ULTRAFILTERS IN SET THEORY

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Abstract. We survey applications of ultrafilters and ultrafilter constructions in two set theoretic contexts. In the first setting, that of large cardinals, we explore a number of large cardinal properties and give an ultrafilter characterization of measurability. We then use a generalized ultrapower to give an elegant proof of Scott's theorem. In the second setting, that of forcing, we explicate the forcing process before describing an interaction between forcing and large cardinals that is mediated by a normal ultrafilter.

Contents

1. Introduction

Different mathematical settings demand different notions of largeness. In analysis, where it is natural to inquire about the size of the set on which a function exhibits a particular mathematical trait or behavior — for instance, continuity — measure serves as the right notion of largeness. However, measure cannot play the same role for set theory, where it is pertinent to ask how sets compare to one another with regard to size; here, cardinality, which gives rise to an infinite linear hierarchy into which any set may be placed, serves as the right notion of largeness.

In order to speak in general about what is large and to distinguish in general what is large from what is small, we need a more versatile notion of largeness. The

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criteria for largeness determined by this notion must be flexible enough to suit a variety of different mathematical settings. One notion of largeness that is adaptable in this sense is the notion provided to us by ultrafilters. Ultrafilters are collections of large sets, where the meaning of the word 'large' can be adjusted for the context. In this paper, we explore ultrafilters and their applications in set theory.

We introduce ultrafilters in Section 2 as special filters and describe an important ultrafilter construction called the ultraproduct, a generalization of the ordinary Cartesian product. We then give a model theoretic characterization of ultraproducts and state a fundamental result known as Łoś's theorem (Theorem 2.3.1). This result can be used to give straightforward proofs of many different theorems, including the compactness theorem of first-order logic (Corollary 2.3.2).

In Section 3, we turn to applications of ultrafilters and ultrafilter constructions to large cardinals (Definition 3.0.1). Many large cardinal properties can be expressed in terms of ultrafilters; one large cardinal property that possesses an ultrafilter characterization is measurability (Definition 3.2.6), which we consider at length. The section concludes with a brisk overview of inner model theory, culminating in a proof of Scott's theorem (Theorem 3.3.12) that involves an ultrapower of a proper class.

Section 4 deals with forcing. The forcing procedure depends on a generalized notion of filter, which we use to build proper extensions of small models of Zermelo-Fraenkel set theory in which the continuum hypothesis fails. We also consider forcing extensions in which the generalized continuum hypothesis fails at infinitely many regular cardinals. Ultrafilters return at the end of the section, where we consider a conjecture known as the singular cardinals hypothesis (Definition 4.4.1).

We assume familiarity with model theory, on the level of Chapters 1 and 2 of [1], and with Gödel's incompleteness theorems, on the level of Chapter 3 of [3]. We also assume familiarity with set theory on the level of Chapter 1 of [10]; in particular, the reader should understand cardinal arithmetic and know the axioms of Zermelo-Fraenkel set theory. We shall recall some concepts as needed. Throughout the paper, in keeping with standard set theoretic practice, we use the symbol ω for the cardinal $\aleph_0$.

2. Filters and Products

In this section, we introduce the tools we shall use throughout the paper.

2.1. Filters and Ultrafilters.

Definition 2.1.1. A *filter* $\mathcal{F}$ on a nonempty set I is a collection of subsets of I such that the following hold:

(1) $I \in \mathcal{F}$.
(2) $\mathcal{F}$ is closed under finite intersections, that is, for all $X, Y \in \mathcal{F}$, $X \cap Y \in \mathcal{F}$.
(3) $\mathcal{F}$ is upwards closed, that is, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

The set I is called the *index set* of $\mathcal{F}$.

For any nonempty set I, $\mathcal{P}(I)$ is a filter on I. Notice that, if $\mathcal{F}$ is any filter containing $\emptyset \subseteq I$, then, by condition (3) of the definition, $\mathcal{F} = \mathcal{P}(I)$. However, $\mathcal{P}(I)$ is uninteresting as a filter (see Example 2.2.2). Therefore, we restrict our attention to *proper filters*, which are filters that do not contain $\emptyset \subseteq I$.

We give some additional examples of filters. Fix a nonempty set I.

Example 2.1.2. Let $\mathcal{F} = \{I\}$. Then $\mathcal{F}$ is the *trivial filter* on I. As the reader may expect, there is little that is interesting about trivial filters.

Example 2.1.3. Let $X \subseteq I$. Then $\mathcal{F} = \{Y \subseteq I : X \subseteq Y\}$ is a filter on I, the *principal filter generated by* X. If X is nonempty, then $\mathcal{F}$ is a proper filter. For instance, if $I = \mathbb{N}$, then $\mathcal{F} = \{Y \subseteq \mathbb{N} : \{0, 1, 2\} \subseteq Y\}$ is the (proper) principal filter generated by $\{0, 1, 2\}$.

Principal filters are sometimes useful because they are easy to construct, but we shall often prefer more complicated filters. A filter $\mathcal{F}$ on I is *nonprincipal* if it is not principal.

We mention two convenient facts about principal filters. First, if $\mathcal{F}$ is a filter containing a finite set, then $\mathcal{F}$ is principal; this follows from the fact that filters are closed under finite intersections. In particular, if I is finite, then any filter on I is principal. Second, if $\mathcal{F}$ is a filter such that $\bigcap_{X \in \mathcal{F}} X \in \mathcal{F}$, then $\mathcal{F}$ is principal, and conversely; see [17] for a proof. Before considering an example, we fix some notation.

Notation 2.1.4. We shall, for the remainder of the paper, use $\bigcap \mathcal{F}$ to denote $\bigcap_{X \in \mathcal{F}} X$, and we shall use $\bigcup \mathcal{F}$ to denote $\bigcup_{X \in \mathcal{F}} X$.

Example 2.1.5. Let $\mathcal{F} = \{X \subseteq I : |I \setminus X| < \omega\}$, so that $\mathcal{F}$ is the set of all cofinite subsets of I. Then $\mathcal{F}$ is called the *Fréchet filter* on I. If I is finite, then the Fréchet filter on I is $\mathcal{P}(I)$. However, when I is infinite, the Fréchet filter is nonprincipal.

In the next definition, we introduce an important new type of filter.

Definition 2.1.6. An *ultrafilter* $\mathcal{U}$ on a nonempty set I is a proper filter on I such that, for all $X \subseteq I$, either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ therefore "decides", for each set X included in its index set I, whether X is "large" or "small"; if $X \in \mathcal{U}$, then X is "large", and if $I \setminus X \in \mathcal{U}$, then X is "small". Thus, when $X \in \mathcal{U}$, we shall sometimes say that X is *$\mathcal{U}$-large*. Except in trivial cases, the notion of $\mathcal{U}$-largeness agrees with the intuitive notion of largeness; in particular, interesting ultrafilters never treat finite sets as large.

The following proposition provides us with a convenient alternative characterization of ultrafilters; see Chapter 4, Section 1 of [1] for a proof.

Proposition 2.1.7. *Let I be an index set. A set $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter if and only if $\mathcal{U}$ is a maximal proper filter.*

We give some examples and non-examples of ultrafilters. Fix a nonempty set I.

Example 2.1.8. Unless I is a singleton, the trivial filter $\{I\}$ is not an ultrafilter. Furthermore, if I is infinite, then the Fréchet filter $\mathcal{F}$ on I is not an ultrafilter, since there exist infinite coinfinite subsets of I; for instance, if $I = \mathbb{N}$, then neither the set of even natural numbers nor its complement is contained in $\mathcal{F}$.

Example 2.1.9. A *principal ultrafilter* is an ultrafilter that is principal. Principal ultrafilters are always generated by singletons: Suppose $\mathcal{U}$ is a principal ultrafilter on I generated by a nonempty set $X \subseteq I$, and assume for contradiction that X is not a singleton. Then for any $x \in X$, we have $I \setminus \{x\} \in \mathcal{U}$; however, $I \setminus \{x\}$ does not include X.

Thus, principal ultrafilters are easy to construct. Nonprincipal ultrafilters are not so straightforward. To conclude that nonprincipal ultrafilters exist at all, we require[1] the axiom of choice (AC). The proof of the following theorem uses Zorn's lemma, which is equivalent over the collection ZF of Zermelo-Fraenkel axioms to AC.

Theorem 2.1.10. *Let I be an infinite set, and let $\mathcal{F}$ be a proper filter on I. Then there exists an ultrafilter $\mathcal{U}$ on I such that $\mathcal{F} \subseteq \mathcal{U}$. If, in addition, the set $\bigcap \mathcal{F}$ is empty, then $\mathcal{U}$ is nonprincipal.*

Proof. Let

$$\mathcal{S} = \{\mathcal{G} \subseteq \mathcal{P}(I) : \mathcal{G} \supseteq \mathcal{F} \text{ and } \mathcal{G} \text{ is a proper filter}\}.$$

Let $\mathcal{C}$ be a chain of elements of $\mathcal{S}$. Then $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$ that is contained in $\mathcal{S}$; hence, Zorn's lemma applies, and we obtain a maximal proper filter $\mathcal{U}$ including $\mathcal{F}$. By Proposition 2.1.7, $\mathcal{U}$ is an ultrafilter.

It remains to show that $\mathcal{U}$ is nonprincipal if $\bigcap \mathcal{F}$ is empty. In this case, because $\mathcal{F} \subseteq \mathcal{U}$, it follows that $\bigcap \mathcal{U} \subseteq \bigcap \mathcal{F} = \emptyset$; hence, $\bigcap \mathcal{U} = \emptyset$. Since $\mathcal{U}$ is proper, $\bigcap \mathcal{U}$ is not contained in $\mathcal{U}$, and thus by the discussion following Example 2.1.3, $\mathcal{U}$ is nonprincipal. □

We shall consider several different kinds of nonprincipal ultrafilter when characterizing large cardinals.

2.2. **Ultraproducts and Ultrapowers.** We now apply filters and ultrafilters towards the construction of new tools.

Definition 2.2.1. Let $\mathcal{F}$ be a filter on a nonempty set I, and let $\{A_i : i \in I\}$ be a collection of sets indexed by I. Let $f, g \in \prod_{i \in I} A_i$. We shall say that f is equivalent to g modulo $\mathcal{F}$, written $f \sim_{\mathcal{F}} g$, if $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$. Then the *reduced product of* $\{A_i : i \in I\}$ *by* $\mathcal{F}$, denoted $\prod_{\mathcal{F}} A_i$, is the collection of $\sim_F$-equivalence classes. That is,

$$\prod_{\mathcal{F}} A_i = \left(\prod_{i \in I} A_i\right) / \sim_{\mathcal{F}} .$$

The use of the word 'equivalent' is justified by the fact that $\sim_{\mathcal{F}}$ is indeed an equivalence relation on $\prod_{i \in I} A_i$; this fact follows straightforwardly from the properties of filters. Notice that, if $f \in \prod_{i \in I} A_i$, then f is a choice function for $\{A_i : i \in I\}$. Hence, when I is infinite, AC is needed to ensure that such an f exists.

Our use of the symbols f and g to denote elements of $\prod_{i \in I} A_i$ perhaps indicates to the reader that we shall always treat $\prod_{i \in I} A_i$ as a collection of functions. In truth, we shall alternate without comment between treating elements of $\prod_{i \in I} A_i$ as functions with domain I and as sequences indexed by I.

Let us consider an example.

Example 2.2.2. Fix a nonempty set I and a collection of sets $\{A_i : i \in I\}$. We shall show that some reduced products may be canonically identified with ordinary Cartesian products.

Suppose $\mathcal{F}_1$ is the trivial filter $\{I\}$. Then for any $f, g \in \prod_{\mathcal{F}_1} A_i$, $f \sim_{\mathcal{F}_1} g$ if and only if $f(i) = g(i)$ for all $i \in I$, that is, if and only if f and g are equal as functions.

[1] See the introduction to Section 3.

Hence, the reduced product by $\mathcal{F}_1$ can be canonically identified with the ordinary Cartesian product $\prod_{i\in I} A_i$ via the map

$$[f]_{\sim_{\mathcal{F}_1}} \mapsto f.$$

Notice that the trivial filter $\{I\}$ is merely the principal filter generated by the set I. Similarly, the nonproper filter $\mathcal{P}(I)$ is the principal filter generated by $\emptyset \subseteq I$. Let $\mathcal{F}_2 = \mathcal{P}(I)$. Then for any $f, g \in \prod_{\mathcal{F}_2} A_i$, $f \sim_{\mathcal{F}_2} g$ if and only if there exists some (possibly empty) subset of I on which f and g agree. Hence, the reduced product of $\{A_i : i \in I\}$ by $\mathcal{F}_2$ can be canonically identified with a singleton, namely the empty product $\prod_{i\in\emptyset} A_i$.

