AN INTRODUCTION TO THE JACOBIAN CONJECTURE

WILLIAM GARLAND

ABSTRACT. The Jacobian Conjecture is a criterion that relates the invertibility of polynomial mappings of $\mathbb{C}$ to a condition on their Jacobian matrix. It arises naturally and is simple to state in elementary terms, but attempted proofs of it have required advanced techniques in algebraic geometry and topology, and it remains unproven for dimensions greater than one. This paper surveys the current state of the conjecture, and offers possible methods of proof and construction.

CONTENTS

1. INTRODUCTION

A common challenge arising in many areas of mathematics is to determine when a function obeying certain properties is invertible, and whether the inverse function obeys similar properties as the original. For example, we might ask when a continuous function has a continuous inverse, or when a group homomorphism has an inverse which is also a group homomorphism. In some cases this question has a clean answer: for example, given any smooth function between manifolds which has an inverse, we are guaranteed that the inverse is also smooth. However, there exist certain continuous functions between topological spaces which have an inverse function, but that inverse is not continuous.

Viewed in this respect, the Jacobian Conjecture can be seen as an attempt to answer this question for functions which are described by polynomial equations. This is particularly important in the field of algebraic geometry, where such polynomial functions are intrinsically linked to morphisms of varieties. This paper seeks to state the history of the Jacobian Conjecture, to present the conjecture and many of the known results about it in a form accessible to those without a background in algebraic geometry, and finally to outline a possible approach to proving the conjecture as well as a constructive method of finding the inverse of a given polynomial mapping.

Date: September 26, 2018.

2. The Conjecture, Definitions, and Motivation

The Jacobian Conjecture gives a condition for when a function from $\mathbb{C}^n$ to $\mathbb{C}^n$ which restricts to a polynomial function in each coordinate has an inverse which is also given by polynomials. We will give a careful definition of this idea below.

Conjecture 2.1 (The Jacobian Conjecture over $\mathbb{C}$)**.** *Let F be a polynomial mapping of $\mathbb{C}^n$. Then F is a polynomial automorphism if and only if the Jacobian of F is constant and nonzero.*

If the Conjecture is true, it provides a simple way to check whether a given polynomial mapping has an inverse. In the remainder of this section, I will define the terms used in the Conjecture and provide some motivation behind why we might expect it to be true.

Definition 2.2. Let $\mathbb{C}^n$ represent the usual complex vector space in n variables. We define the *coordinates* or *coordinate functions* $x_i : \mathbb{C}^n \to \mathbb{C}$ for each $i = 1..n$ to be the natural projection maps. That is, for $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$x_i(\mathbf{z}) = z_i \qquad \text{for } 1 \leq i \leq n.$$

The function x_i is called the "ith coordinate of $\mathbb{C}^n$".

Definition 2.3. Given a function $F : \mathbb{C}^n \to \mathbb{C}^n$, for each i from 1 to n we define the *ith component of F* to be the function $F_i : \mathbb{C}^n \to \mathbb{C}$ defined as the composition of F with the ith coordinate x_i.

$$F_i := x_i \circ F$$

We often write $F = (F_1, \dots, F_n)$, corresponding to the fact that for any $\mathbf{z} \in \mathbb{C}^n$,

$$F(\mathbf{z}) = (F_1(\mathbf{z}), \dots, F_n(\mathbf{z})).$$

Remark 2.4. A function $F : \mathbb{C}^n \to \mathbb{C}^n$ is completely determined by its component functions. Therefore we can study the properties of the function F by studying all of its components.

Definitions 2.5. Let $G : \mathbb{C}^n \to \mathbb{C}$. We call G a *polynomial function* if there exists a complex polynomial g in n variables such that for each $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$G(\mathbf{z}) = g(z_1, \dots, z_n).$$

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a function such that each component F_i of F is a polynomial function, we call F a *polynomial mapping of* $\mathbb{C}^n$ (or just a polynomial mapping, if n is understood).

