STOCHASTIC CALCULUS AND BLACK-SCHOLES MODEL

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Abstract. Ito's lemma is often used in Ito calculus to find the differentials of a stochastic process that depends on time. This paper will introduce the concepts in stochastic calculus and derive Ito's lemma. Then, the paper will discuss Black-Scholes model as one of the applications of Ito's lemma. Both Black-Scholes formula for calculating the price of European options and Black-Scholes partial differential equation for describing the price of option over time will be derived and discussed.

Contents

1. Introduction

Ito's lemma is used to find the derivative of a time-dependent function of a stochastic process. Under the stochastic setting that deals with random variables, Ito's lemma plays a role analogous to chain rule in ordinary differential calculus. It states that, if f is a C^2 function and B_t is a standard Brownian motion, then for every t,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

This paper will introduce the concepts in stochastic calculus to build foundations for Ito's lemma. Then, we will derive Ito's lemma using the process similar to Riemann integration in ordinary calculus.

Since Ito's lemma deals with time and random variables, it has a broad applications in economics and quantitative finance. One of the most famous applications is Black-Scholes Model, derived by Fischer Black and Myron Scholes in 1973. We will first discuss Black-Scholes formula, which is used to compute the value of an European call option (C_0) given its stock price (S_0), exercise price (X), time to expiration (T), standard deviation of log returns (σ), and risk-free interest rate (r).

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It states that, for an option that satisfies seven conditions which will be introduced in detail in section 4 of this paper, its value can be calculated by

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2),$$

where

$$d_1 = \frac{ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, d_2 = \frac{ln(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

We will derive Black-Scholes formula and provide some examples of how it is used in finance to evaluate option prices. We will also discuss limitations of Black-Scholes formula by comparing the computed results with historical option prices in markets.

On the other hand, Black-Scholes equation describes the price of option over time. It states that, given the value of an option ($f(t, S_t)$), stock price (S_t), time to expiration (t), standard deviation of log returns (σ), and risk-free interest rate (r), they satisfy

$$\frac{\partial f(t, S_t)}{\partial t} + rS_t \frac{\partial f(t, S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f(t, S_t)}{\partial S_t^2} = rf(t, S_t).$$

We will derive Black-Scholes equation as well using Ito's lemma from stochastic calculus. The natural question that arises is whether solving for f in Black-Scholes equation gives the same result as the Black-Scholes formula. Solving the equation with boundary condition $f(t, S_t) = max(S - X, 0)$, which depicts a European call option with exercise price X, indeed gives a Black-Scholes formula. This completes the Black-Scholes model.

2. Stochastic Calculus

Definition 2.1. A **stochastic process** is a process that can be described by the change of some random variables over time.

Definition 2.2. Stationary increments means that for any $0 < s, t < \infty$, the distribution of the increment $W_{t+s} - W_s$ has the same distribution as $W_t - W_0 = W_t$.

Definition 2.3. Independent increments means that for every choice of nonnegative real numbers $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n < \infty$, the increment random variables $W_{t_1} - W_{s_1}, W_{t_2} - W_{s_2}, ..., W_{t_n} - W_{s_n}$ are jointly independent.

Definition 2.4. A **standard Brownian motion (Weiner process)** is a stochastic process $\{W_t\}, t \geq 0^+$ with the following properties:

(1) $W_0 = 0$,
(2) the function $t \to W_t$ is continuous in t,
(3) the process $\{W_t\}, t \geq 0$ has stationary, independent increments,
(4) the increment $W_{t+s} - W_s$ has the Normal$(0, t)$ distribution.

Definition 2.5. A variable x is said to follow a **Weiner process with drift** if it satisfies $dx = a\, dt + b\, dW(t)$, where a, b are constants and $W(t)$ is a Weiner process.

Notice that there is no uncertainty in $dx = a\ dt$, and it can easily be integrated to $x = x_0 + at$ where x_0 is the initial value. A constant a represents the magnitude of certain change in x as t varies. On the other hand, $b\ dW(t)$ represents the variability of the path followed by x as t changes. A constant b represents the magnitude of uncertainty.

However, the magnitudes of expected drift and volatility are not constant in most real-life models. Instead, they often depend on when the value of x is evaluated (t) and the value of x at time t (X_t). For example, an expected change in stock price and its volatility are often estimated using the current stock price and the time when it is estimated. Such a motivation naturally leads to the following generalization of Weiner process.

Definition 2.6. An **n-dimensional Ito process** is a process that satisfies

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t.$$

where W is an m-dimensional standard Brownian motion for some number m, a and b are n-dimensional and $n * m$-dimensional adapted processes, respectively.

Note that n-dimensional Ito process is an example of a stochastic differential equation where X_t evolves like a Brownian motion with drift $a(t, X_t)$ and standard deviation $b(t, X_t)$. Moreover, we say that X_t is a solution to such a stochastic differential equation if it satisfies

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s,$$

where X_0 is a constant. Integrating constant X_0 and the ds integral can easily be done using ordinary calculus. The only problem is the term that involves dW_s integral. We solve this issue by introducing stochastic integration.

Definition 2.7. A process A_t is a **simple process** if there exist times $0 = t_0 < t_1 < ... < t_n < \infty$ and random variables Y_j for $j = 0, 1, 2, ..., n$ that are F_{tj}-measurable such that $A_t = Y_j, t_j \leq t \leq t_{j+1}$.

