ELEMENTARY HOMOLOGY THEORY WITH COMPUTATIONS

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Abstract. This paper aims to expose the reader unfamiliar with algebraic topology to elementary topics in homology theory. In particular, we intend to briefly motivate homology and introduce the basic definitions and properties, namely invariance. We conclude with some low-dimensional computations to reinforce the ideas presented.

Contents

1. Introduction and Motivation

In algebraic topology, the basic aim is to construct algebraic invariants classifying topological spaces. Such invariants include the homotopy, homology, and cohomology groups; this paper focuses on introducing homology, which we reinforce with explicit computations for S^1, S^2, and $\mathbb{R}P^2$. It assumes some linear algebra, group theory, and point-set topology.

Though we present homology in isolation, it would be remiss to ignore the historical roots underpinning its more formal treatment. Very informally, homology begins with the intuitive notion of distinguishing between spaces by examining the presence of $n-$dimensional holes; since such holes persist in a space under continuous deformation, they might reasonably amount to an invariant. Euler's polyhedron formula and Riemann's definition for a genus are some early instances of this concept at play. It would not be until the end of the nineteenth century, however, that the theory would assume the semblances of its modern form, when Henri Poincaré introduced homology classes in his seminal *Analysis Situs*.

As we venture into the more formal developments presented in the subsequent section, bear in mind the intuition. Indeed, by rendering a more precise formulation from this notion, we will have established an invariant whose value lies in part in its computability (in comparison to the homotopy groups).

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2. Simplicial Homology

Our discussion begins with the standard $n-$simplex, a generalization of the triangle to n dimensions. In order to lend a more arithmetical formulation to the boundary of a simplex, the ordered $n-$simplex is introduced. From there we specify a manner by which to combine simplexes by defining a simplicial complex.

Definition 2.1. The standard $n-$simplex Δ^n with affinely independent vertices $v_0, ..., v_n$ (i.e. $v_1 - v_0, ..., v_n - v_0$ are linearly independent) is the convex hull of the vertices represented as vectors in $\mathbb{R}^n$:

$$\Delta^n = \{x_0v_0 + ... + x_nv_n \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1 \text{ and for all } i, x_i \geq 0\}$$

Note that each Δ^n is completely determined by its vertices, allowing us to represent them as their set of vertices.

Removing all but some arbitrary choice of $k \leq n$ vertices leaves a $k-$face.

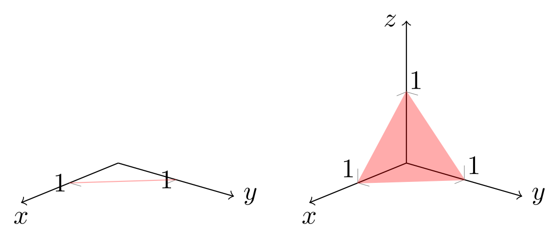

Figure 1: Graphical representation of the standard 1- and $2-$simplexes

Definition 2.2. An ordered $n-$simplex is an $n-$simplex equipped with an ordering on its vertices, denoted $[v_0, ..., v_n]$. This induces an orientation on the $k-$faces $[v_{i_1}, ..., v_{i_k}]$ of the simplex.

Definition 2.3. Consider a space X and a family Σ of continuous maps from a (ordered) simplex Δ^k to X indexed by $\mathcal{A}$, for which each λ determines some k:

$$\Sigma = \{\sigma_\lambda | \sigma_\lambda : \Delta^k \to X \text{ is cts. and } \lambda \in \mathcal{A}\}$$

We call Σ a simplicial complex structure on the space X if its members satisfy

(1) The restriction of any map to the simplex's interior is injective.
(2) The restriction of any map to any k-face is also an element of Σ, particularly one whose index determines a map having a domain of dimension k.
(3) A set U is open in X iff each preimage $\sigma_\lambda^{-1}(U)$ is open.