Definition 2.6. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. We say that F is a *polynomial automorphism* if there exists a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ obeying

$$F \circ G = G \circ F = \mathrm{Id}_{\mathbb{C}^n}.$$

When is a polynomial mapping F a polynomial automorphism? In order to come up with an answer to this question, we might begin by determining what properties such a polynomial automorphism must have. For instance, we note that a polynomial mapping is a particular example of a smooth function from $\mathbb{C}^n \to \mathbb{C}^n$. Smooth functions have derivatives, and by the Chain Rule, we know how these derivatives behave with respect to composition. The following definitions and proposition make this precise.

Definitions 2.7. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a smooth function, $\mathbf{z} \in \mathbb{C}^n$. Then the *total derivative of F at $\boldsymbol{z}$* (or *Jacobian matrix of F at $\boldsymbol{z}$*), denoted $(DF)_{\mathbf{z}}$, is the $n \times n$ matrix defined by

$$(DF_{\mathbf{z}})_{i,j} = \frac{\partial F_i}{\partial x_j}(\mathbf{z}).$$

The *Jacobian of F at $\boldsymbol{z}$* is defined to be the determinant of this matrix:

$$J(F)(\mathbf{z}) := \det((DF)_{\mathbf{z}}).$$

Remark 2.8. If F is a polynomial mapping, then each component F_i of F is a polynomial function. Therefore, we know from calculus that the partial derivative of a polynomial function is also a polynomial function, so each $\partial F_i/\partial x_j$ is also a polynomial function in $\mathbf{z}$. Furthermore, $\det(A)$ is a polynomial in the entries of the matrix A, and the composition of polynomials is a polynomial. Therefore, we note that for F a polynomial mapping, $J(F)$ is a polynomial function in $\mathbf{z}$.

Proposition 2.9. *Let F be a polynomial automorphism. Then $J(F)(\boldsymbol{z})$ is a constant nonzero function of $\boldsymbol{z}$.*

Proof. Since F is a polynomial automorphism, we know there exists a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ which satisfies

$$G \circ F = \mathrm{Id}_{\mathbb{C}^n}.$$

Therefore, note

$$1 = J(\mathrm{Id}_{\mathbb{C}^n})(\mathbf{z}) = J(G \circ F)(\mathbf{z}) = J(G)(F(\mathbf{z})) \cdot J(F)(\mathbf{z})$$

by the Chain Rule. Therefore, $J(F)$ is given by a polynomial which has a multiplicative inverse. Therefore, the degree of $J(F)$ must be 0, so $J(F)$ must be a constant nonzero polynomial. □

This proposition shows that for a polynomial mapping F to be a polynomial automorphism, a *necessary* condition is for $J(F)$ to be constant. The Jacobian Conjecture asserts that this is also a *sufficient* condition.
Finally, we should note the following definition.

Definition 2.10. A function of topological spaces $f : X \to Y$ is called *proper* if for any $K \subset Y$ which is compact, $f^{-1}(K)$ is compact.

Remark 2.11. Recall that a subset of $\mathbb{C}^n$ is compact if and only if it is closed and bounded.

As later sections will show, whether or not a polynomial mapping is proper has a great significance on whether it is a polynomial automorphism.

3. A Short History of the Jacobian Conjecture

The Jacobian Conjecture is attributed to O. H. Keller, who first posed it in 1939 [4]. He considered polynomial mappings with integer coefficients and showed that the conjecture was true given that the mapping F was birational. A number of extensions to the conjecture have been proposed, and many conditions on a polynomial mapping with constant Jacobian have been shown to be equivalent to invertibility. Many of these are summarized well in Theorem (2.1) of [2]; however, we will review those which are most relevant here.

Theorem 3.1 (Bass, Connell, Wright [2]). *Let F be a polynomial mapping of $\mathbb{C}^n$ with constant nonzero Jacobian. Then the following statements are equivalent:*

(1) F is invertible and F^{-1} is a polynomial mapping.
(2) F is injective.
(3) F is proper.

These reductions address some concerns we may have. For instance, it shows that if F is invertible, its inverse will automatically be a polynomial mapping. It also gives conditions for proving that F is invertible which are hopefully easier to prove.