This reasoning generalizes to principal filters in general. If $\mathcal{F}_3$ is a principal filter generated by a nonempty set $X \subsetneq I$, then for any $f, g \in \prod_{\mathcal{F}_3} A_i$, $f \sim_{\mathcal{F}_3} g$ if and only if f and g agree on X. Thus, the reduced product by $\mathcal{F}_3$ can be canonically identified with the Cartesian product $\prod_{i\in X} A_i$.

We shall often prefer for the filter $\mathcal{F}$ of Definition 2.2.1 to be an ultrafilter.

Definition 2.2.3. Let $\mathcal{F}, I$, and $\{A_i : i \in I\}$ be as in Definition 2.2.1. Then the reduced product of $\{A_i : i \in I\}$ by $\mathcal{F}$ is called the *ultraproduct* of $\{A_i : i \in I\}$ by $\mathcal{F}$. Furthermore, if there exists a set A such that, for all $i \in I$, $A_i = A$, then the ultraproduct is called the *ultrapower* of A by $\mathcal{F}$.

The reduced product of Definition 2.2.1 is a product of sets without structure. It is useful also to consider, given a first-order language $\mathcal{L}$, a reduced product of models for $\mathcal{L}$. Such a product is itself an $\mathcal{L}$-model, having as its universe the reduced product of the universes of the factor models; its interpretation of a symbol of $\mathcal{L}$ depends in a natural way on the interpretation of that symbol in each factor model.

Before giving the definition, we fix some notation.

Notation 2.2.4. Let $\mathcal{L}$ be a first-order language, let $\mathcal{F}$, I, and $\{A_i : i \in I\}$ be as in Definition 2.2.1, and let $\{\mathcal{A}_i : i \in I\}$ be a collection of $\mathcal{L}$-models such that, for each $i \in I$, A_i is the universe of $\mathcal{A}_i$.

For C a constant symbol of $\mathcal{L}$, denote by D_i the interpretation of C in $\mathcal{A}_i$; for F an n-ary function symbol of $\mathcal{L}$, denote by G_i the interpretation of F in $\mathcal{A}_i$; and for R an n-ary relation symbol of $\mathcal{L}$, denote by S_i the interpretation of R in $\mathcal{A}_i$.

If $f \in \prod_{i\in I} A_i$, then we write $[f]$ for $[f]_{\sim_{\mathcal{F}}}$.

Definition 2.2.5. The *reduced product of* $\{\mathcal{A}_i : i \in I\}$ *by* $\mathcal{F}$, denoted $\prod_{\mathcal{F}} \mathcal{A}_i$, is the model for $\mathcal{L}$ with the following properties:

(1) The universe set of $\prod_{\mathcal{F}} \mathcal{A}_i$ is $\prod_{\mathcal{F}} A_i$.
(2) For C a constant symbol of $\mathcal{L}$, $\prod_{\mathcal{F}} \mathcal{A}_i$ interprets C as the constant $E = [\langle D_i : i \in I\rangle]$.
(3) For F an n-ary function symbol of $\mathcal{L}$, $\prod_{\mathcal{F}} \mathcal{A}_i$ interprets F as the function H such that, if $f_0, f_1, \ldots, f_{n-1} \in \prod_{i\in I} A_i$, then
$$H([f_0], [f_1], \ldots, [f_{n-1}]) = [\langle G_i(f_0(i), f_1(i), \ldots, f_{n-1}(i)) : i \in I\rangle].$$
(4) For R an n-ary relation symbol of $\mathcal{L}$, $\prod_{\mathcal{F}} \mathcal{A}_i$ interprets R as the relation T such that, if $f_0, f_1, \ldots, f_{n-1} \in \prod_{i\in I} A_i$, then
$$T([f_0], [f_1], \ldots, [f_{n-1}]) \text{ if and only if } \{i \in I : S_i(f_0(i), f_1(i), \ldots, f_{n-1}(i))\} \in \mathcal{F}.$$

Because these interpretations are defined in terms of equivalence classes, we must check that they do not depend on the choices of representative; this follows easily from the fact that filters are closed under finite intersections.

2.3. **Some Applications of Ultrafilters and Ultraproducts.** Given our model theoretic characterization of reduced products, the reader may wonder whether the first-order statements satisfied by the reduced product relate somehow to the first-order statements satisfied by the factor models. For reduced products in general, this relationship is murky. For ultraproducts, however, we have the following elegant result of Łoś; see Chapter 4, Section 1 of [1] for a proof.

Theorem 2.3.1 (Łoś's theorem)**.** *Let $\mathcal{L}$ be a first-order language, let $\mathcal{U}$ be an ultrafilter on a nonempty set I, let $\{\mathcal{A}_i : i \in I\}$ be a collection of models for $\mathcal{L}$ such that the universe of $\mathcal{A}_i$ is a set A_i, and let $\varphi(x_0, x_1, \ldots, x_{n-1})$ be a formula of $\mathcal{L}$ with n free variables. Then, for any $f_0, f_1, \ldots, f_{n-1} \in \prod_{i\in I} A_i$,*

$$\prod_{\mathcal{U}} \mathcal{A}_i \models \varphi([f_0], [f_1], \ldots, [f_{n-1}])$$

if and only if

$$\{i \in I : \mathcal{A}_i \models \varphi(f_0(i), f_1(i), \ldots, f_{n-1}(i))\} \in \mathcal{U}.$$

Thus, for a sentence φ of $\mathcal{L}$, $\prod_{\mathcal{U}} \mathcal{A}_i \models \varphi$ if and only if $\{i \in I : \mathcal{A}_i \models \varphi\} \in \mathcal{U}$; that is, φ is true in $\prod_{\mathcal{U}} \mathcal{A}_i$ if and only if the collection of models $\mathcal{A}_i$ in which φ is true is $\mathcal{U}$-large. The ultraproduct thus follows the $\mathcal{U}$-majority when deciding the truth of a sentence.[2]

If the ultraproduct is an ultrapower, then we have that $\prod_{\mathcal{U}} \mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \varphi$, so that the ultrapower and the factor model are elementarily equivalent; that is, for any sentence of the language $\mathcal{L}$, the ultrapower and the factor model agree on the truth of the sentence.

The "ultraness" of ultrafilters is necessary for the conclusion of Łoś's theorem to be sensible, and this is why the theorem does not apply for reduced products in general: If we replace $\mathcal{U}$ with a "non-ultra" filter $\mathcal{F}$, then we risk encountering a sentence φ of $\mathcal{L}$ such that neither the set $\{i \in I : \mathcal{A}_i \models \varphi\}$ nor its complement $\{i \in I : \mathcal{A}_i \not\models \varphi\}$ is contained in $\mathcal{F}$. Then the reduced product $\prod_{\mathcal{F}} \mathcal{A}_i$ fails to decide the truth of φ.

The proofs of the following two corollaries demonstrate the power of Łoś's theorem. The first corollary is the compactness theorem of first-order logic.

Corollary 2.3.2. *Let Σ be a collection of $\mathcal{L}$-sentences. Then Σ is satisfiable if and only if it is finitely satisfiable, that is, if and only if every finite subset of Σ is satisfiable.*

Proof. ($\Rightarrow$) This is immediate.

($\Leftarrow$) Let $I = \{i \subseteq \Sigma : |i| < \omega\}$ be the set of all finite subsets of Σ. For each $i \in I$, let $\mathcal{A}_i$ be an $\mathcal{L}$-model of i, and for each $\sigma \in \Sigma$, let $\Gamma_\sigma = \{i \in I : \sigma \in i\}$ be the set of all finite subsets of Σ in which σ is contained. Define $\Gamma = \{\Gamma_\sigma : \sigma \in \Sigma\}$. If $\Gamma_{\sigma_0}, \Gamma_{\sigma_1} \in \Gamma$, then $\Gamma_{\sigma_0} \cap \Gamma_{\sigma_1} \neq \emptyset$; in particular, the set $\{\sigma_0, \sigma_1\} \in \Gamma_{\sigma_0} \cap \Gamma_{\sigma_1}$.

Now, define

$$\mathcal{F} = \bigcap \{\mathcal{D} \subseteq \mathcal{P}(I) : \mathcal{D} \supseteq \Gamma \text{ and } \mathcal{D} \text{ is a filter}\}.$$

Then $\mathcal{F}$ is a proper filter on I; the properness of $\mathcal{F}$ follows from the fact that any two elements of Γ have nonempty intersection. By Theorem 2.1.10, $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{U}$ on I. Now we shall show that, for any $\sigma \in \Sigma$, the set

[2] Sadly, ultraproducts do not think for themselves.

$S_\sigma = \{i \in I : \mathcal{A}_i \models \sigma\}$ is $\mathcal{U}$-large. Since ultrafilters are upwards closed and since $\Gamma \subseteq \mathcal{U}$ implies $\Gamma_\sigma \in \mathcal{U}$, it is enough to show that S_σ includes Γ_σ. Thus, let $i \in \Gamma_\sigma$. Then $\mathcal{A}_i \models i$, and since $\sigma \in i$, $\mathcal{A}_i \models \sigma$, as desired. It follows from Łoś's theorem that $\prod_\mathcal{U} \mathcal{A}_i \models \Sigma$. □

Łoś's theorem can also be used to prove the following algebraic result.

Corollary 2.3.3. *Let $I = \mathbb{N}$, and let p_i be the ith prime. Suppose that, for each $i \in I$, $\mathcal{A}_i$ is the algebraic closure of the field of p_i elements, and suppose $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Then the ultraproduct $\prod_\mathcal{U} \mathcal{A}_i$ is an algebraically closed field of characteristic zero.*

Proof. Let $\mathcal{L}_f$ be the language of fields, and expand $\mathcal{L}_f$ to a language $\mathcal{L}'_f$ in which there is a constant symbol P_j for each $j \in \mathbb{N}$. We stipulate that, for each $i \in I$, $\mathcal{A}_i$ interprets P_j as follows:

$$P_j^{\mathcal{A}_i} = \begin{cases} p_j & j < i \\ 0 & j = i \\ p_j \text{ (modulo } p_i) & j > i \end{cases}.$$

By the assumptions on the fields $\mathcal{A}_i$, the $\mathcal{L}'_f$-sentence σ_j expressing that P_j is invertible holds in each field except $\mathcal{A}_j$, in which $P_j^{\mathcal{A}_j} = 0$. Hence, the set $\{i \in I : \mathcal{A}_i \not\models \sigma_j\} = \{j\}$. Since $\mathcal{U}$ is nonprincipal, it does not contain any finite sets, and so $\{j\} \notin \mathcal{U}$. Therefore, $\mathbb{N} \setminus \{j\} \in \mathcal{U}$; then the set of fields in which p_j is invertible is $\mathcal{U}$-large. By Łoś's theorem and the fact that each $\mathcal{A}_i$ satisfies all of the axioms for an algebraically closed field, the ultraproduct $\prod_\mathcal{U} \mathcal{A}_i$ is an algebraically closed field satisfying σ_j for all $j \in \mathbb{N}$; thus, for any prime p_j, the characteristic of $\prod_\mathcal{U} \mathcal{A}_i$ is not p_j. It follows that the characteristic of the ultraproduct is zero. □

With some additional effort, one can show that, in fact, $\prod_\mathcal{U} \mathcal{A}_i$ is isomorphic to the field $\mathbb{C}$; see [18] for a proof.

Having concluded our study of the essentials of ultrafilters and ultraproducts, we may proceed to more advanced considerations. In the next section, we take an ultrafilter approach to large cardinals.

3. Ultrafilter Characterizations of Large Cardinals

We mentioned in Subsection 2.1 that AC is needed to guarantee the existence of nonprincipal ultrafilters. Use of the word 'needed' here is justified; the existence of nonprincipal ultrafilters is in fact independent of ZF. That is, it is consistent with ZF that all ultrafilters are principal, and it is also consistent with ZF that not all ultrafilters are principal. From this result, we may conclude that ZF alone is too weak to settle the question of the existence of nonprincipal ultrafilters. To secure an (affirmative) answer to this question, we must ascend to a stronger system, namely ZFC, the system consisting of all the axioms of ZF together with AC.

There is no reason to cease this sort of inquiry at the level of nonprincipality, because there are many ultrafilter properties of greater sophistication; see Definition 3.2.6 for an example of such a property. It is natural to ask, then, whether ZFC alone is strong enough to decide whether ultrafilters possessing these properties exist.