We abuse notation by not representing simplexes as maps.

While the first two conditions are straightforward, the final condition is not as clear. It is included because it requires the finest topology possible on the space X, in order to rule out trivialities.

Intuitively, a simplicial complex on X is a nice decomposition of the space into simplexes with some deformation, hence the continuous maps. It moreover lends itself to a clear geometric interpretation: to each map we associate one simplex and hence obtain a collection of simplexes, from which we can construct our complex by gluing together their faces in a manner adherent to the aforementioned requirements, the result of which is homeomorphic to the space (this justifies our aforementioned abuse of notation). The disjoint union of the simplexes associated to each map, quotiented by the face identifications, offers another representation.

Now that we have specified a manner by which to combine simplexes, we are prepared to introduce the more technical mechanisms of chain groups, boundary maps, and the chain complex. Chain groups capture the traversals of a series of $k-$simplexes as formal linear combinations.

Definition 2.4. For a simplicial complex, the $k-$th chain group C_k is the free abelian group generated by its k-simplexes. Its members are called $k-$chains.

Definition 2.5. The boundary operator of an ordered $n-$simplex $[v_0, ..., v_n]$ is the alternating sum of its $(n-1)-$faces:

$$\partial_n[v_0, ..., v_n] = \sum_{i=0}^{n}(-1)^i[v_0, ..., \hat{v_i}, ..., v_n]$$

Each boundary homomorphism $\partial_k : C_k \to C_{k-1}$ is defined in the expected way:

$$\partial_k(a_1\Delta_1^k + ... + a_j\Delta_j^k) = \sum_{i=0}^{j} a_i\partial_k\Delta_i^k$$

Definition 2.6. For a simplicial complex, the chain complex is a diagram consisting of the chain groups of the complex, where successive chain groups connect via the appropriate boundary maps; it terminates at the trivial group:

$$...C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Defining the boundary map as an alternating sum of the $(n-1)-$faces of a simplex admits a crucial property.

Proposition 2.7. *For any $k-$chain $\mathcal{K}$, the boundary of its boundary vanishes:*

$$(\partial_{k-1} \circ \partial_k)(\mathcal{K}) = 0$$

Hence we have that each image Im ∂_{k+1} is a subgroup of the corresponding kernel Ker ∂_k for all k.

Proof. Each map ∂_k is a homomorphism, so it suffices to check its behavior on an ordered $k-$simplex $[v_0, ..., v_k]$. From the definition we have

$$\partial_k[v_0, ..., v_k] = \sum_{i=1}^{k}(-1)^i[v_0, ..., \hat{v_i}, ..., v_k]$$

Let us examine the behavior of each term in the summation under the $(k-1)-$th boundary map:

$$\partial_{k-1}[v_0, ..., \hat{v_i}, ..., v_k] = [\hat{v_0}, ..., \hat{v_i}, ..., v_k] + ... + (-1)^{i-1}[v_0, ..., \hat{v}_{i-1}, \hat{v_i}, ..., v_k]$$
$$+(-1)^i[v_0, ..., \hat{v_i}, \hat{v}_{i+1}, ..., v_n] + ... + (-1)^{k-1}[v_0, ..., \hat{v_i}, ..., \hat{v_n}]$$

But observe that for the terms in the last expansion whose index j exceeds i, the coefficient becomes $(-1)^{j-1}$ since the $i-$th term is removed. Hence we can write

$$\partial_{k-1}[v_0, ...\hat{v_i}, ..., v_k] = \sum_{j=0}^{i-1}(-1)^j[v_0, ...\hat{v_j}, ..., \hat{v_i}, ..., v_k] + \sum_{j=i+1}^{k}(-1)^{j-1}[v_0, ...\hat{v_i}, ...\hat{v_j}, .., v_k]$$

Then we have two instances of every $[v_0, ..., \hat{v_i}, ...\hat{v_j}, ...v_k] \in \partial_{k-1} \circ \partial_k[v_0, ..., v_k]$: one from $\partial_{k-1}[v_0, ..., \hat{v_i}, ..., v_k]$ with coefficient $(-1)^{i+j}$, and another from $\partial_{k-1}[v_0, ...\hat{v_j}, ..., v_k]$

with coefficient $(-1)^{i+j-1}$. These pairs annihilate one another, and we obtain $(\partial_{k-1} \circ \partial_k)(\mathcal{K}) = 0$ for any $k-$chain $\mathcal{K}$.

