $\mathbb{R}^N$ for a sufficiently large N.*

Bibliography

[1] P.S. Alexandroff. Diskrete Räume. MatematiceskiiSbornik (N.S.) 2(1937), 501–518.

[2] P.S. Alexandroff and H. Hopf. Topologie I. Berlin, 1935.

[3] F.G. Arenas. Alexandroff spaces. Acta Math. Univ. Comenianae 68(1999), 17–25.

[4] J. Anusiak and K.P., Shum. Remarks on finite topological spaces. Colloquium Mathematicum 23(1971), 217–223.

[5] J.A. Barmak. Algebraic topology of finite spaces and applications. Lecture Notes in Mathematics 2032. Springer 2011.

[6] J.A. Barmak and E.G. Minian. Minimal finite models. J. Homotopy Relat. Struct. 2(2007), 127–140.

[7] J.A. Barmak and E.G. Minian. Simple homotopy types and finite spaces. Adv. Math. 218(2008), 87–104.

[8] R. Bumby, R. Fisher, H. Levinson and R. Silverman. Topologies on Finite Sets. Proc. 9th S-E Conf. Combinatorics, Graph Theory, and Computing (1978), 163–170.

[9] J.C. Culberson and G.J.E. Rawlins. New Results from an Algorithm for Counting Posets. Order 7 (90/91), no 4, p 361-374. dy

[10] A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische Produkte. Annals of Math. 67(1958), 239–281.

[11] M. Erne and K. Stege. Counting Finite Posets and Topologies. Order 8 (91), p 247-265.

[12] J.W. Evans, F. Harary, and M.S. Lynn. On the computer enumeration of finite topologies. Communications of the ACM 10(1967), 295–297, 313.

[13] R. H. Fox. On topologies for function spaces. Bull. Amer. Math. Soc. 51(1945), 429-432.

[14] R. Fritsch and R.A. Piccinini. Cellular structures in algebraic topology. Cambridge University Press. 1990.

[15] D. Gorenstein. Finite Groups. Harper and Row. 1968.

[16] K.A. Hardie and J.J.C. Vermeulen. Homotopy theory of finite and locally finite T_0-spaces. Expositiones Math. 11(1993), 331–341.

[17] K.A. Hardie, J.J.C. Vermeulen, and P.J. Witbooi. A nontrivial pairing of finite T_0-spaces. Topology and its Applications 125 (2002), 533–542.

[18] A. Hatcher. Algebraic topology. Cambridge University Press. 2002.

[19] R. D. Helmstutler and R.S. Higginbottom. Finite topological spaces as a pedagorical tool. PRIMUS, 22(1)(2012), 64–72.

[20] P.J. Hilton and S. Wylie. Homology theory. Cambridge University Press. 1960.

[21] D. Kleitman and B. Rothschild. The number of finite topologies. Proc. AMS, 25, 1970, 276-282.

[22] D. Kleitman and B. Rothschild. Asymptotic enumeration of partial orders on a finite set. Trans. Amer. Math. Soc. 205 (1975), 205–220.

[23] Y. Koda. The numbers of finite lattices and finite topologies. Bulletin of the Institute of Combinatorics and its applications 10 (1994), 83–89.

[24] T.Y. Kong and E. Khalimsky. Polyhedral analogs of locally finite topological spaces. Lecture Notes in Pure and Applied Mathematics Vol 123, General Topology and Applications, 153–164. Marcel Dekker. 1990.

[25] V. Krishnamurthy. On the number of topologies on a finite set. Amer. Math. Monthly 73(1966), 154-157.

[26] M.J. Kukiela. On homotopy types of Alexandroff spaces. Order 27(2010), no. 1, 921.

[27] J.P. May. Simplicial objects in algebraic topology. The University of Chicago Press. 1967; reprinted 1982.

[28] J.P. May. Classifying spaces and fibrations. Memoirs Amer. Math. Soc. 155. 1975.

[29] J.P. May. Weak equivalences and quasifibrations. In: R. Piccinini, editor, Groups of self-equivalences and related topics. Lecture Notes in Mathematics Vol. 1425. Springer–Verlag, 1990, 91–101.
[30] J.P. May. A concise course in algebraic topology. The University of Chicago Press. 1999.
[31] J.P. May and K. Ponto. More concise algebraic topology: localization, completion, and model categories. University of Chicago Press. 2012.
[32] M.C. McCord. Singular homology groups and homotopy groups of finite topological spaces. Duke Mathematical Journal 33(1966), 465–474.
[33] J.R. Munkres. Topology. Second edition. Prentice–Hall. 1975.
[34] T. Osaki. Reduction of finite topological spaces. Interdisciplinary Information Sciences 5(1999), 149–155.
[35] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. Advances in Mathematics 28(1978), 101–128.
[36] P. Renteln. On the enumeration of finite topologies. Journal of Combinatorics, Information, and System Sciences 19(1994), 201-206.
[37] P. Renteln. Geometrical Approaches to the Enumeration of Finite Posets: An Introductory Survey. Nieuw Archief voor Wiskunde 14 (1996), 349-371.
[38] H. Seifert and W. Threlfall. A textbook of topology.
[39] E.H. Spanier. Algebraic topology. McGraw-Hill. 1966.
[40] R.E. Stong. Finite topological spaces. Trans. Amer. Math. Soc. 123(1966), 325–340.
[41] R.E. Stong. Group actions on finite spaces. Discrete Mathematics 49(1984), 95–100.
[42] J.H.C. Whitehead. Combinatorial homotopy I, II. Bulletin Amer. Math. Soc. 55(1948), 213–245 and 453–496.
[43] P.J. Witbooi. Globalizing weak homotopy equivalences. Topology and its Applications 100(2000), 229–240.