In many cases, the answer is an emphatic 'no'. As it happens, if P is a sufficiently sophisticated ultrafilter property, then the formal statement A asserting the existence of an ultrafilter possessing the property P is exceptionally strong, to the degree that it proves the consistency of ZFC. That is,

$$\mathsf{ZFC} + \mathsf{A} \vdash \mathrm{Con}(\mathsf{ZFC}).$$

Thus, if ZFC is consistent, then ZFC does not prove A, and so the existence of an ultrafilter possessing the property P is not guaranteed by ZFC alone.

Fascinatingly, the study of sophisticated ultrafilter properties aligns closely with the study of some particular set theoretic properties.

Definition 3.0.1. A *large cardinal property* is a property $P(x)$ such that

$$\mathsf{ZFC} + \text{“there exists a cardinal } \kappa \text{ such that } P(\kappa)\text{”} \vdash \mathrm{Con}(\mathsf{ZFC}).$$

If $P(x)$ is a large cardinal property and κ is a cardinal such that $P(\kappa)$, then κ is said to be a *large cardinal*, and the axiom asserting the existence of a cardinal κ such that $P(\kappa)$ is called a *large cardinal axiom*.

The reader should be aware that there is no universally accepted definition of the term ‘large cardinal’; the definition we have just provided is the one that is most useful for our purposes. Furthermore, we shall frequently conflate the three terms introduced in the definition and use the phrase ‘large cardinal’ for the cardinal, the associated large cardinal property, and the associated large cardinal axiom. For some examples of large cardinal properties, see Subsection 3.2 and also [9] and [15].

Large cardinal axioms may be viewed as statements that give us more information about the set theoretic world (see Definition 3.1.9) than ZFC does alone. Partly for this reason, set theorists once considered large cardinal axioms candidates for “dream solutions” to the problem of the continuum hypothesis (CH), which was shown to be independent of ZFC by work of Gödel in the 1930s and Cohen in the 1960s. More explicitly, the goal was to find a large cardinal axiom A agreeing with human intuition about mathematics such that $\mathsf{ZFC} + \mathsf{A}$ would decide CH; see [5] for a more complete history. To date, no such large cardinal axiom has been found. However, large cardinals have proved to be deeply intriguing mathematical objects in their own right.

A number of large cardinal properties can be stated in terms of ultrafilters. Hence, we here consider some interactions between ultrafilters and large cardinals. Using the ultrafilter characterization of one large cardinal in particular enables us to give a neat proof of Scott’s theorem in Subsection 3.3.

For the remainder of this section, assume that ZFC is consistent.

3.1. **Set Theory Preliminaries.** We begin by recalling three definitions from set theory. Examples follow after some discussion.

Definition 3.1.1. A set α is an *ordinal* if each of the following holds:

(1) The set α is transitive, that is, if $\beta \in \alpha$, then $\beta \subseteq \alpha$.
(2) The set α is linearly ordered by the membership relation $\in$.

We say that α is a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$, where $\beta + 1 = \beta \cup \{\beta\}$; if α is neither the empty set nor a successor ordinal, then α is a *limit ordinal*.

Denote by $\mathbf{ON}$ the proper class of all ordinals. This class is well-ordered by the relation $\in$. If $\alpha, \beta \in \mathbf{ON}$ and $\beta \in \alpha$, then we write $\beta < \alpha$.

Definition 3.1.2. Suppose κ is an ordinal such that, for all ordinals $\lambda < \kappa$, there does not exist a surjection from λ onto κ. Then κ is a *cardinal*. We say that κ is a *successor cardinal* if, for some cardinal $\lambda < \kappa$, the least cardinal greater than λ is κ. In this case, we write $\kappa = \lambda^+$. If κ is neither the empty set nor a successor cardinal, then it is a *limit cardinal*.

For any set X, the *cardinality* of X, denoted $|X|$, is the unique cardinal μ such that there exists a bijection between μ and X.

We must justify the definition of the term 'cardinality'. We first need the following fact: If $(X, \preceq)$ is a well-order, then there exists a unique ordinal α such that $(X, \preceq) \cong (\alpha, \in)$. The proof of this fact can be found in Chapter I, Section 7 of [10].

Proposition 3.1.3. *For any set X, $|X|$ exists.*

Proof. First, we show that the cardinality of any ordinal exists. Let α be an ordinal, and let

$$S_\alpha = \{\beta \in \mathbf{ON} : \beta \leq \alpha \text{ and there exists a bijection } f : \beta \to \alpha\}.$$

Since S_α is a set of ordinals, it has an $\in$-least element; call this element β_0. Then β_0 is a cardinal: Suppose not. Then there exist an ordinal $\gamma < \beta_0$ and a surjection $f : \gamma \to \beta_0$. By definition, there exists a bijection $g : \beta_0 \to \alpha$. Hence, the map $g \circ f : \gamma \to \alpha$ is a surjection. Since $\gamma < \beta_0 \leq \alpha$ and thus $\gamma \subseteq \alpha$, the inclusion map from γ to α is an injection. It follows from the Cantor-Schröder-Bernstein theorem that there exists a bijection between γ and α. This contradicts the choice of β_0 as the least ordinal having a bijection to α. Thus, β_0 is a cardinal. That β_0 is unique follows immediately from its definition and the fact that **ON** is linearly ordered by $\in$.

Now, let X be any set. Use AC to endow X with a well-ordering $\preceq$. By the fact above, there exists a unique ordinal α such that $(X, \preceq) \cong (\alpha, \in)$. Since α is an ordinal, its cardinality $|\alpha|$ exists; it is then easy to show that $|X| = |\alpha|$. $\square$

We remark that that an ordinal κ is a cardinal if and only if $|\kappa| = \kappa$. Furthermore, since the identity map from any ordinal α to itself is bijective, $|\alpha|$ never exceeds α.

Definition 3.1.4. Let $\alpha, \beta \in \mathbf{ON}$ with β a limit ordinal. Suppose $f : \alpha \to \beta$ is such that, for each $\delta \in \beta$, there exists an ordinal $\gamma \in \alpha$ such that $f(\gamma) > \delta$. Then f is a *cofinal map*, and α is *cofinal* in β. If α is the least ordinal such that there exists a cofinal map from α to β, then we say that α is the *cofinality* of β, and we write $\alpha = \mathrm{cf}(\beta)$. If $\mathrm{cf}(\beta) = \beta$, then β is *regular*. Otherwise, β is *singular*.

If β is not a limit ordinal, then either $\beta = 0$ or β is a successor ordinal. In the first case, define $\mathrm{cf}(\beta) = 0$; in the second, define $\mathrm{cf}(\beta) = 1$.

For any ordinal α, $\mathrm{cf}(\alpha)$ exists. Let

$$S_\alpha = \{\beta \in \mathbf{ON} : \text{there exists a cofinal map from } \beta \text{ to } \alpha\}.$$

Then S_α has a unique $\in$-least element β_0, and $\beta_0 = \mathrm{cf}(\alpha)$; the proof is similar to that of Proposition 3.1.3. For any limit ordinal α, the identity map from α to itself is a cofinal map, and thus the cofinality of α never exceeds α.

Notice the following distinction between cardinality and cofinality on the one hand and ordinality on the other: In order to determine whether an ordinal α is a cardinal, it is necessary to search the set theoretic world outside of α for an appropriate surjective function (or the lack of such a function). The same is true if we replace 'cardinal' with 'regular cardinal' and 'surjective' with 'cofinal'. However, in order to determine whether α is an ordinal in the first place, no such search needs to be conducted; we need only see what happens internally to α. We shall make this distinction formal in Section 4.

We now give explicit examples of ordinals, cardinals, and cofinalities and elucidate some similarities and differences among these concepts.

Example 3.1.5. Define the finite ordinals recursively as follows:

$$\begin{aligned} 0 &= \emptyset, \\ n &= n-1 \cup \{n-1\} . \end{aligned}$$

Then each such n is both an ordinal and a cardinal. Furthermore, the first infinite countable ordinal ω, which is the set $\{0, 1, 2, \dots\}$ of all finite ordinals, is both an ordinal and a cardinal. However, $\omega+1 = \omega \cup \{\omega\}$ is an ordinal that is not a cardinal, since there exists a mapping onto $\omega + 1$ from ω (send 0 to ω; if $n \neq 0$, send n to $n - 1$).

Example 3.1.6. For any nonzero $n \in \omega$, the cofinality of n is 1, since each such n is a successor ordinal. Hence, if n is neither 0 nor 1, then n is singular. It is clear that the cofinality of ω is not finite, and so $\mathrm{cf}(\omega) = \omega$; thus, ω is regular.

Example 3.1.7. Define the cardinals ω_α by transfinite recursion on $\alpha \in \mathbf{ON}$ as follows: Let $\omega_0 = \omega$. If α is a successor ordinal, then let $\beta \in \mathbf{ON}$ be such that $\alpha = \beta + 1$; now define $\omega_\alpha = \omega_\beta^+$, the least cardinal greater than ω_β. If α is a limit ordinal, then let $\omega_\alpha = \bigcup_{\gamma<\alpha} \omega_\beta$. Then each ω_n is a regular cardinal, but $\omega_\omega = \bigcup_{n\in\omega} \omega_n$ is an uncountable singular cardinal; to see that ω_ω has cofinality ω, observe that $f : \omega \to \omega_\omega$ defined by $f(n) = \omega_n$ is cofinal.

By definition, every cardinal is also an ordinal. Example 3.1.7 shows us that not every cardinal is regular; however, every regular ordinal is a cardinal. Also, the cofinality of any ordinal is a cardinal. We summarize these facts in the following proposition.

Proposition 3.1.8. *Every cardinal is an ordinal, and every regular ordinal is a cardinal. Furthermore, if $\alpha, \beta \in \mathbf{ON}$ and $\beta = cf(\alpha)$, then β is a cardinal.*

Proof. The first claim follows directly from the definitions.

For the second claim, suppose α is a regular (limit) ordinal, and let $\beta < \alpha$. Then there does not exist a cofinal map from β to α. It follows that there is no surjection from β onto α, since a surjective map is cofinal when its target is a limit ordinal. Hence, α is a cardinal.

For the third claim, let α, β be as stated. Assume for contradiction that β is not a cardinal. Then there exist a cardinal γ such that $\gamma < \beta$ and a bijection $f : \gamma \to \beta$. Since $\beta = \mathrm{cf}(\alpha)$, there exists a cofinal map $g : \beta \to \alpha$. However, the map $g \circ f : \gamma \to \alpha$ is also cofinal; since $\gamma < \beta$, this contradicts that $\beta = \mathrm{cf}(\alpha)$. □

Let us now recall the construction of the set theoretic (von Neumann) universe.

Definition 3.1.9. We define the *cumulative hierarchy* by transfinite recursion as follows: Let $V_0 = \emptyset$. If $\alpha, \beta \in \mathbf{ON}$ such that $\alpha = \beta \cup \{\beta\} = \beta + 1$, then let $V_\alpha = \mathcal{P}(V_\beta)$. If α is a limit ordinal, then let $V_\alpha = \bigcup_{\gamma<\alpha} V_\gamma$. For each ordinal α, we call V_α the αth *stage* or *level* of the cumulative hierarchy. The *set theoretic universe*, denoted $\mathbf{V}$, is the proper class $\bigcup_{\alpha \in \mathbf{ON}} V_\alpha$. We sometimes call $\mathbf{V}$ the collection of all sets.

It is a fact that, for each $\alpha \in \mathbf{ON}$, V_α is a set and V_α is transitive. The universe $\mathbf{V}$ itself is transitive as well (although, of course, it is not a set). If $\alpha < \beta$, then $V_\alpha \subseteq V_\beta$. We may think of V_α as the result of "capping off the universe" at its αth stage, so that, for any ordinal β such that $\beta > \alpha$, no set belonging to $V_\beta \setminus V_\alpha$ appears in V_α. We note that V_α itself is not an element of V_α.

Every set $x \in \mathbf{V}$ appears at some stage of the cumulative hierarchy; hence, there is a least such stage.

Definition 3.1.10. For any $x \in \mathbf{V}$, the *rank of* x, denoted $\mathrm{rk}(x)$, is the least ordinal β such that $x \in V_{\beta+1}$.

The cumulative hierarchy and its stages are crucial in the study of large cardinals. For most large cardinal properties $P(x)$, if κ is a cardinal such that $P(\kappa)$, then V_κ is a (set[3]) model of ZFC; that is, each axiom of ZFC is true in V_κ. It follows from Gödel's second incompleteness theorem and our assumption that ZFC is consistent that ZFC does not prove the existence of such a κ. If it did, then it would prove the existence of a model V_κ of ZFC, from which it would follow by the soundness theorem of first-order logic that ZFC proves its own consistency.

This concludes our review of set theory. For more details, see Chapter I of [10].