If $\mathcal{K}$ is the boundary of a $(k+1)-$chain $\mathcal{L}$ ($\mathcal{K} \in \text{Im } \partial_{k+1}$ and $\partial_{k+1}(\mathcal{L}) = \mathcal{K}$), then

$$\partial_k(\mathcal{K}) = (\partial_k \circ \partial_{k+1})(\mathcal{L}) = 0$$

It follows that $\mathcal{K} \in \text{Ker } \partial_k$, hence $\text{Im } \partial_{k+1} \subset \text{Ker } \partial_k$ for all $k \geq 0$. Now because each map ∂_k is a homomorphism, the $k-$th kernel and $(k+1)-$th image are both subgroups of each $k-$th chain group:

$$\text{Ker } \partial_k, \text{Im } \partial_{k+1} \leq C_k \text{ for all } k \geq 0$$

But provided the inclusion of each $(k+1)-$image in each $k-$kernel, we have

$$\text{Im } \partial_{k+1} \leq \text{Ker } \partial_k \text{ for all } k \geq 0$$

$\square$

For homology, we want to count holes, not boundaries; defining it as the quotient of $k-$cycles ($\text{Ker } \partial_k$) by $k-$boundaries ($\text{Im } \partial_{k+1}$)—whose cosets are equivalence classes of $k-$cycles identified by their corresponding $k-$hole—accomplishes this.

Definition 2.8. Consider a space X along with an associated simplicial complex and chain complex. We define the $k-$th (simplicial) homology group $H_k^{\Delta}(X)$ of X by the quotient of the $k-$th kernel modulo the $(k+1)-$th image under the boundary maps:

$$H_k^{\Delta}(X) = \text{Ker } \partial_k \Big/ \text{Im } \partial_{k+1}$$

Elements of $H_k^{\Delta}(X)$ are referred to as (simplicial) homology classes. Two cycles in the same homology class are called homologous.

Definition 2.9. We regard a chain complex as exact if the $k-$th kernel and the $(k+1)-$th image coincide for all $k \geq 0$. Observe that any space whose simplicial complex yields an exact chain complex has trivial homology groups for all $k \geq 0$.

In light of this definition, we can succinctly refer to homology as a measure of the inexactness of the chain complex associated with a space.

3. Homology as an Invariant

Since several different simplicial complexes may represent the same space, how can we guarantee that they all admit the same homology? We do not answer this question in this paper; Prerna Nadathur's 2007 REU paper, titled "An Introduction to Homology," completes with a proof of this. It involves the more abstract formulation of singular homology, which begins with the definition of a singular simplex (to follow), demonstrating its equivalence to simplicial homology for a space admitting a suitable simplicial complex.

Definition 3.1. A singular $k-$simplex in X is a continuous map $\sigma_k : \Delta^k \to X$. In singular homology, we consider all continuous maps from a $k-$simplex into X:

$$\{\sigma_k : \Delta^k \to X | \sigma_k \text{ is cts.}\}$$

Chain groups, $k-$chains, boundary maps/homomorphisms, and the singular homology groups are defined analogously to the simplicial definitions.

It remains to demonstrate the invariance of homology. We will characterize it as a functor from the category of topological spaces **Top** to that of abelian groups **Ab**. That functors preserve isomorphisms between categories means that a homeomorphism amongst spaces implies an isomorphism in homology.