In the years since the conjecture was posed, there have been two common approaches to reducing it. We will briefly look at each approach and discuss its merits and challenges. The first approach, which I will refer to as Reduction of Dimension, attempts to solve the conjecture for small values of n, that is, looking at polynomial mappings of $\mathbb{C}^2$ and $\mathbb{C}^3$. The case $n = 1$ is trivial, as it can be immediately seen that polynomial mappings of $\mathbb{C}$ are exactly the polynomial functions, and it is well known that the invertible polynomial functions on $\mathbb{C}$ are exactly those of degree 1, which are also exactly those whose determinants are a nonzero scalar. However, this approach becomes much more difficult at higher dimensions. The case $n = 2$ has been documented by a great many authors, but remains unproven. Furthermore, as n increases, a computational approach becomes very difficult, because the Jacobian of a function is fundamentally a determinant, and determinants grow factorially more difficult to compute as n grows large.

A second approach of restricting the degrees of the coordinate functions has found more success. I will refer to this method as Reduction of Degree. Note that each F_i is a polynomial function, so it has a polynomial degree. Let d denote the largest of the degrees of $F_1, \ldots, F_n$. It was first noticed by Wang [6] and later by Oda [5] that the Conjecture could be proven for $d = 2$.

Theorem 3.2 (S. Wang [6]). *Let F be a polynomial mapping of $\mathbb{C}^n$ with $\deg(F_i) \leq 2$ for each $1 \leq i \leq n$, and $J(F)$ a nonzero constant. Then F is injective.*

By Theorem 3.1, we see that F is then invertible with polynomial inverse. An elementary proof of this theorem is reproduced in both [3] and [2]. While this theorem covers only a small number of polynomial mappings, in fact it is close to proving everything that we need, thanks to the main result of [2].

Theorem 3.3 (Bass, Connell, Wright [2], paraphrased). *Suppose that the Jacobian Conjecture is true for any polynomial mapping of degree less than or equal to 3. Then the Jacobian Conjecture is true.*

Thus we have that the Jacobian Conjecture is true for polynomial mappings of degree at most 2, and we need only show that it is also true for mappings of degree at most 3 to prove that it is true for every mapping. Despite this, no proof for mappings of degree 3 has been found, so the conjecture remains unproven.

One can also ask whether the conjecture holds over other fields. Proposition 2.9 certainly is true for polynomial mappings with other coefficients, since the argument used holds over any field (since all concerned functions are given by polynomials, we can use the polynomial definition of derivative to avoid difficulty over other fields). This proposition gives the intuition for the Jacobian Conjecture, so it is reasonable to wonder whether its converse could hold in other fields. It turns out that we can say something about characteristic 0 fields, but for fields with nonzero characteristic, the conjecture is false as stated, and it is not clear what amendments may be needed to fix it.

Let k be any field of characteristic 0. By defining a "polynomial mapping" and Jacobian analogously to Definitions 2.5 and 2.7, we can ask whether the Jacobian Conjecture holds over k. We might wonder whether it matters if the field is algebraically closed, or even if it matters if k is a field at all (for example, what if k were an integral domain, such as the integers $\mathbb{Z}$)? According to a theorem of Bass, Connell, and Wright, it turns out that the answer entirely depends on whether the conjecture holds over $\mathbb{C}$.

Theorem 3.4 (Bass, Connell, Wright [2]). *Let k be an integral domain of characteristic 0. If the Jacobian Conjecture holds for polynomial mappings over $\mathbb{C}$, then it also holds for polynomial mappings over k.*

Remark 3.5. This theorem is why we have otherwise restricted our attention to polynomials over $\mathbb{C}$.

When we refer to the Jacobian Conjecture over other domains, we specifically mean that a polynomial mapping with nonzero, constant Jacobian must have a polynomial inverse. Over $\mathbb{C}$, a polynomial mapping has a nonzero constant Jacobian if and only if its Jacobian is always nonzero. This is a consequence of the Fundamental Theorem of Algebra, and so is true for any algebraically closed field. However, over other domains such as the real numbers $\mathbb{R}$, a polynomial can have no zeros and yet not be constant. A polynomial mapping of $\mathbb{R}$ with such a nonvanishing, nonconstant Jacobian is *not* expected to have an inverse, or its inverse will not be a polynomial mapping.