3.2. **Inaccessible, Measurable, and Reinhardt Cardinals.** We begin our survey of large cardinals by briefly considering a cardinal that is not large at all — namely, ω.

The reason for this diversion is that, in many ways, ω "comes close" to being a large cardinal; see [7] for a detailed explication of this idea. We shall see that, if two of the large cardinal properties we consider here did not require uncountability, then ω itself would possess those properties. Of course, ω is not a large cardinal, since the statement that ω exists is not strong enough to prove the consistency of ZFC. Indeed, V_ω, the collection of all finite sets, is not a model of ZFC; the axiom of infinity fails in V_ω because no infinite sets appear in V_ω. However, we have the following theorem; see Chapter IV, Section 3 of [10] (note that this source writes $R(\omega)$ for V_ω).

Theorem 3.2.1. *Let ZFC − Inf denote the collection of all axioms of ZFC except the axiom of infinity. Then* $V_\omega \models \mathsf{ZFC} - \mathsf{Inf}$.

In this sense, V_ω is a microcosm of $\mathbf{V}$ in which every set is finite.

We identify two reasons why V_ω is special compared to V_α for an arbitrary ordinal α. First, V_ω easily satisfies the power set axiom; since the power set of any finite set is also finite and since V_ω is the collection of all finite sets, V_ω is closed under the power set operation. Above ω, however, the power set operation behaves unpredictably. Given infinite cardinals κ, λ with $\lambda < \kappa$, it is entirely possible for 2^λ to jump above κ. For instance, if CH fails, then $2^\omega > \omega_1$ even though $\omega < \omega_1$.

There are cardinals κ other than ω, however, such that V_κ is closed under the taking of power sets.

[3]If V_κ were a proper class, then it would not be a legitimate model of ZFC. We consider proper class "models" in Subsection 3.3.

Definition 3.2.2. Let κ be a limit cardinal. We say that κ is a *strong limit cardinal* if, for all $\lambda < \kappa$, we have $2^\lambda < \kappa$.

Therefore, if κ is an uncountable strong limit cardinal, then V_κ satisfies both the power set axiom and the axiom of infinity. However, without an additional assumption, we cannot be certain that V_κ satisfies all of ZFC; in particular, it may fail to satisfy the replacement scheme (see [6]).

The second reason why V_ω is special, then, is that it easily satisfies the replacement scheme; if $X \in V_\omega$ and $f : X \to V_\omega$ is a function, then $\operatorname{ran}(f)$ is finite, and so $\operatorname{ran}(f) \in V_\omega$. Thus, V_ω is closed under replacement. Replacement becomes more slippery above ω. For instance, consider ω_ω. Although $\omega \in V_{\omega_\omega}$ and the function $f : \omega \to V_{\omega_\omega}$ defined by $f(n) = \omega_n$ has range included in V_{ω_ω}, the set $\operatorname{ran}(f)$ is not an element of V_{ω_ω} (see Example 3.1.7).

To remedy this situation, we need our strong limit cardinals to possess one additional property — regularity.

Definition 3.2.3. Let κ be an uncountable strong limit cardinal. If in addition κ is regular, then κ is said to be *strongly inaccessible.*

Often we say merely 'inaccessible' for 'strongly inaccessible'. Notice that, if the word 'uncountable' were removed from the definition, then ω would be an inaccessible cardinal.

Happily, this solution is effective; if κ is inaccessible, then $V_\kappa \models \mathsf{ZFC}$. The axiom of infinity holds in V_κ because κ is uncountable, and our two troublemakers are handled summarily; the power set axiom is satisfied because κ is a strong limit cardinal, and the replacement scheme is satisfied because κ is regular. That V_κ also satisfies the other axioms of ZFC is shown in the standard way. See Chapter 1 of [9] for the full details.

Because the existence of an inaccessible cardinal provides us with a model of ZFC, it follows from the soundness theorem of first-order logic that

$$\mathsf{ZFC} + \text{“there exists an inaccessible cardinal”} \vdash \operatorname{Con}(\mathsf{ZFC}).$$

Hence, inaccessibility is a large cardinal property (see Definition 3.0.1).

We said earlier that large cardinals provide us with more information about $\mathbf{V}$ than ZFC does alone. When we assume the existence of an inaccessible cardinal, we learn that $\mathbf{V}$ contains a model of ZFC, which ZFC by itself cannot tell us, but we do not learn much more than that. To acquire a significantly greater amount of information about $\mathbf{V}$, we must seek stronger large cardinal axioms.

Although there are many ways in which to describe the strength of a large cardinal axiom, we shall restrict our attention to the following definition.

Definition 3.2.4. Suppose $P(x), Q(x)$ are two large cardinal properties, and suppose that

$$\operatorname{Con}(\mathsf{ZFC} + \text{“there exists a cardinal } \kappa \text{ such that } P(\kappa)\text{”})$$

proves

$$\operatorname{Con}(\mathsf{ZFC} + \text{“there exists a cardinal } \lambda \text{ such that } Q(\lambda)\text{”}).$$

Then we may say that $P(x)$ is *stronger than* $Q(x)$. Furthermore, if κ is a cardinal such that $P(\kappa)$ and λ is a cardinal such that $Q(\lambda)$, then we say that κ is stronger than λ.

Our next large cardinal property will be (much) stronger than inaccessibility. Its original formulation was motivated by measure theoretic considerations, but the characterization we give here is in terms of a sophisticated ultrafilter property. An ultrafilter possessing this property satisfies a strong closure condition.

Definition 3.2.5. Let κ be a cardinal. An ultrafilter $\mathcal{U}$ on an index set I is *κ-complete* if it is closed under intersections of size less than κ, that is, if, for any collection $\mathcal{C}$ of subsets of I such that $|\mathcal{C}| < \kappa$ and each $C \in \mathcal{C}$ is $\mathcal{U}$-large, then $\bigcap \mathcal{C} \in \mathcal{U}$.

Notice that any ultrafilter is ω-complete, since ultrafilters are closed under finite intersections. Also, for any cardinal κ, if $\mathcal{U}$ is a principal ultrafilter on I, then $\mathcal{U}$ is κ-complete: For any collection $\mathcal{C}$ of elements of $\mathcal{U}$, the singleton that generates $\mathcal{U}$ (see Example 2.1.9) is included in $\bigcap \mathcal{C}$. However, when κ is uncountable, the existence of a nonprincipal κ-complete ultrafilter cannot be proved from ZFC. In fact, the existence of such an ultrafilter coincides with the existence of a cardinal with a certain large cardinal property.

Definition 3.2.6. An uncountable cardinal κ is *measurable* if there exists a nonprincipal κ-complete ultrafilter on κ.

Notice that, again, if the word 'uncountable' were removed from the definition, then ω would be measurable: By AC, any nonprincipal filter on ω can be extended to a nonprincipal ultrafilter $\mathcal{U}$, and since all ultrafilters are ω-complete, $\mathcal{U}$ realizes the "measurability" of ω.

Measurable cardinals are large cardinals. In fact, every measurable cardinal κ is (strongly) inaccessible (see Chapter 4, Section 2 of [1]), and thus by the discussion following Definition 3.2.3, $V_\kappa \models \mathsf{ZFC}$. However, measurability is a much stronger property than inaccessibility, and so the axiom asserting the existence of a measurable cardinal provides us with more information about $\mathbf{V}$ than does each of ZFC and ZFC + "there exists an inaccessible cardinal". For instance, under the assumption that a measurable cardinal exists, we have the following result, which provides information about the number of inaccessibles in $\mathbf{V}$; see Chapter 4, Section 2 of [1] for a proof.

Theorem 3.2.7. *Suppose κ is a measurable cardinal. Then κ is the κth inaccessible cardinal.*

We shall see in Subsection 3.3 that the assumption that a measurable cardinal exists allows us to draw some very profound conclusions about $\mathbf{V}$ indeed.

There are plenty of large cardinal properties stronger than measurability; see, for instance, [9] or [15]. The reader may well wonder how high (that is, how strong) we can go. Can we ascend through the cardinal ranks indefinitely, forever obtaining stronger and stronger large cardinals? The answer is 'no'. Because strong large cardinals provide us with vast amounts of information about the set theoretic universe, it becomes "likelier" as strength increases that some of the information we are given by large cardinals is incompatible with the information that ZFC gives us. We may worry, then, that the axiom asserting the existence of a wildly strong large cardinal is inconsistent with ZFC.

This concern proves justified, as we shall see in a moment. First, we recall the following model theoretic definition.

Definition 3.2.8. Let $\mathcal{L}$ be a first-order language, and let $\mathcal{M}, \mathcal{N}$ be models for $\mathcal{L}$. An *elementary embedding* from $\mathcal{M}$ into $\mathcal{N}$ is an injective function $j : M \to N$ such that, for all $\mathcal{L}$-formulas $\varphi(x_0, x_1, \ldots, x_{n-1})$ and for all $y_0, y_1, \ldots, y_{n-1} \in M$,

$$\mathcal{M} \models \varphi(y_0, y_1, \ldots, y_{n-1})$$

if and only if

$$\mathcal{N} \models \varphi(j(y_0), j(y_1), \ldots, j(y_{n-1})).$$

Many large cardinal properties can be stated in terms of elementary embeddings; see, for instance, [2] and [7].

Definition 3.2.9. Let κ be a cardinal. Then κ is a *Reinhardt cardinal* if there exists an elementary embedding $j : \mathbf{V} \to \mathbf{V}$ such that $j(\kappa) > \kappa$.

Thus, if there exists a Reinhardt cardinal, then there is a non-trivial (that is, non-identity) elementary embedding of the universe $\mathbf{V}$ into itself. The axiom asserting the existence of a Reinhardt cardinal is exceptionally strong — so strong that it is incompatible with AC. It remains open whether the existence of a Reinhardt cardinal is compatible with ZF without choice.

3.3. **A Detour into Inner Model Theory.** We have seen that large cardinals provide us with information about models of ZFC in $\mathbf{V}$. In addition, strong large cardinals such as measurables provide information about the number of weaker large cardinals contained in $\mathbf{V}$. However, we can glean much more meaningful information about $\mathbf{V}$ from strong large cardinals; this is the focus of the present subsection.

The following definition requires knowledge of definability: Briefly, if X, Y are sets, then Y is *definable from* X if there exists some formula $\varphi(x)$ in the language of set theory (possibly with parameters from X) such that $Y = \{y \in X : \varphi(y)\}$. See the appendix of [12] for a more general characterization of definability.

Definition 3.3.1. We define the *constructible hierarchy* by transfinite recursion on ordinals as follows: Let $L_0 = \emptyset$. If $\alpha, \beta \in \mathbf{ON}$ such that $\alpha = \beta + 1$, then let $L_\alpha = \{X \subseteq L_\beta : X \text{ is definable from } L_\beta\}$. If α is a limit ordinal, then let $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$. The *constructible universe*, denoted $\mathbf{L}$, is the proper class $\bigcup_{\alpha\in\mathbf{ON}} L_\alpha$.

For each $\alpha \in \mathbf{ON}$, L_α is a transitive set; also, $\mathbf{L}$ is a transitive class, and $\mathbf{L}$ includes $\mathbf{ON}$ as a subclass.

It can be shown that $\mathbf{L}$ is a legitimate arena in which to study set theory, in the sense that each axiom of ZFC is true in $\mathbf{L}$ (see Chapter VI, Sections 2 and 4 of [10]). It is not quite correct to say that $\mathbf{L}$ is a model of ZFC, since $\mathbf{L}$ is not a set. However, $\mathbf{L}$ behaves in many ways like a model of ZFC. The following definition allows us to speak unproblematically about the model theoretic properties of $\mathbf{L}$.

Definition 3.3.2. Let $\mathbf{M}$ be a transitive proper class such that each axiom of ZFC holds true in $\mathbf{M}$ and $\mathbf{ON}$ is a subclass of $\mathbf{M}$. Then $\mathbf{M}$ is an *inner model* of ZFC.

Hence, $\mathbf{L}$ is an inner model of ZFC. Another example of an inner model of ZFC is $\mathbf{V}$.

The similarities between $\mathbf{L}$ and $\mathbf{V}$, as well as the similarities between the constructible and cumulative hierarchies, are apparent from the definitions. The two hierarchies differ at successor stages; for any ordinal β, whereas all subsets of V_β are contained in $V_{\beta+1}$, only the definable subsets of L_β are contained in $L_{\beta+1}$. This makes $\mathbf{L}$ "thinner" than $\mathbf{V}$. In fact, the following is true; see Chapter VI, Section 3 of [10] for a proof.

Theorem 3.3.3. *Let* $\mathbf{M}$ *be any inner model of ZFC. Then* $\mathbf{L}$ *is a subclass of* $\mathbf{M}$.