Definition 3.2. A category $\mathcal{C}$ consists of a class of objects $\mathrm{Obj}(\mathcal{C})$, a set of morphisms $\mathrm{Hom}(A, B)$ for any $A, B \in \mathrm{Obj}(\mathcal{C})$, and a composition law $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ denoted $(g, f) \mapsto f \circ g$. It satisfies

(1) Each morphism $f \in \mathrm{Hom}(A, B)$ of $\mathcal{C}$ has a unique domain A and unique target B.
(2) For every object A, there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$
(3) The composition law is associative.

The objects and morphisms of **Top** are topological spaces and continuous functions; those of **Ab** are abelian groups and group homomorphisms.

Definition 3.3. In a category $\mathcal{C}$, a morphism $f \in \mathrm{Hom}(A, B)$ is an isomorphism if there exists a morphism $f^{-1} \in \mathrm{Hom}(B, A)$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Isomorphisms in **Top** and **Ab** are homeomorphisms and group isomorphisms.

Definition 3.4. For categories $\mathcal{C}$ and $\mathcal{D}$, a map $F : \mathcal{C} \to \mathcal{D}$ is a functor if

(1) If A is an object of $\mathcal{C}$, then $F(A)$ is an object of $\mathcal{D}$.
(2) If $f \in \mathrm{Hom}(A, B)$, then $F(f) \in \mathrm{Hom}(F(A), F(B))$.
(3) If $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, then $F(g \circ f) = F(g) \circ F(f)$
(4) For every object $A \in \mathcal{C}$, we have $F(1_A) = 1_{F(A)}$.

Proposition 3.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.*

Proof. If $f \in \mathrm{Hom}(A, B)$ is an isomorphism in $\mathcal{C}$, it has an inverse $f^{-1} \in \mathrm{Hom}(B, A)$. Using (3) and (4) from Definition 3.4, one can show that $F(f)$ and $F(f^{-1})$ satisfy Definition 3.3.

□

To characterize homology as a functor, a mechanism by which to relate the chain groups of two spaces given a continuous map is required.

Definition 3.6. For any singular $n-$simplex $\sigma \in X$ and continuous map $f : X \to Y$, $f \circ \sigma$ is a singular $n-$simplex in Y by closure. To extend this to singular $n-$chains we define the chain map $f_\# : C_n(X) \to C_n(Y)$ as

$$f_\#(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma f \circ \sigma$$

We omit references to n, X, and Y in this definition to avoid excessive notation; the next lemma justifies this.

Lemma 3.7. *If $f : X \to Y$ is continuous, then $\partial_n \circ f_\# = f_\# \circ \partial_n$. Here, it is implicit that the first boundary map refers to Y while the second refers to X.*

Proof. Checking this for a basis element σ suffices for homomorphisms $f_\#$ and ∂_n:

$$(f_\# \circ \partial_n)(\sigma) = f_\#(\sum (-1)^i \sigma) = \sum (-1)^i f \circ \sigma = \partial_n(f \circ \sigma) = (\partial_n \circ f_\#)(\sigma)$$

□

Theorem 3.8. *Homology is an invariant of a space.*

Proof. To show $H_n : \mathbf{Top} \to \mathbf{Ab}$ is a functor, we define a well-defined action on the objects and morphisms of **Top** while checking the functorial properties.

Obtaining the homology group $H_n(X)$ from a space X is well-defined, since consider all continuous simplicial maps in singular homology.