Warning 3.6. Let $F : \mathbb{R} \to \mathbb{R}$ be a polynomial mapping defined by

$$F(x) = \frac{1}{3}x^3 + x.$$

Note $J(F)$ is nonvanishing and nonconstant since

$$J(F)(x) = x^2 + 1 > 0.$$

F is invertible, however F^{-1} is certainly not a polynomial.

We may also ask whether the conjecture may hold over fields with nonzero characteristic. However, the following counterexample holds for $n = 1$.

Warning 3.7. Let k be a field of characteristic p with $p \neq 0$. Consider the polynomial mapping $F : k \to k$ defined by

$$F(x) = x^p + x.$$

Note

$$J(F)(x) = px^{p-1} + 1 = 1 \qquad \text{since } k \text{ has characteristic } p.$$

However F is not invertible in general.

4. Properness of a Polynomial Map

As mentioned in the previous section in Theorem 3.1, one possible method to prove the Jacobian Conjecture is to prove that any polynomial mapping F with constant nonzero Jacobian is a proper map. We begin this approach with the following proposition.

Proposition 4.1. *Let F be a polynomial mapping with constant nonzero Jacobian. Then for each $\boldsymbol{y} \in \mathbb{C}^n$, $F^{-1}(\boldsymbol{y})$ is finite.*

Proof. By assumption, we know that $(DF)_{\mathbf{x}}$ has constant nonzero determinant, so in particular $(DF)_{\mathbf{x}}$ is invertible for all $\mathbf{x} \in \mathbb{C}^n$. Therefore, F is a submersion. One definition of a manifold states that a manifold of dimension d is locally the preimage of a point under a smooth submersion from $\mathbb{C}^n$ to $\mathbb{C}^{n-d}$. Since $F : \mathbb{C}^n \to \mathbb{C}^n$ defines such a submersion, we know that the preimage of any point $\mathbf{y}$ under F will be a complex manifold of dimension 0. Therefore, $F^{-1}(\mathbf{y})$ is always discrete.

To further prove that $F^{-1}(\mathbf{y})$ is finite, we use a result of algebraic geometry. Without loss of generality, let $\mathbf{y} = \mathbf{0}$, since each of our assumptions is invariant under translation. Then, note that $\mathbf{x} \in F^{-1}(\mathbf{0})$ if and only if for each component function F_i, $F_i(\mathbf{x}) = 0$. Therefore,

$$F^{-1}(\mathbf{0}) = Z(F_1, \dots, F_n)$$

where Z denotes the common zero locus of a set of polynomials, i.e. the set of points where all of the polynomials vanish. By algebraic geometry, we know that $Z(F_1, \dots, F_n)$ can be written as a finite union of "irreducible" closed sets, each of which has a well defined dimension not greater than its dimension as a complex manifold. Since the manifold dimension of $Z(F_1, \dots, F_n)$ is zero, each irreducible component must also have dimension zero, which implies that each is a single point. Thus $Z(F_1, \dots, F_n)$ is a finite union of points, and thus is a finite set. □

Thus, we know that each preimage is finite, so the map F must be finite-to-one. If the map F is proper, then since F is nonsingular at each point, we would conclude that F has a continuous degree, defined by

$$\deg(F, \mathbf{y}) = \sum_{\mathbf{x} \in F^{-1}(\mathbf{y})} \operatorname{sgn}(J(F)(\mathbf{x})) = |F^{-1}(\mathbf{y})|. \tag{4.2}$$

Furthermore, this degree would be locally constant, and thus (since $\mathbb{C}^n$ is connected) globally constant. Therefore, we would conclude that every preimage would have exactly the same size. However, I prove that the converse is also true, which offers a possible route to proving the conjecture overall.

Theorem 4.3. *If F is a polynomial mapping with nonzero constant Jacobian, such that for each $\boldsymbol{y} \in \operatorname{im}(F)$, $|F^{-1}(\boldsymbol{y})| = r$ for some fixed $r \in \mathbb{N}$, then F is proper onto its image. That is, $F : \mathbb{C}^n \to \operatorname{im}(F)$ is a proper map.*

The usefulness of this result is shown by the following result by [1].