So $\mathbf{L}$ is "thinner" than any other inner model of ZFC.

Now, we must provide a disclaimer: It is entirely possible that $\mathbf{V}$ and $\mathbf{L}$ are one and the same, so that these comments about "thinness" are not quite correct; if $\mathbf{V}$ were equal to $\mathbf{L}$, then all inner models of ZFC would be equally "thin" by the previous theorem. By itself, ZFC does not give us enough information about the universe to determine whether or not all sets are constructible. It is thus consistent with ZFC that $\mathbf{V} = \mathbf{L}$. Indeed, it is sometimes useful to work under this assumption, because it follows from $\mathbf{V} = \mathbf{L}$ that the (generalized) continuum hypothesis is true (see Chapter VI, Section 4 of [10]). However, that $\mathbf{V} = \mathbf{L}$ does not seem "likely". We are straying here dangerously deep into a philosophical minefield, and we shall not pursue the issue any further. The interested reader should consult [11] for more details.

By appending certain axioms to ZFC, we can prove either $\mathbf{V} = \mathbf{L}$ or $\mathbf{V} \neq \mathbf{L}$. In particular, if we assume the existence of a measurable cardinal, then we can show that $\mathbf{V} \neq \mathbf{L}$; the presence of a measurable cardinal implies that $\mathbf{V}$ is "too wide" to align with $\mathbf{L}$. The proof of this remarkable fact, which is known as *Scott's theorem* (Theorem 3.3.12), makes use of an ultrapower of a proper class. Such an ultrapower behaves similarly to an ordinary ultrapower. However, the equivalence classes we ordinarily use to form the product (see Definition 2.2.1) might in this context be too large to be sets. Hence, we replace each equivalence class with a pseudo-equivalence class having a restriction that guarantees that it is a set, as follows.

Definition 3.3.4. Let κ be a measurable cardinal, and let $\mathcal{U}$ be a nonprincipal κ-complete ultrafilter on κ. Let $\prod_{i\in\kappa}\mathbf{V}$ be the collection of all functions (in $\mathbf{V}$) with domain κ, that is, $\mathbf{V}^\kappa$.[4] As in Definition 2.2.1, for any $f, g \in \mathbf{V}^\kappa$, say that $f \sim_{\mathcal{U}} g$ if $\{i \in \kappa : f(i) = g(i)\} \in \mathcal{U}$. Define the restricted equivalence class of f modulo $\sim_{\mathcal{U}}$ by

$$[f] = \{g \in \mathbf{V}^\kappa : g \sim_{\mathcal{U}} f \text{ and if } h \in \mathbf{V}^\kappa \text{ such that } f \sim_{\mathcal{U}} h, \text{ then } \mathrm{rk}(g) \leq \mathrm{rk}(h)\}.$$

Then the *ultrapower of* $\mathbf{V}$ *by* $\mathcal{U}$, denoted $\prod_{\mathcal{U}}\mathbf{V}$, is the proper class $\{[f] : f \in \mathbf{V}^\kappa\}$ of $\sim_{\mathcal{U}}$-(restricted) equivalence classes.

We interpret $\in$ in $\prod_{\mathcal{U}}\mathbf{V}$ as the binary relation $\mathbf{E}$ such that, for any $f, g \in \mathbf{V}^\kappa$,

$$[f]\mathbf{E}[g] \text{ if and only if } \{i \in \kappa : f(i) \in g(i)\} \in \mathcal{U}.$$

Since, for any $f \in \mathbf{V}^\kappa$, the ranks (see Definition 3.1.10) of the elements of $[f]$ are bounded, $[f]$ is guaranteed to be a set. This fact is needed for the proof of Lemma 3.3.9.

It is a fact that the definition of $\mathbf{E}$ is independent of the choices of representative. With this interpretation of $\in$, we have an analogue of Łoś's Theorem (see Theorem 2.3.1) for the ultrapower of $\mathbf{V}$ by $\mathcal{U}$. We shall abuse notation and write merely $\prod_{\mathcal{U}}\mathbf{V}$ in place of $(\prod_{\mathcal{U}}\mathbf{V}, \mathbf{E})$ and $\mathbf{V}$ in place of $(\mathbf{V}, \in)$.

[4] The reader may wonder whether the collection $\mathbf{V}^\kappa$ exists, since $\mathbf{V}$ is the universe of all sets and $\mathbf{V}^\kappa$ appears to be an "expansion" of $\mathbf{V}$. However, $\mathbf{V}^\kappa$ is in fact a subclass of $\mathbf{V}$; there are plenty of sets in $\mathbf{V}$ that are not functions with domain κ.

Theorem 3.3.5 (Łoś's theorem for proper classes)**.** *Let κ and $\mathcal{U}$ be as above. Then for any formula $\varphi(x_0, x_1, \ldots, x_{n-1})$ of the language of set theory with n free variables and for any $f_0, f_1, \ldots, f_{n-1} \in \mathbf{V}^\kappa$,*

$$\prod_{\mathcal{U}} \mathbf{V} \models \varphi([f_0], [f_1], \ldots, [f_{n-1}])$$

if and only if

$$\{i \in \kappa : \mathbf{V} \models \varphi(f_0(i), f_1(i), \ldots, f_{n-1}(i))\} \in \mathcal{U}.$$

(Here our use of the symbol $\models$ might be viewed as an abuse of notation, since the modeling relation is only defined when the modeling object is a set. For a sentence φ, read "$\mathbf{V} \models \varphi$" as "$\varphi$ is true in $\mathbf{V}$", not "$\mathbf{V}$ models φ".)

It follows immediately from Theorem 3.3.5 that, if φ is a sentence of the language of set theory, then $\prod_{\mathcal{U}} \mathbf{V} \models \varphi$ if and only if $\mathbf{V} \models \varphi$. Hence, the universe $\mathbf{V}$ and the ultrapower $\prod_{\mathcal{U}} \mathbf{V}$ are elementarily equivalent.

In fact, $\mathbf{V}$ and $\prod_{\mathcal{U}} \mathbf{V}$ are more than merely elementarily equivalent; it is also true that $\mathbf{V}$ elementarily embeds in $\prod_{\mathcal{U}} \mathbf{V}$. See [13] for a proof of the following corollary.

Corollary 3.3.6. *Let κ and $\mathcal{U}$ be as in the statement of Theorem 3.3.5, and let $\mathbf{E}$ be the binary relation defined in Definition 3.3.4. There exists an elementary embedding $i : (\mathbf{V}, \in) \to (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E})$ defined by $i(x) = [c_x]$, where c_x is the constant function from κ to $\mathbf{V}$ with range $\{x\}$.*

We shall need the following definitions and lemma.

Definitions 3.3.7. Let $\mathbf{A}$ be a proper class, and let $\mathbf{R}$ be a relation on $\mathbf{A}$.

(1) We say that $\mathbf{R}$ is *well-founded* on $\mathbf{A}$ if, for all nonempty sets $X \subseteq \mathbf{A}$, there exists an element $x \in X$ such that there is no element $y \in X$ with $y\mathbf{R}x$.
(2) We say that $\mathbf{R}$ is *set-like* on $\mathbf{A}$ if, for all $x \in \mathbf{A}$, the collection $\{y \in \mathbf{A} : y\mathbf{R}x\}$ is a set.
(3) We say that $\mathbf{R}$ is *extensional* on $\mathbf{A}$ if

$$\forall x, y \in \mathbf{A}(\forall z \in \mathbf{A}(z\mathbf{R}x \leftrightarrow z\mathbf{R}y) \to x = y).$$

Equivalently, $\mathbf{R}$ is extensional if, whenever $x, y \in \mathbf{A}$ and $x \neq y$, then there exists $z \in \mathbf{A}$ such that either

$$z\mathbf{R}x \text{ and } \neg(z\mathbf{R}y)$$

or

$$z\mathbf{R}y \text{ and } \neg(z\mathbf{R}x).$$

Thus, if $\mathbf{R}$ is well-founded on $\mathbf{A}$, then every nonempty set $X \subseteq \mathbf{A}$ has an $\mathbf{R}$-minimal element. If in addition $\mathbf{R}$ is set-like on $\mathbf{A}$, then every nonempty class $\mathbf{X}$ included in $\mathbf{A}$ has an $\mathbf{R}$-minimal element (see Chapter III, Section 5 of [10]).

Example 3.3.8. By the axiom of foundation, the relation $\in$ is well-founded on $\mathbf{V}$; by the axiom of extensionality, it is also extensional. It can be shown that $\in$ is set-like on $\mathbf{V}$ as well.

The relation $\mathbf{E}$ described in Definition 3.3.4 is very similar to the membership relation, as the following lemma shows; the proof uses the fact that the restricted equivalence classes of the relation $\sim_{\mathcal{U}}$ are sets (and not proper classes). See [13] for some additional details.

Lemma 3.3.9. *If κ, $\mathcal{U}$, and* $\mathbf{E}$ *are as in Definition 3.3.4, then the relation* $\mathbf{E}$ *is well-founded, set-like, and extensional on* $\prod_{\mathcal{U}} \mathbf{V}$.

We now state an important lemma that allows us to canonically identify a set equipped with a relation that behaves like the membership relation with a transitive set equipped with the "true"[5] membership relation. This lemma will be crucial for the proof of Scott's theorem. See Chapter III, Section 5 of [10] for a proof.

Lemma 3.3.10 (Mostowski's collapsing theorem)**.** *Let* $\mathbf{A}$ *be a proper class, and let* $\mathbf{R}$ *be a well-founded, set-like, extensional relation on* $\mathbf{A}$. *Then there exists a unique transitive proper class* $\mathbf{M}$ *and a unique isomorphism* $\mathbf{G} : (\mathbf{A}, \mathbf{R}) \to (\mathbf{M}, \in)$.

For $\mathbf{A}, \mathbf{M}$, and $\mathbf{G}$ as in the statement of the theorem, we call $\mathbf{M}$ the *Mostowski collapse* of $\mathbf{A}$ and $\mathbf{G}$ the *collapsing isomorphism.*

We now quote a rich result relating the ultrafilter characterization of measurability to a model theoretic characterization. Before stating the result, we note that, since the relation $\mathbf{E}$ is well-founded, set-like, and extensional on $\prod_{\mathcal{U}} \mathbf{V}$ (see Lemma 3.3.9), it follows from Lemma 3.3.10 that there exist a transitive proper class $\mathbf{M}$ and an isomorphism $\mathbf{G} : (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E}) \to (\mathbf{M}, \in)$.

Lemma 3.3.11. *Let κ be a measurable cardinal, let $\mathcal{U}$ be a κ-complete ultrafilter on κ, let* $i : (\mathbf{V}, \in) \to (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E})$ *be as in the statement of Corollary 3.3.6, and let* $\mathbf{M}$ *and* $\mathbf{G} : (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E}) \to (\mathbf{M}, \in)$ *be as above. If* $j = \mathbf{G} \circ i$, *then* $j(\kappa) > \kappa$.[6]

At last, we have the tools needed to prove that measurable cardinals imply that $\mathbf{V}$ is very large.

Theorem 3.3.12 (Scott's theorem)**.** *Suppose there exists a measurable cardinal. Then* $\mathbf{V} \neq \mathbf{L}$.

Proof. We follow the proof in [13]. Let κ be the least measurable cardinal of $\mathbf{V}$, and let $\mathcal{U}$ be a nonprincipal κ-complete ultrafilter on κ. Form the ultrapower $\prod_{\mathcal{U}} \mathbf{V}$ of $\mathbf{V}$ by $\mathcal{U}$, and interpret $\in$ in $\prod_{\mathcal{U}} \mathbf{V}$ as the relation $\mathbf{E}$ of Definition 3.3.4. Further, let $\mathbf{M}$ and $\mathbf{G} : (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E}) \to (\mathbf{M}, \in)$ be as in the statement of Lemma 3.3.11.

Now, assume for contradiction that $\mathbf{V} = \mathbf{L}$. Define $i : (\mathbf{V}, \in) \to (\prod_{\mathcal{U}} \mathbf{V}, \mathbf{E})$ as in Corollary 3.3.6. Then $j = \mathbf{G} \circ i$ is an elementary embedding carrying $(\mathbf{V}, \in)$ into $(\mathbf{M}, \in)$. Since $\mathbf{V} = \mathbf{L}$, it follows from Theorem 3.3.3 that $\mathbf{V} = \mathbf{M}$ as well.