For this proof, let $Z_n(X) = \text{Ker } \partial_n$ and $B_n(X) = \text{Im } \partial_{n+1}$. Consider a continuous map $f : X \to Y$. Define $H_n(f) : H_n(X) \to H_n(Y)$ by sending a coset in $H_n(X)$ to the corresponding coset in $H_n(Y)$ under the induced map $f_\#$. Here, z_n is a cycle in $Z_n(X)$:

$$H_n(f)(z_n + B_n(X)) = f_\#(z_n) + B_n(Y)$$

To see that it is well-defined, first note that $f_\#(z_n) \in Z_n(Y)$, given z_n has zero boundary. We also check that cosets are sent to corresponding cosets. If $\partial_{n+1}(u) \in B_n(X)$, then by Lemma 3.7 we have

$$(f_\# \circ \partial_{n+1})(u) = (\partial_{n+1} \circ f_\#)(u)$$

Because $f_\#(u)$ is a $(n+1)-$chain for Y, $(f_\# \circ \partial_{n+1})(u) \in B_n(Y)$ follows. Recall $y_n + B_n(X) = z_n + B_n(X)$ iff $y_n - z_n \in B_n(X)$. Then

$$f_\#(y_n - z_n) = f_\#(y_n) - f_\#(z_n) \in B_n(Y)$$

Thus $f_\#(y_n) + B_n(Y) = f_\#(z_n) + B_n(Y)$.

Now we check that H_n satisfies the functorial properties. First $H_n(f)$ should be a group homomorphism. Take any two $y_n + B_n(X), z_n + B_n(X) \in H_n(X)$. Then

$$H_n(f)[(y_n + B_n(X)) + (z_n + B_n(X))] = H_n(f)[(y_n + z_n) + B_n(X)]$$
$$= f_\#(y_n + z_n) + B_n(Y) = [f_\#(y_n) + B_n(Y)] + [f_\#(z_n) + B_n(Y)]$$
$$= H_n(f)(y_n + B_n(X)) + H_n(f)(z_n + B_n(X))$$

We also want $H_n(1_X)$ is the identity morphism. Let $z_n = \sum n_\sigma \sigma$. Observe:

$$H_n(1_X)(z_n + B_n(X)) = 1_{X_\#}(z_n) + B_n(X)$$
$$= \sum n_\sigma 1_X \circ \sigma + B_n(X) = z_n + B_n(X) = 1_{H_n(X)}(z_n + B_n(X))$$

Hence $H_n(1_X) = 1_{H_n(X)}$. Finally we check that H_n obeys composition:

$$H_n(g \circ f)(z_n + B_n(X)) = (g \circ f)_\#(z_n) + B_n(Z) = \sum n_\sigma (g \circ f) \circ \sigma + B_n(Z)$$
$$= \sum n_\sigma g \circ (f \circ \sigma) + B_n(Z) = g_\#(\sum n_\sigma f \circ \sigma) + B_n(Z)$$
$$= H_n(g)(f_\#(z_n) + B_n(Y)) = (H_n(g) \circ H_n(f))(z_n + B_n(X))$$

And so we have $H_n(g \circ f) = H_n(g) \circ H_n(f)$. Thus $H_n : \mathbf{Top} \to \mathbf{Ab}$ is a functor. □

4. Some Computations

Now that we have characterized homology as an invariant, we will compute the homology groups of some elementary spaces—namely, the sphere and real projective space in 1 and 2 dimensions, some properties of which follow.

Recall that S^n is a compact $n-$manifold; refer to [3] for a proof.

Real projective $n-$space $\mathbb{R}P^n$ consists of all lines in $\mathbb{R}^{n+1}$ passing through the origin. Its topology is generated by the open cones with the cone point on the origin. It can be represented as a quotient space with the inherited topology:

$$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ (x \sim \lambda x \text{ for } \lambda \in \mathbb{R})$$

Similarly to S^n, $\mathbb{R}P^n$ is a compact $n-$manifold. ([2] shows compactness, [4] demonstrates that it is a manifold.) Moreover, $\mathbb{R}P^1$ is homeomorphic to the circle S^1 and $\mathbb{R}P^2$ is homeomorphic to the $2-$sphere S^2 with its antipodal points identified, which is in turn homeomorphic to the closed $2-$disk with antipodal points on the boundary identified:

$$\mathbb{R}P^2 \cong S^2 \Big/ (x \sim -x) \cong D^2 \Big/ (x \sim -x \text{ when } x \in \partial D^2)$$

Three homology computations are to follow.