Theorem 4.4 (Byrnes and Lindquist [1], Main Result)**.** *If $F : X \to Y$ is a regular (i.e. polynomial) map with constant nonzero Jacobian, then F is **biregular** (i.e. has a polynomial inverse) if and only if $F : X \to \operatorname{im}(F)$ is proper.*

Proof of Theorem 4.3. Let K be a compact subset of $\operatorname{im}(F)$. We will show that $F^{-1}(K)$ is compact. We will do this by showing that any sequence in $F^{-1}(K)$ admits a convergent subsequence. First, note that if $F^{-1}(K) = \emptyset$, we are done. Now, take a sequence $\{\mathbf{x}_n\}$ on $F^{-1}(K)$. Let $\{\mathbf{y}_n\} = F\{\mathbf{x}_n\}$. $\{\mathbf{y}_n\}$ is a sequence on K, so since K is compact, we can pass to a convergent subsequence of $\{\mathbf{y}_n\}$ which converges to an element $\mathbf{y}_\infty \in K$. Since $\mathbf{y}_\infty \in K \subset \operatorname{im}(F)$, by assumption we have

$$F^{-1}(\mathbf{y}_\infty) = \{p_1, \dots, p_r\} \subset F^{-1}(K).$$

Note also that $(DF)_{p_i}$ is invertible for each i from 1 to r, so by the inverse function theorem, there are open neighborhoods U_i of p_i and V_i of $\mathbf{y}_\infty$ such that $F : U_i \to V_i$ is a diffeomorphism, with inverse maps $G_i : V_i \to U_i$. First, restrict each U_i so that the collection is pairwise disjoint, and restrict the V_i's accordingly. By taking

$$V := \bigcap_{i=1}^{r} V_i, \qquad U_i' := G_i(V)$$

we restrict to the case where each local diffeomorphism has the same image. Now, V is an open neighborhood of $\mathbf{y}_\infty$, and $\{\mathbf{y}_n\} \to \mathbf{y}_\infty$, so therefore we may pass to a convergent subsequence lying entirely inside V. Now, for each $n \in \mathbb{N}$, $\mathbf{y}_n \in V$, so for each $i = 1..r$, we have

$$F(G_i(\mathbf{y}_n)) = \mathbf{y}_n \implies G_i(\mathbf{y}_n) \in F^{-1}(\mathbf{y}_n).$$

Furthermore since the U_i's are pairwise disjoint, the $G_i(\mathbf{y}_n)$ are distinct elements, so we have enumerated r distinct elements of $F^{-1}(\mathbf{y}_n)$. However, by assumption, $F^{-1}(\mathbf{y}_n)$ has only r elements, so therefore the points $G_i(\mathbf{y}_n)$ account for all of the preimages of $\mathbf{y}_n$. By construction, $\mathbf{x}_n$ is also a preimage of $\mathbf{y}_n$, so it follows that

$$\text{For every } n \in \mathbb{N} \text{ there exists } i_n \text{ such that } \mathbf{x}_n = G_{i_n}(\mathbf{y}_n).$$

Note that there are infinitely many $\mathbf{x}_n$'s but only finitely many i's, so at least one j in $1, \dots, r$ must be repeated infinitely many times. Passing to the subsequence of points corresponding to this index j, where each $\mathbf{x}_n$ is precisely $G_j(\mathbf{y}_n)$, we see that since $\{\mathbf{y}_n\} \to \mathbf{y}_\infty$ and G_j is a differentiable, and thus continuous, function, it must be the case that

$$\{\mathbf{x}_n\} = G_j\{\mathbf{y}_n\} \to G_j(\mathbf{y}_\infty) = p_j$$

and thus we have constructed a convergent subsequence of our original sequence. Therefore $F^{-1}(K)$ is compact. □

5. Computing Polynomial Inverses

Suppose we know that the Jacobian Conjecture holds. How can we effectively find the inverse map G? One possible answer to this question comes from a closer look at the method by which we proved Proposition 2.9. In the proof we noted that if F was a polynomial automorphism with inverse G, then $J(G) \circ F$ and $J(F)$ were inverse polynomials. In fact, noting that J is the determinant of the total derivative, we can see the even stronger condition

$$(DF)_{\mathbf{z}}(DG)_{F(\mathbf{z})} = I_n = (DG)_{\mathbf{z}}(DF)_{G(\mathbf{z})}$$