Let $\varphi(x)$ be the formula in the language of set theory expressing that x is the least measurable cardinal; by assumption, $\mathbf{V} \models \varphi(\kappa)$. By elementarity of j, $\mathbf{M} \models \varphi(j(\kappa))$, and by Lemma 3.3.11, $j(\kappa) > \kappa$. However, since $\mathbf{V} = \mathbf{M}$ and $\mathbf{M} \models \varphi(j(\kappa))$, then also $\mathbf{V} \models \varphi(j(\kappa))$; that is, in $\mathbf{V}$, $j(\kappa)$ is the least measurable cardinal. But $\kappa < j(\kappa)$ is a smaller measurable cardinal.

Hence, $\mathbf{V} \neq \mathbf{L}$. □

There are theorems of the same flavor for other large cardinals. One particularly interesting such theorem involves the large cardinal axiom $0^{\#}$ (pronounced 'zero-sharp'), which concerns indiscernible sequences. The statement of this axiom alone is beyond the scope of this paper, but the interested reader should see Chapter 2 of [9].

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[5]With apologies to formalist readers.

[6]We say in this situation that κ is the critical point of j.

We mentioned at the beginning of this section that no presently defined large cardinal property decides CH. The reader might wonder, then, how to arrive at the independence of the continuum hypothesis. Gödel proved that CH is true in $\mathbf{L}$, and since it is consistent with ZFC that $\mathbf{V} = \mathbf{L}$ (see the discussion following Theorem 3.3.3), $\mathsf{ZFC} + \mathsf{CH}$ is consistent. In the next section, we show a powerful method by which to obtain the consistency of $\mathsf{ZFC} + \neg\mathsf{CH}$.

4. A Crash Course in Forcing

When we work with large cardinals, we begin with assumptions about the universe $\mathbf{V}$ and use these assumptions to conclude the existence of models — "sub-universes" of $\mathbf{V}$ — in which various statements of interest hold true. For instance, given the assumption that a measurable cardinal κ exists, we conclude that there is a set-sized sub-universe, namely V_κ, in which each axiom of ZFC, together with the statement "there exists an inaccessible cardinal", is true.

With *forcing*, we achieve similar results in more-or-less the opposite way: We begin with a set-sized model M of ZFC and extend M to a larger model in which various statements of interest, such as $\neg$CH, are "forced" to be true. As we shall see, the method of forcing is quite powerful, to the extent that it can be used to prove that spectacular failures of the generalized continuum hypothesis (GCH) are consistent with ZFC.

In this section, we encounter a generalization of the filter notion that is crucial in forcing. When we consider interactions between forcing and large cardinals in Subsection 4.4, we shall use both these generalized filters and true ultrafilters (that is, ultrafilters in the sense of Definition 2.1.6) to study a deep result about singular cardinals; see Theorem 4.4.9.

For the remainder of this section, assume ZFC is consistent, and fix a countable transitive model M of ZFC.[7]

4.1. Essentials of Forcing.

The main ingredient of any forcing argument is a cleverly chosen partial order $\mathbb{P}$ in M. Some subsets of $\mathbb{P}$ may not be contained in M. If this seems counterintuitive, recall that, from the perspective of $\mathbf{V}$, M is countable while $\mathcal{P}(M)$ is uncountable. Thus, M cannot possibly contain all of its subsets. In fact, we shall prove (see Lemma 4.1.11) that, as long as $\mathbb{P}$ meets some mild conditions, we shall always be able to choose a subset G of $\mathbb{P}$ that falls outside of M. By adjoining G to M, we thus create a proper extension $M[G] \supsetneq M$, where $M[G]$ is the smallest transitive model of ZFC that includes M and contains G.

On the basis of the previous paragraph, the reader may already have intuited that we must appreciate the limited perspective M has as a small sub-universe of $\mathbf{V}$; for instance, as we have just remarked, M does not witness all of its subsets. To understand what M sees, does not see, and sees but in the wrong light, we first need some familiarity with the notion of *absoluteness*.

Definition 4.1.1. Let N_0, N_1 be (set) models of ZFC such that $N_0 \subseteq N_1$. A formula $\varphi(x_0, x_1, \ldots, x_{n-1})$ of the language of set theory with n free variables is

[7] Such an M exists because, under the assumption that ZFC is consistent, the completeness theorem ensures that there exists a (set) model of ZFC. By the Löwenheim-Skolem theorem, there then exists a countable model of ZFC, which can be Mostowski-collapsed (see Lemma 3.3.10) to a transitive model.

absolute for N_0 *and* N_1 if, whenever $y_0, y_1, \dots, y_{n-1} \in N_0$,

$$N_0 \models \varphi(y_0, y_1, \dots, y_{n-1})$$

if and only if

$$N_1 \models \varphi(y_0, y_1, \dots, y_{n-1}).$$

Furthermore, we say that $\varphi(x_0, x_1, \dots, x_{n-1})$ is *absolute for* N_0 if, whenever $y_0, y_1, \dots, y_{n-1} \in N_0$,

$$N_0 \models \varphi(y_0, y_1, \dots, y_{n-1})$$

if and only if

$$\mathbf{V} \models \varphi(y_0, y_1, \dots, y_{n-1}).$$

If a formula is absolute for our fixed model M, we shall often say merely that the formula is absolute.

Suppose N is a transitive model of ZFC such that $M \subseteq N$. When a formula is absolute for M and N, then from the perspective of N, M is "always right" about that formula. For instance, let $\varphi(x)$ be the (formalization of the) expression 'x is uncountable', and suppose $M \models \varphi(y)$, where $y \in M$. If $\varphi(x)$ is absolute for M and N, then $N \models \varphi(y)$ as well, and so, from the perspective of N, M is right to believe that y is uncountable. However, it is possible that, from the perspective of the universe $\mathbf{V}$, both M and N are wrong in believing that $\varphi(y)$. That is, although M and N think that y is uncountable, they are mistaken; "in reality"[8], y is countable.

We can be assured, however, that M is genuinely right about a formula (that is, that M is right about a formula from the perspective of $\mathbf{V}$) if that formula is absolute for M. For instance, suppose $\varphi(x)$ is the (formalization of the) expression 'x is a transitive set'. Then $\varphi(x)$ is absolute for M (see Lemma 4.1.4), and so if $y \in M$ such that $M \models \varphi(y)$, then $\mathbf{V} \models \varphi(y)$ as well. Thus, M is genuinely right to believe that y is a transitive set.

If $s \in \mathbf{V}$ and $\varphi(x)$ is the formula $x = s$, then we write s^M, or $(s)^M$ when appearance demands, for the element $y \in M$ such that $M \models \varphi(y)$. For instance, consider the ordinal ω_2, and let $\varphi(x)$ be the formula $x = \omega_2$. We write $(\omega_2)^M$ for the element y of M such that $M \models \varphi(y)$.

There is a helpful condition for absoluteness of a formula that depends on the complexity of the formula in question. Very simple formulas are absolute for any transitive models of ZFC. We capture a notion of simplicity with the following definitions.

Definitions 4.1.2. An existential quantifier is *bounded* if it ranges over a restricted domain, that is, if it appears in an expression of the form $(\exists x \in y)$, where y is a set. Similarly, a universal quantifier is *bounded* if it appears in an expression of the form $(\forall x \in y)$, where y is a set. A quantifier that is not bounded is said to be *unbounded*.

Say that a formula $\varphi(x_0, x_1, \dots, x_{n-1})$ in the language of set theory with n free variables is Δ_0^0 if either of the following holds:

(1) It possesses no quantifiers.
(2) Each of its quantifiers is bounded.

[8]With further apologies to formalist readers.

In the symbol Δ_0^0, the subscript 0 is used to mean that the formula has no unbounded quantifiers, while the superscript 0 is used to mean that the formula does not involve quantification over higher-order objects. None of our formulas features quantifiers that range over higher-order objects.

If a set theoretic object can be described via a Δ_0^0 formula $\varphi(x)$ with one free variable (possibly with parameters), then given any object y in the universe $\mathbf{V}$, we need not look outside of y to determine whether $\varphi(y)$; because all of the quantifiers in $\varphi(x)$ are bounded, the state of the universe outside of y is irrelevant to the truth of $\varphi(y)$.

Example 4.1.3. We mentioned after Definition 3.1.4 that we can determine whether a set x is an ordinal without looking outside of x, whereas we must look outside of x to determine whether x is a cardinal and to determine whether x is regular. Now we can formalize this distinction; the formula expressing that x is an ordinal is Δ_0^0, while the formula expressing that x is a cardinal and the formula expressing that x is regular are not Δ_0^0.

Notice that, although ordinality — that is, "being an ordinal" — is absolute, ordinals themselves are in general not absolute. For instance, if N is a transitive model of ZFC including M, then N believes that $(\omega_2)^M$ is an ordinal, but it is not necessarily the case that $(\omega_2)^M$ is equal to $(\omega_2)^N$.

We now state a lemma that relates Δ_0^0 formulas and absoluteness; see Chapter IV, Section 3 of [10] for proofs of the lemma and the corollary that follows.

Lemma 4.1.4. *Let N_0, N_1 be transitive models of ZFC such that $N_0 \subseteq N_1$, and let $\varphi(x_0, x_1, \ldots, x_{n-1})$ be a Δ_0^0 formula with n free variables. Then $\varphi(x_0, x_1, \ldots, x_{n-1})$ is absolute for N_0 and N_1.*

Corollary 4.1.5. *The (formalizations of the) following formulas are Δ_0^0 and hence are absolute for any transitive model N of ZFC:*

(1) $y = \bigcup x$.
(2) $y = x \setminus z$.
(3) α *is an ordinal.*
(4) x *is finite.*
(5) $x = 0$, $x = 1$, $x = 2$, $\ldots$.
(6) $x = \omega$.

It follows from clause (6) of this corollary that, from the perspective of $\mathbf{V}$, what M believes to be ω is in fact ω, that is, $\omega^M = \omega$. This corollary therefore permits us to suppress the superscripts for ω. We shall also suppress superscripts for finite sets and for the cardinals $0, 1, 2, \ldots$.

Notably absent from the above corollary is mention of absoluteness for cardinals. Indeed, cardinality is *not* absolute (see Example 4.1.3). In general, if N is a model of ZFC including M, then N may see that some sets thought by M to be cardinals are merely ordinals. Because the purpose of forcing is to effect changes in the realm of the non-absolute, forcing can greatly alter the cardinalities of sets, as we shall see in Subsection 4.2.

Recall that partial orders are the main ingredients of forcing. We now develop some terminology associated with partial orders.

Definition 4.1.6. Let $\mathbb{P} = (P, \leq)$ be a partial order. If $p, q \in P$ such that there does not exist $r \in P$ where $r \leq p$ and $r \leq q$, then p and q are $\mathbb{P}$-*incompatible*, denoted $p \perp_{\mathbb{P}} q$. Otherwise, p and q are $\mathbb{P}$-*compatible*, denoted $p \parallel_{\mathbb{P}} q$.

When the partial order $\mathbb{P}$ is understood, we say merely 'incompatible' and 'compatible' and drop the subscript $\mathbb{P}$.

Note that, if $\mathbb{P}$ is a linear order, then any two elements of P are $\mathbb{P}$-compatible. Because we shall prefer for our partial orders to exhibit properties dependent on the presence of incompatible elements (see Definition 4.1.10), we shall disregard linear orders.

We now give a generalization of the filter concept.

Definition 4.1.7. Let $\mathbb{P} = (P, \leq)$ be a partial order in M. A nonempty subset F of P is a *filter on* $\mathbb{P}$ if the following hold:

(1) For all $p, q \in F$, there exists $r \in F$ such that $r \leq p$ and $r \leq q$.
(2) Whenever $p \in F$ and $q \in P$ such that $p \leq q$, $q \in F$.

Notice that condition (1) implies that any two elements of a filter are compatible. Furthermore, for any set X, $(\mathcal{P}(X), \subseteq)$ is a partial order; over this order, filters in the sense of Definition 4.1.7 are also filters in the sense of Definition 2.1.1. Thus, our use of the word 'generalization' is justified. Here, we use the symbol F instead of $\mathcal{F}$ to emphasize that the filters we are considering are sets, not proper classes.

If $\mathbb{P}$ is a partial order in M and F is a filter on $\mathbb{P}$, then $F \subseteq M$. Thus, by the discussion at the beginning of this subsection, F need not be contained in M; in fact, we shall often prefer that $F \notin M$. Our next two definitions will allow us to distinguish a collection of nontrivial filters that typically fall outside of M (see Lemma 4.1.11).

Definition 4.1.8. Let $\mathbb{P} = (P, \leq)$ be a partial order. A set $D \subseteq P$ is *dense in* $\mathbb{P}$ if, for all $p \in P$, there exists $q \in D$ such that $q \leq p$.