Computation 4.1. To compute the homology of the circle S^1, we begin with a simplicial complex with proper identifications, yielding an object homeomorphic to the circle. This simplifies the problem, since the simplicial maps for the former figure amount to nothing more than identity maps, so we do not need to deal with them explicitly. Start with an oriented $1-$simplex $[x_1, x_2]$, which we denote a. Identify the vertices x_1 and x_2, referring to them as x to reduce notation. The figure below summarizes this procedure:

$$x \xrightarrow[a]{\hspace{3cm}} x$$

Figure 2: simplicial complex for S^1

It admits the following chain complex. Here, $\langle p_1, ..., p_n\rangle$ denotes the free abelian group generated by the set $\{p_i\}$ and 0 denotes the trivial group:

$$...0 \xrightarrow{\partial_2} \langle a\rangle \xrightarrow{\partial_1} \langle x\rangle \xrightarrow{\partial_0} 0$$

Since the zeroth boundary homomorphism maps all 0-simplexes to 0, we immediately get Ker $\partial_0 = \langle x\rangle$. To compute Im ∂_1, we must observe its behavior on some arbitrary $1-$chain na, with $n \in \mathbb{Z}$:

$$\partial_1(na) = n\partial_1(a) = n(x - x) = 0$$

Thus Im ∂_1 is trivial. We now compute the zeroth homology group:

$$H_0 = \langle x\rangle / 0 = \langle x\rangle$$

Hence $H_0 \cong \mathbb{Z}$.

Since the function ∂_1 maps every $1-$chain to zero, it follows that the simplex a generates Ker ∂_1, i.e. Ker $\partial_1 = \langle a\rangle$. The domain of the map ∂_2 is trivial, implying the triviality of Im ∂_2 in turn. Similarly to before, we can characterize the first homology group:

$$H_1 = \langle a\rangle / 0 = \langle a\rangle$$

Thus $H_1 \cong \mathbb{Z}$. As with Im ∂_2, each higher image and kernel amount to nothing more than the trivial group and hence admit trivial homology groups. This completes the computation for the homology of the circle. Moreover, because the real projective line is homeomorphic to the circle, we have also computed its homology.

Computation 4.2. Similarly to the previous computation, we begin by constructing a simplicial complex with identifications rendering it homeomorphic to the $2-$sphere S^2. Begin with two oriented 2-simplexes $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$. First, attach the simplexes by identifying one of the $1-$faces, denoted c, which connects vertices x_1 to z_1 and x_2 to z_2. To indicate the subsequent vertex identifications, we refer to them as x and z. We will identify the edges connecting vertices x and $y_{1,2}$ and denote the resulting edge a. Similarly, we will identify the edges connecting vertices $y_{1,2}$ and z and denote the resulting edge b. Under these identifications, it makes sense to refer to the former vertices $y_{1,2}$ as y. Finally, we orient the actual simplexes A counter-clockwise (bottom) and B clockwise (top), respectively, to respect the orientation of the vertices. See the figure below:

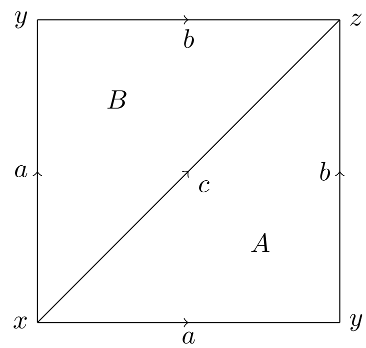

Figure 3: simplicial complex for S^2

Provided this, we can characterize the chain complex:

$$...0 \xrightarrow{\partial_3} \langle A, B\rangle \xrightarrow{\partial_2} \langle a, b, c\rangle \xrightarrow{\partial_1} \langle x, y, z\rangle \xrightarrow{\partial_0} 0$$