holds by the chain rule, since $G \circ F = F \circ G = \mathrm{Id}$. Therefore, considering $DF_{\mathbf{z}}$ as a matrix with coefficients that are polynomials in $\mathbf{z}$, we have

$$(DG)_{\mathbf{z}} = (DF)^{-1}_{G(\mathbf{z})}.$$

This is a system of partial differential equations in the components of G. We know that $G = F^{-1}$ satisfies this equation. Furthermore, the inverse of a square matrix is unique, so this relation uniquely characterizes the partial derivatives of the components of G. Suppose some function $H : \mathbb{C}^n \to \mathbb{C}^n$ satisfies the above system. It need not be the case that $H = F^{-1}$, however I claim that from H we can easily derive G. Note that for such an H,

$$\begin{aligned} I_n = (D\,\mathrm{Id})_{\mathbf{z}} = (DF)_{H(\mathbf{z})}(DH)_{\mathbf{z}} = D(F \circ H)_{\mathbf{z}} \\ \implies D(\mathrm{Id} - F \circ H)_{\mathbf{z}} = 0 \\ \implies F \circ H = \mathrm{Id} + \mathbf{c} \end{aligned}$$

for some constant $\mathbf{c} \in \mathbb{C}^n$. Therefore, $F \circ H$ represents translation by some fixed $\mathbf{c}$. This is an invertible transformation, and precomposing by its inverse yields the following identity:

$$(F \circ H)(\mathbf{z} - \mathbf{c}) = \mathbf{z} - \mathbf{c} + \mathbf{c} = \mathbf{z}.$$

Therefore, $H(\mathbf{z} - \mathbf{c}) = F^{-1}(\mathbf{z})$, so F^{-1} is easily found from H. Note that $\mathbf{c}$ is just $F(H(0))$.

Therefore, we have the following proposition, which summarizes the discussion above.

Proposition 5.1. *Suppose the Jacobian Conjecture is true. Then if F is a polynomial map such that $J(F)$ is a nonzero constant, then computing F^{-1} is equivalent to finding a solution to the system of partial differential equations*

$$(DG)_z = (DF)^{-1}_{G(z)}.$$

Expressed another way, the system is given by

$$\frac{\partial G_i}{\partial x_j} = \frac{(-1)^{i+j}}{J(F)} \det((DF)^{j,i}_G)$$

for each i, j ranging from 1 to n. ($(DF)^{j,i}_G$ denotes the j,ith minor of $(DF)_G$, which is essentially the submatrix of $(DF)_G$ obtained by omitting row j and column i).

Remark 5.2. From this presentation, it is not immediately obvious that such a system would always have solutions, or that those solutions would be polynomials. However, the Jacobian Conjecture asserts that there is a polynomial solution to these equations (namely F^{-1}) and that furthermore, from our discussion above, any solution H must be the composition of a polynomial with a translation, which is again a polynomial. Therefore, if there exists some F a polynomial map with constant nonzero Jacobian, such that the above system has a solution defined everywhere which is *not* a polynomial mapping, then the Jacobian Conjecture would be proven false. So this approach could provide a method for finding a counterexample, should such a counterexample exist.

Acknowledgments

I would like to thank my advisor, Ronno Das, for his support and advice; Dr. Peter May, for organizing an excellent REU program; and everyone in the University of Michigan Mathematics Department for their generosity and support. Thank you!

References

[1] Christopher I. Byrnes and Anders Lindquist *A Note on the Jacobian Conjecture* AMS **136**, (2008), no. 9, 30073011, S 0002-9939(08)09245-9
[2] Hyman Bass, Edwin H. Connell and David Wright *The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse* AMS Bulletin **7**, (1982), no. 2, 0273-0979/82/0000-0754/
[3] Arno van den Essen *Polynomial Automorphisms and the Jacobian Conjecture* Socit Mathmatique de France (1985), 14E05
[4] O. H. Keller *Ganze Cremona-Transformationen* Monats. Math. Physik **47**, (1939), 299-306

[5] Susumu Oda *The Jacobian problem and the simply-connectedness of A" over a field k of characteristic zero* Osaka Univ., preprint, (1980)
[6] S. Wang *A jacobian criterion for separability* J. Algebra **65**, (1980), 453-494