Dense sets are therefore found "at the bottom" of the order. Notice that, for any partial order $\mathbb{P} = (P, \leq)$, P itself is dense in $\mathbb{P}$. Also, for any $p \in P$, the set $D = \{q \in P : q \leq p \text{ or } q \perp p\}$ is dense in $\mathbb{P}$.

Definition 4.1.9. Let $\mathbb{P} = (P, \leq)$ be a partial order, and let G be a filter on $\mathbb{P}$. Suppose that, for all dense sets $D \subseteq P$ such that $D \in M$, $G \cap D$ is nonempty. Then G is said to be a $\mathbb{P}$-*generic filter over* M.

Because generic filters are guaranteed to meet dense sets, we can extract a great deal of information about generic filters and generic objects associated with them; see the discussion following Definition 4.2.1.

Now, we show that, as long as the partial order on which a filter G is generic over M meets a mild condition, then $G \notin M$.

Definition 4.1.10. Let $\mathbb{P} = (P, \leq)$ be a partial order. Suppose that, for all $p \in P$, there exist $q, r \in P$ such that $q \leq p$, $r \leq p$, and $q \perp r$. Then $\mathbb{P}$ is said to be *separative*.

Lemma 4.1.11. *Let $\mathbb{P} = (P, \leq)$ be a separative partial order. If G is a $\mathbb{P}$-generic filter over M, then $G \notin M$.*

Proof. Suppose that G is a filter and $G \in M$. By Corollary 4.1.5(2), the set $(\mathbb{P} \backslash G)^M$ is the true $\mathbb{P} \setminus G$, and so $\mathbb{P} \setminus G$ is contained in M. Since $\mathbb{P}$ is separative, and since any two elements of G are compatible, the set $\mathbb{P} \setminus G$ is dense in $\mathbb{P}$. However, $G \cap (\mathbb{P} \setminus G) = \emptyset$. It follows that G is not $\mathbb{P}$-generic over M. □

We remark that, if $\mathbb{P}$ is not separative, then there exists $p \in P$ such that, for all $q, r \in P$, if $q \leq p$ and $r \leq p$, then $q \parallel r$. It follows that $G = \{q \in P : q \parallel p\}$ is a $\mathbb{P}$-generic filter over M that intersects every dense subset D of P, regardless of whether $D \in M$. Thus, $G \in M$.

Now we know that it is possible to find filters G on partial orders in M such that $G \notin M$; therefore, by adjoining G to M, we obtain a proper extension $M[G]$ of M. We sometimes call $M[G]$ a *generic extension* of M. One may worry that $M[G]$ fails to satisfy some of ZFC; happily, it can be shown that $M[G]$ is a countable transitive model of ZFC (see Chapter VII, Section 4 of [10]).

4.2. **Cohen Forcing and the Continuum Hypothesis.** We now define a partial order that will enable us to produce a model of ZFC in which $\neg$CH holds. When considered in a forcing context, partial orders are called *notions*.

Recall that a *partial function* $f : X \to Y$ is a function such that $\mathrm{dom}(f) \subseteq X$.

Definition 4.2.1. Let $\mathrm{Fn}(I, J)$ be the set of all partial functions $f : I \to J$ such that $|f| < \omega$. Define a partial ordering $\preceq$ of $\mathrm{Fn}(I, J)$ by $f \preceq g$ if g extends f as a function, that is, if $g \supseteq f$. Then the partial order $(\mathrm{Fn}(I, J), \preceq)$ is called the *Cohen notion*.

A word of caution: Ordering relations are often "backwards" in forcing. Notice that $f \preceq g$ if g, the $\preceq$-greater element, is included in f. In some texts, this convention is reversed.

When the sets I, J belong to M, the set $\mathrm{Fn}(I, J)$ is absolute for M. This follows from Corollary 4.1.5(4) and the fact that any element of $\mathrm{Fn}(I, J)$ is finite.

Following the treatment in [10], we shall take a cardinal κ in M and let $I = \kappa \times \omega$, $J = \{0, 1\} = 2$. Then we shall be able to produce a generic extension of M in which $2^\omega \geq \kappa$. Semi-formally, the reason is that, if G is $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over M, then $\bigcup G$ is a function from $\kappa \times \omega$ to 2. We may then treat $\bigcup G$ as a function that assigns, to each $\alpha \in \kappa$, a function f_α from ω to 2; genericity of G then allows us to conclude that, if $\alpha, \beta \in \kappa$ and $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$. This is because the set of all elements $p \in \mathrm{Fn}(\kappa \times \omega, 2)$ such that there exists $n \in \omega$ for which the following hold:

(1) The ordered pair $\langle \alpha, n \rangle$ is contained in $\mathrm{dom}(p)$.
(2) The ordered pair $\langle \beta, n \rangle$ is contained in $\mathrm{dom}(p)$.
(3) The values $p(\langle \alpha, n \rangle)$ and $p(\langle \beta, n \rangle)$.

is dense in $\mathrm{Fn}(\kappa \times \omega, 2)$. Now, since $G \in M[G]$, $\bigcup G \in M[G]$ (see Corollary 4.1.5(1)), and so we very nearly have at least κ-many distinct functions from ω to 2 in $M[G]$.

However, we must not be too hasty. Recall that cardinality is not absolute. Thus, what we really have are $|\kappa|^{M[G]}$-many distinct functions from ω to 2. Although, in M, the cardinality of κ is κ since κ is a cardinal in M, $|\kappa|^{M[G]}$ may in general fail to be κ; κ will remain an ordinal in $M[G]$, but it may not be a cardinal.[9]

To ensure that $M[G]$ never believes that M is wrong about which ordinals are cardinals, we shall require that $M[G]$ be built from M using a partial order that does not shrink the cardinals of M. In this way, we guarantee that (so to speak) the

[9] As we shall see, this does not happen in the case of Cohen forcing, but in general, extensions of M will collapse cardinals in this way.

problem with M is not that it has the wrong "measuring stick" (cardinals) entirely. Instead, if M satisfies CH and $M[G]$ is a forcing extension of M in which CH fails, then the problem is that M merely misses some of the sets that contribute to the "length" (cardinality) of 2^ω.

Definition 4.2.2. Let $\mathbb{P}$ be a partial order in M, and let G be a $\mathbb{P}$-generic filter over M. Then $\mathbb{P}$ *preserves cardinals* if, whenever κ is a cardinal in M, then κ is also a cardinal in $M[G]$.

When we know that M and $M[G]$ have the same cardinals, we write merely $|\kappa|$ for both $|\kappa|^M$ and $|\kappa|^{M[G]}$.

Cofinality is not absolute, either, and so we must make similar comments about it. The following definition, while useful, is not strictly necessary at this point. However, we shall be interested in the cofinalities of singular cardinals in Subsection 4.4.

Definition 4.2.3. Let $\mathbb{P}$ be a partial order in M, and let G be a $\mathbb{P}$-generic filter over M. Then $\mathbb{P}$ *preserves cofinalities* if, for all cardinals κ, $(\mathrm{cf}(\kappa))^M = (\mathrm{cf}(\kappa))^{M[G]}$.

The following lemma provides a useful relationship between these previous two definitions. See Chapter VII, Section 5 of [10] for a proof.

Lemma 4.2.4. *Let $\mathbb{P}$ be a partial order in M. If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.*

There is a convenient condition guaranteeing that a partial order preserves cofinalities and thus cardinals. We first need the following two definitions.

Definition 4.2.5. Let $\mathbb{P} = (P, \leq)$ be a partial order. If $A \subseteq P$ such that any two elements of A are incompatible, then A is an *antichain* of $\mathbb{P}$. The *length* of an antichain is its cardinality.

Definition 4.2.6. A partial order $\mathbb{P}$ meets the *countable antichain condition* if no antichain in $\mathbb{P}$ has length exceeding ω.

Any countable partial order therefore meets the countable antichain condition. Relevant for us is the fact that $\mathrm{Fn}(\kappa \times \omega, 2)$ meets this condition as well; the proof uses a combinatorial lemma from which it follows that uncountable subsets of this partial order contain (uncountably) many compatible elements, so that long antichains are forbidden. See Chapter VII, Section 5 of [10] for the full details.

Lemma 4.2.7. *Let $\mathbb{P} = (P, \leq)$ be a partial order in M that meets the countable antichain condition. Then $\mathbb{P}$ preserves cofinalities.*

Now, since we know that $\mathrm{Fn}(\kappa \times \omega, 2)$ meets the countable antichain condition, we can conclude that it preserves cofinalities; therefore, it preserves cardinals. Taking $\kappa = (\omega_2)^M$ and letting G be $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over M, we find by revisiting the discussion following Definition 4.2.1 that $M[G]$ contains at least $|\kappa|^{M[G]} = \kappa = \omega_2$ distinct functions from ω to 2. Hence, in $M[G]$, 2^ω is at least ω_2, and we have a model of ZFC in which CH fails.

By developing the theory of so-called 'names' and placing one additional condition on κ, we can force equality; that is, we can force $2^\omega = \omega_2$. Details can be found in Chapter VII, Section 5 of [10].

4.3. **Easton Forcing and the Generalized Continuum Hypothesis.** Armed with the techniques of the last section, we now make good on our promise to cause failure of not only CH but also GCH. In order to force failures of GCH with great generality, we shall need a new partial order. The definition of this partial order, however, involves a modification of the Cohen notion.

Definition 4.3.1. Let λ be an infinite cardinal. Then $\mathrm{Fn}(I, J, \lambda)$ is the set of all partial functions $f : I \to J$ such that $|f| < \lambda$. Define the ordering relation $\preceq$ on $\mathrm{Fn}(I, J, \lambda)$ as in Definition 4.2.1.

Thus, $\mathrm{Fn}(I, J, \omega)$ is the same as $\mathrm{Fn}(I, J)$ (see Definition 4.2.1). Notice that, when the λ of the above definition is greater than ω, we cannot apply Corollary 4.1.5(4) as we did earlier to conclude that $\mathrm{Fn}(I, J, \lambda)$ is absolute for M. In fact, $\mathrm{Fn}(I, J, \lambda)$ is *not* absolute for M when $\lambda > \omega$. But this will not inconvenience us.

We now define our new forcing notion.

Definition 4.3.2. Let E be a function such that the following hold:

(1) The domain of E is a set of regular cardinals.
(2) For all $\kappa \in \mathrm{dom}(E)$, $E(\kappa)$ is a cardinal and $\mathrm{cf}(E(\kappa)) > \kappa$.
(3) For all $\kappa, \lambda \in \mathrm{dom}(E)$, if $\kappa < \lambda$, then $E(\kappa) \leq E(\lambda)$.

Then E is said to be an *Easton index function*.

Definition 4.3.3. Let E be an Easton index function. Define the *Easton notion of* E to be the partial order $\mathbb{P}(E) = (P(E), \trianglelefteq)$, where $P(E)$ is the set of all functions p such that the following hold:

(1) The domain of p is $\mathrm{dom}(E)$.
(2) For all $\kappa \in \mathrm{dom}(p)$, $p(\kappa) \in \mathrm{Fn}(E(\kappa), 2, \kappa)$.
(3) For every regular cardinal λ, $|\{\kappa \in \lambda \cap \mathrm{dom}(E) : p(\kappa) \neq 0\}| < \lambda$.

The relation $\trianglelefteq$ is defined as follows: For all $p, q \in P(E)$, $p \trianglelefteq q$ if and only if, for all $\kappa \in \mathrm{dom}(E)$, $q(\kappa) \subseteq p(\kappa)$.

Notice again that the ordering relation is "backwards".

If E is an Easton index function, then the domain of E is the set of regular cardinals at which we would like GCH to fail in the generic extension of M. With this in mind, we may draw a similarity between Cohen forcing and Easton forcing when the domain of E is the singleton $\{\omega\}$. Recall the discussion following Definition 4.2.1; with Cohen forcing, we produce a generic extension of M in which new functions from ω to 2 are witnessed. This inflates 2^ω such that CH fails. When we apply Easton forcing with an Easton index function having domain $\{\omega\}$, we produce a generic extension in which new subsets of ω, rather than functions from ω to 2, are witnessed. Of course, because $\mathcal{P}(\omega)$ can be identified with 2^ω, both Cohen forcing and Easton forcing will produce failures of CH.

The advantage of Easton forcing over Cohen forcing is that an Easton index function E may have an infinite domain. In this case, the generic extension obtained by Easton forcing witnesses new elements of $\mathcal{P}(\kappa)$ for each $\kappa \in \mathrm{dom}(E)$, such that GCH does not hold for any element in the domain of E. We therefore obtain infinitely many failures of GCH.

Let us look more closely at Definition 4.3.2, following the treatment in [10]. It is first important to note that the domain of any Easton index function consists entirely of regular cardinals; we consider singular cardinals in the next subsection. If E is an Easton index function, then we may treat E as the function "2-to-the".