Again we already have Ker $\partial_0 = \langle x, y, z\rangle$. In order to determine Im ∂_1, we must examine the behavior of the map ∂_1 on any arbitrary $1-$chain $la+mb+nc$. Observe:

$$\partial_1(la + mb + nc) = l\partial_1(a) + m\partial_1(b) + n\partial_1(c)$$

$$= l(y - x) + m(z - y) + n(z - x) = (-l - n)x + (l - m)y + (m + n)z$$

Since any finitely-generated free abelian group with rank n is isomorphic to $\mathbb{Z}^n$, we can represent this chain as a $3-$tuple $[l, m, n]^T \in \mathbb{Z}^3$. Then we observe the following behavior of the vector under ∂_1:

$$[l, m, n]^T \mapsto [-l - n, l - m, m + n]^T$$

We can represent the linear transformation ∂_1 as a matrix:

$$[\partial_1] = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Performing (integral) row reduction yields

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Elementary techniques from linear algebra give us the basis vectors for Im $[\partial_1]$:

$$[-1, 1, 0]^T, [0, -1, 1]^T$$

They correspond to $y - x$ and $z - y$, so we will compute the following quotient to obtain the zeroth homology group:

$$H_0 = \langle x, y, z\rangle \Big/ \langle y - x, z - y\rangle$$

By identifying $y - x$ and $z - y$, the quotient homomorphism q : Ker $\partial_0 \to H_0$ behaves as follows:

$$q(y - x) = q(y) - q(x) = 0 \Longrightarrow q(y) = q(x)$$

$$q(z-y) = q(z) - q(y) = 0 \Longrightarrow q(z) = q(y)$$

We obtain the quotient by first swapping instances of y by x:

$$H_0 = \langle x, x, z\rangle \Big/ \langle 0, z-x\rangle = \langle x, z\rangle \Big/ \langle z-x\rangle$$

Subsequently we swap instances of z with y, i.e. x:

$$H_0 = \langle x, x\rangle / 0 = \langle x\rangle$$

Thus $H_0 \cong \mathbb{Z}$.

With our matrix representation for ∂_1, we can compute the basis for Ker ∂_1 to be $[-1,-1,1]^T$ i.e. $-a-b+c$ by recalling that the row-reduced matrix and the original matrix have the same kernels. We implement a similar procedure as described before to determine Im ∂_2 by considering some arbitrary 2−chain $mA+nB$:

$$\partial_2(mA+nB) = m\partial_2(A) + n\partial_2(B)$$

$$= m(a-c+b) + n(a-c+b) = (m+n)a + (m+n)b + (-m-n)c$$

This admits a matrix representation:

$$[\partial_2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then its row-reduced form is the following:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The basis set for of this matrix is $[1,1,-1]^T$, which corresponds to the chain $a+b-c$. But then $-a-b+c$ is another suitable basis for. From this information we can compute the first homology group:

$$H_1 = \langle -a-b+c\rangle \Big/ \langle -a-b+c\rangle = 0$$

So H_1 is trivial.

Using the row-reduced form of the matrix representation of the map ∂_2, we compute the basis of Ker ∂_2 to be $[-1,1]^T$, i.e. $A-B$. Im ∂_3 is trivial, since the map ∂_3 has domain 0. We now have the second homology group:

$$H_2 = \langle A-B\rangle / 0$$

Thus we see $H_2 \cong \mathbb{Z}$. All of the higher homology groups are trivial, so we are done.

Computation 4.3. We begin by considering a more tractable representation of the real projective plane than the original definition provided. Namely, we consider the closed 2−disk with antipodal points on the boundary identified. Our construction of a suitable simplicial complex begins by noting that the closed square is homeomorphic to the closed 2−disk. To begin accounting for the boundary identifications, we identify opposite vertices on the square and denote them x and y. Beginning with the left 1−face and moving clockwise, we denote the first edge connecting x and y by a and the second edge connecting y and the subsequent x by b. We will continue by identifying the subsequent to edges with a and b, respectively. Since antipodal points are identified on the boundary, it makes sense that the orientation of the remaining faces be oriented oppositely to their identifications. But, we must separate the square into two 2−simplexes, which we achieve by connecting the opposite x vertices by an edge c and orienting it diagonally upwards. To orient the actual 2−simplexes, which we denote by A (top) and B (bottom), we take into account the directionality of the faces a and b. Hence both A and B are oriented clockwise. The included figure summarizes this construction:

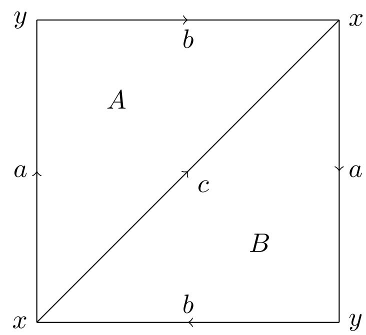

Figure 4: simplicial complex for $\mathbb{R}P^2$

This construction admits the following chain complex:

$$...0 \xrightarrow{\partial_3} \langle A, B\rangle \xrightarrow{\partial_2} \langle a, b, c\rangle \xrightarrow{\partial_1} \langle x, y\rangle \xrightarrow{\partial_0} 0$$

We immediately get that Ker $\partial_0 = \langle x, y\rangle$. We now characterize Im ∂_1:

$$\partial_1(la + mb + nc) = l\partial_1(a) + m\partial_1(b) + n\partial_1(c)$$

$$= l(y - x) + m(x - y) + n(x - x) = (m - l)x + (l - m)y$$

Provided this we can represent the map ∂_1 as a matrix:

$$[\partial_1] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Its row-reduced form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we compute the basis of Im $[\partial_1]$ to be $[-1, 1]^T$, which corresponds to the chain $y - x$. Hence Im $\partial_1 = \langle y - x\rangle$. Now we can compute the zeroth homology group:

$$H_0 \cong \langle x, y\rangle \Big/ \langle y - x\rangle = \langle x, x\rangle / 0 = \langle x\rangle$$

Thus $H_0 \cong \mathbb{Z}$.

Using the row-reduced of the map $[\partial_1]$, we can compute the basis of Ker ∂_1 to be $[1, 1, 0]^T$ and $[0, 0, 1]$, so Ker $\partial_1 = \langle a + b, c\rangle$. Now consider some 2−chain $mA + nB$ and its behavior under the map ∂_2:

$$\partial_2(mA + nB) = m\partial_2(A) + n\partial_2(B)$$

$$= m(b - c + a) + n(b + c + a) = (m + n)a + (m + n)b + (n - m)c$$

Then its matrix representation is

$$[\partial_2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Under row-reduction we obtain

$$\begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

The basis vectors for Im $[\partial_2]$ are then $[1,1,-1]$ and $[1,1,1]$, which means Im ∂_2 is generated by $a+b-c$ and $a+b+c$. We can now compute the quotient H_1, following a similar procedure to that in the previous computation; in particular, we replace instances of $a+b$ with c, since they correspond to the same thing under the quotient map:

$$H_1 = \langle a+b, c\rangle \Big/ \langle a+b-c, a+b+c = \langle c, c\rangle \Big/ \langle 0, 2c\rangle = \langle c\rangle \Big/ \langle 2c\rangle$$

Hence $H_1 \cong \mathbb{Z}_2$.

With the row-reduced form, we can quickly compute that Ker ∂_2 is trivial. Since we know Im ∂_3 is already trivial, we deduce that H_2 is also trivial. Of course, so are the higher homology groups; hence we have computed the homology of $\mathbb{R}P^2$.

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