That is, if $\kappa \in \text{dom}(E)$, then $E(\kappa)$ is what 2^κ will be forced to be in the generic extension. Thus, conditions (2) and (3) of Definition 4.3.2 are needed to ensure that the extension satisfies each of the following:

(1) König's lemma, which is the statement that, for any cardinal κ, $\text{cf}(2^\kappa)$ is greater than κ (condition (2)).
(2) The monotonicity property, which is the statement that, if κ, λ are cardinals and $\kappa < \lambda$, then $2^\kappa \leq 2^\lambda$ (condition (3)).

As for the Easton notion of E, we require condition (3) to ensure that long antichains do not appear in $\mathbb{P}(E)$. We prefer not to have long antichains because we would like for $\mathbb{P}(E)$ to preserve cofinalities; an analogue of Lemma 4.2.7 holds for partial orders that permit longer-than-countable antichains by Lemma 4.4.6. Once we know that $\mathbb{P}(E)$ preserves cofinalities, we can conclude that $\mathbb{P}(E)$ preserves cardinals as well (see Lemma 4.2.4). However, if λ is as in the statement of condition (3) of Definition 4.3.3, then potentially problematic long antichains do not arise unless λ is a large cardinal; see Chapter VIII, Section 4 of [10] for a more complete explanation. This provides some intimation of the fact that forcing and large cardinals interact in meaningful ways. We explore this idea in the next subsection.

For now, we quote a theorem from which the consistency of failures of GCH at infinitely many regular cardinals follows. The proof of the second half of this theorem depends on the fact that the Easton notion preserves cofinalities. See Chapter VIII, Section 4 of [10] for the full details.

Theorem 4.3.4. *Suppose that GCH holds in M, and let E be an Easton index function in M. Then $\mathbb{P}(E)$ preserves cofinalities. Furthermore, if G is $\mathbb{P}(E)$-generic over M, then for all $\kappa \in \text{dom}(E)$, $2^\kappa = E(\kappa)$ in the generic extension $M[G]$.*

We present in the following example a situation in which Easton forcing secures infinitely many failures of GCH.

Example 4.3.5. Let E be a function with domain $\{\omega_n : n \in \omega\}$ (see Example 3.1.7). Suppose that, for each ω_n, $E(\omega_n) = \omega_{n+5}$. Since each ω_n is regular, and since E meets conditions (2) and (3) of Definition 4.3.2, E is an Easton index function. If G is $\mathbb{P}(E)$-generic over M, then by the previous theorem, it is true in $M[G]$ that $2^{\omega_n} = E(\omega_n) = \omega_{n+5}$ for all ω_n. Hence, in $M[G]$, GCH fails at infinitely many regular cardinals.

4.4. **Forcing in the Presence of Large Cardinals.** The forcing notions we have considered thus far — the Cohen and Easton notions — have strong preservation properties; in particular, both preserve cofinalities. Because any cofinality-preserving notion also preserves cardinals, both of these notions are excellent tools for the manipulation of cardinalities of sets.

However, in some situations we prefer not to preserve certain properties. In particular, there are times when we wish to collapse the cofinalities of cardinals. Closely related to the generalized continuum hypothesis is the *singular cardinals hypothesis.*

Definition 4.4.1. The singular cardinals hypothesis (SCH) is the statement that, for any singular strong limit cardinal κ, $2^\kappa = \kappa^+$.

Recall that a regular strong limit cardinal is a strongly inaccessible cardinal (see Definition 3.2.3). Here, however, we focus our attention on singular strong limit cardinals. Since GCH implies SCH and we have assumed that ZFC is consistent, $\mathsf{ZFC} + \mathsf{SCH}$ is also consistent. In fact, SCH is independent of ZFC, but it is very difficult to prove that $\mathsf{ZFC} + \neg\mathsf{SCH}$ is consistent. For instance, consider the Easton forcing notion: Its purpose is to make GCH fail at regular cardinals, and so we know that $\neg\mathsf{GCH}$ is consistent with ZFC. However, it could be the case that GCH only ever fails at regular cardinals — never at singular cardinals. As it happens, it is possible to find a model of ZFC in which $\neg\mathsf{SCH}$ holds, but we shall need the assistance of large cardinals.

First, we require a new forcing notion that does not preserve cofinalities. To develop this notion, we state a new ultrafilter property in terms of which an alternative characterization of measurability (see Definition 3.2.6) can be given.

Definition 4.4.2. Let κ be an uncountable cardinal. Suppose $\mathcal{U}$ is a nonprincipal κ-complete ultrafilter on κ such that, for every function $f \in \kappa^\kappa$ for which the set $\{\beta : f(\beta) < \beta\}$ is in $\mathcal{U}$, there exists $\gamma < \kappa$ such that the set $\{i \in \kappa : f(i) = \gamma\}$ is in $\mathcal{U}$. Then $\mathcal{U}$ is said to be *normal*.

Thus, an ultrafilter $\mathcal{U}$ is normal if, whenever $f \in \kappa^\kappa$ and there is a $\mathcal{U}$-large set on which f is dominated by the identity function on κ, then there is a $\mathcal{U}$-large set on which f is constant. Notice that, if we did not require nonprincipality and κ-completeness, then principal ultrafilters would be normal: Let $\mathcal{U}$ be a principal ultrafilter on an uncountable cardinal κ. Then $\mathcal{U}$ is generated by a singleton $\{\alpha\} \subseteq \kappa$ (see Example 2.1.9), where $\alpha < \kappa$. If $f \in \kappa^\kappa$, then let $\gamma = f(\alpha)$; it follows that $\gamma < \kappa$. The set $\{i \in \kappa : f(i) = \gamma\}$ contains α, so $\{i \in \kappa : f(i) = \gamma\} \in \mathcal{U}$.

We now quote a result that shows that every measurable cardinal gives rise to a normal ultrafilter; the proof makes use of an ultrapower. See Chapter 4, Section 2 of [1] for the details.

Lemma 4.4.3. *Suppose κ is a measurable cardinal. Then there exists a normal ultrafilter on κ.*

On the surface, the reason we use normality instead of κ-completeness is that our next forcing notion is defined using a normal ultrafilter. Deeper than this is the fact that use of a normal ultrafilter allows us to apply Rowbottom's theorem, a complicated combinatorial statement associated with so-called Ramsey cardinals[10], in order to show that this forcing notion meets a particular technical condition. We shall not consider these details here and shall instead aim merely to sketch the main ideas. The interested reader should consult Chapter 7 of [9] and Section 1 of [4] for more information.

At last, we define our new forcing notion.

Definition 4.4.4. Let κ be measurable, and let $\mathcal{U}$ be a normal ultrafilter on κ. Define $P_\mathcal{U}$ to be the set of all pairs $\langle p, A\rangle$ such that p is a finite subset of κ, $A \in \mathcal{U}$, and $\min(A) > \max(p)$. Then the *Prikrý notion* $\mathbb{P}_\mathcal{U} = (P_\mathcal{U}, \lesssim)$ *associated with* $\mathcal{U}$ is the partial order on $P_\mathcal{U}$ given by $\lesssim$, where $\langle p, A\rangle \lesssim \langle q, B\rangle$ if the following hold:

(1) p end-extends q, that is, q is an initial segment of p.
(2) $A \subseteq B$.
(3) $p \setminus q \subseteq B$.

[10]These are large cardinals

The minimum of A exists because A is a subset of κ, which is an ordinal and is thus well-ordered by $\in$. The maximum of p exists because p is finite.

If $\mathbb{P}_{\mathcal{U}}$ is the Prikrý notion associated with a normal ultrafilter $\mathcal{U}$ on a measurable cardinal κ and G is a $\mathbb{P}_{\mathcal{U}}$-generic filter over M, then the generic extension $M[G]$ will contain a cofinal function not belonging to M from ω to the measurable cardinal κ. Hence, all we need show towards a failure of SCH is that $\mathbb{P}_{\mathcal{U}}$ preserves cardinals; in this case, κ remains a cardinal in $M[G]$, but it is singular. We shall show first that $\mathbb{P}_{\mathcal{U}}$ preserves cardinals above κ. The proof that $\mathbb{P}_{\mathcal{U}}$ preserves cardinals beneath κ involves a consideration of names. While names are of great importance in forcing, we must omit discussion of them for the sake of space. The reader should consult Chapter VII of [10] for more information.

First, we need a generalization of the countable antichain condition (see Definition 4.2.5).

Definition 4.4.5. Let κ be a cardinal, and let $\mathbb{P}$ be a partial order such that no antichain in $\mathbb{P}$ has length exceeding κ. Then $\mathbb{P}$ is said to meet the κ^+ *antichain condition*.

Hence, a partial order meeting the countable antichain condition also meets the ω^+ antichain condition, and conversely.

We now state an analogue of Lemma 4.2.4 for the κ^+ antichain condition; see Chapter VII, Section 6 of [10] for a proof.

Lemma 4.4.6. *Let κ be a regular cardinal in M, and let $\mathbb{P}$ be a partial order meeting the κ^+ antichain condition. Then $\mathbb{P}$ preserves cardinals greater than κ.*

We can now prove that the Prikrý notion preserves cardinals above κ.

Lemma 4.4.7. *Let κ and $\mathcal{U}$ be as in the statement of Definition 4.4.4, and let $\mathbb{P}_{\mathcal{U}} = (P_{\mathcal{U}}, \lesssim)$ be the associated Prikrý notion. Then $\mathbb{P}_{\mathcal{U}}$ preserves cardinals greater than κ.*

Proof. By the previous lemma, it is enough to show that $\mathbb{P}_{\mathcal{U}}$ meets the κ^+ antichain condition and that κ is regular in M. We follow the proof in [4]: First, it is clear that κ is regular in M since κ is measurable in M. If $\langle p, A\rangle \in P_{\mathcal{U}}$, then p is a finite subset of κ. Each finite subset of κ can be coded as an element of κ. Hence, there are only κ-many finite subsets of κ. Now, let $\langle p, A\rangle, \langle p, B\rangle \in P_{\mathcal{U}}$. Then $\langle p, A\rangle$ and $\langle p, B\rangle$ are compatible; the element $\langle p, A \cap B\rangle$ is a witness for compatibility. Hence, if two elements of P have the same first coordinate, then they are compatible. Since there are only κ-many choices for the first coordinate, there are at most κ incompatible elements in $\mathbb{P}_{\mathcal{U}}$. No antichain in $\mathbb{P}_{\mathcal{U}}$ can thus have length exceeding κ. □

Now we quote the lemma that secures the same result for cardinals below κ. See Chapter VII, Sections 2 and 3 of [10] and Section 1 of [4] for background and a proof.

Lemma 4.4.8. *Let κ and $\mathcal{U}$ be as in the statement of Definition 4.4.4, and let $\mathbb{P}_{\mathcal{U}}$ be the associated Prikrý notion. Then $\mathbb{P}_{\mathcal{U}}$ preserves κ and all cardinals less than κ.*

At last, we obtain the following theorem.

Theorem 4.4.9. *Let κ, $\mathcal{U}$, and $\mathbb{P}_{\mathcal{U}}$ be as in the statement of the above lemma. If G is $\mathbb{P}_{\mathcal{U}}$-generic over M and M satisfies $2^\kappa > \kappa^+$, then $M[G]$ is a model of ZFC in which SCH fails at κ.*

Proof. That the sequence $\bigcup\{p : \text{there exists } A \in \mathcal{U} \text{ such that } \langle p, A\rangle \in G\}$ is cofinal in κ in $M[G]$ follows from the genericity of G. Since $\mathbb{P}_\mathcal{U}$ preserves cardinals, $|\kappa|^{M[G]} = |\kappa|^M = \kappa$. Hence, in $M[G]$, κ is a singular cardinal, and $2^\kappa > \kappa^+$. Thus, SCH fails at κ in $M[G]$. □

If M does not satisfy $2^\kappa > \kappa^+$ (for instance, if $M \models \mathsf{GCH}$), then we can use Easton forcing to obtain a generic extension M' of M in which $2^\kappa > \kappa^+$ and apply Theorem 4.4.9 with M' in place of M.

Although we have here only considered applications of forcing to questions about cardinals and cofinalities, forcing is an exceptionally versatile method that can be used to obtain relative consistency results for many different hypotheses. For instance, Prikrý forcing can also be applied to obtain sophisticated consistency results concerning Suslin trees (see Section 2 of [4]). The consistency of combinatorial statements such as $\Diamond$ with ZFC can be proved using forcing as well (see Chapters 2 and 7 of [10]). We shall not explore such advanced results in this paper, but the interested reader will be able to glean a great deal of knowledge from the sources referenced throughout this section.

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