Now, set $t_{n+1} = \infty$ and assume $\mathbb{E}[Y_j^2] < \infty$ for each j. For simple process A_t, we define

$$Z_t = \int_0^t A_s dB_s$$

by

$$Z_{tj} = \sum_{i=0}^{j-1} Y_i[B_{ti+1} - B_{ti}], \quad Z_t = Z_{tj} + Y_j[B_t - B_{tj}] \tag{2.8}$$

Just like Riemann integration for ordinary calculus, we are making sure that the integral is bounded by setting $\mathbb{E}[Y_j^2] < \infty$ and dividing the domain into partitions to define integral. We now have all necessary concepts in stochastic calculus to derive Ito's lemma.

3. Ito's Lemma

Theorem 3.1 (Ito's Lemma I)**.** *Suppose f is a C^2 function and B_t is a standard Brownian motion. Then, for every t,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

The formula above can also be written in differential form as

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

Proof. For simplicity, let's assume that $t = 1$ so that

$$f(B_1) = f(B_0) + \int_0^1 f'(B_s)dB_s + \frac{1}{2}\int_0^1 f''(B_s)ds.$$

$$f(B_1) = f(B_0) - f(B_0) + f(B_{1/n}) - f(B_{1/n}) + ... + f(B_{(n-1)/n}) - f(B_{(n-1)/n}) + f(B_1)$$

$$= f(B_0) + \sum_{j=1}^{n}[f(B_{j/n}) - f(B_{(j-1)/n})].$$

Therefore,

$$f(B_1) - f(B_0) = \sum_{j=1}^{n}[f(B_{j/n}) - f(B_{(j-1)/n})]. \tag{3.2}$$

Now, using the second degree Taylor approximation, we can write

$$f(B_{j/n}) = f(B_{(j-1)/n}) + f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})$$
$$+\frac{1}{2}f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2)$$

and therefore,

$$f(B_{j/n}) - f(B_{(j-1)/n}) = f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n}) \tag{3.3}$$
$$+\frac{1}{2}f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2).$$

Combining the equations (3.2) and (3.3),

$$f(B_1) - f(B_0) = \sum_{j=1}^{n}[f'(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})$$
$$+\frac{1}{2}f''(B_{(j-1)/n})(B_{j/n} - B_{(j-1)/n})^2 + o((B_{j/n} - B_{(j-1)/n})^2)].$$

Taking limits of $n \to \infty$ to both sides, $f(B_1) - f(B_0)$ is equal to the sum of the following three limits:

$$\lim_{n\to\infty}\sum_{j=1}^{n}[f'(B_{(j-1)/n})(B_{j/n}-B_{(j-1)/n}), \tag{3.4}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{2}f''(B_{(j-1)/n})(B_{j/n}-B_{(j-1)/n})^2, \tag{3.5}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}o((B_{j/n}-B_{(j-1)/n})^2)]. \tag{3.6}$$

Let's first think about the limit 3.4. Comparing the definition of simple process approximation from the equation 2.8, we notice that $f'(B_t)$ is in place of Y_i. Therefore,

$$\lim_{n\to\infty}\sum_{j=1}^{n}[f'(B_{(j-1)/n})(B_{j/n}-B_{(j-1)/n}) = \int_0^1 f'(B_t)dB_t.$$

Now consider the limit 3.5. Let $h(t) = f''(B_t)$. Since f is C^2 function, $h(t)$ is continuous function. Therefore, for every $\epsilon > 0$, there exists a step function $h_\epsilon(t)$ such that, for every t, $|h(t)-h_\epsilon(t)| < \epsilon$. Given an ϵ, consider each interval on which h_ϵ is constant so find

$$\lim_{n\to\infty}\sum_{j=1}^{n}h_\epsilon(t)[B_{j/n}-B_{(j-1)/n}]^2 = \int_0^1 h_\epsilon(t)dt. \tag{3.7}$$

Moreover, for given ϵ,

$$|\sum_{j=1}^{n}[h(t)-h_\epsilon(t)][B_{j/n}-B_{(j-1)/n}]^2| \le \epsilon\sum_{j=1}^{n}[B_{j/n}-B_{(j-1)/n}]^2 \to \epsilon.$$

as $n\to\infty$.

Since the sum of the differences can become smaller that any number ϵ,

$$\int_0^1 h_\epsilon(t)dt = \int_0^1 h(t)dt = \int_0^1 f''(B_t)dt. \tag{3.8}$$

Combining the results of 3.7 and 3.8, we get

$$\lim_{n\to\infty}\frac{1}{2}\sum_{j=1}^{n}f''(B_{(j-1)/n})[B_{j/n}-B_{(j-1)/n}]^2 = \frac{1}{2}\int_0^1 f''(B_t)dt.$$

Lastly, consider the limit 3.6. Since B_t is a standard Brownian motion, $[B_{j/n} - B_{(j-1)/n}]^2$ is approximately $1/n$. Therefore, the limit 3.6 is n terms that are smaller than $1/n$. Therefore, as $n\to\infty$, the limit equals zero.

Therefore,

$$f(B_1)-f(B_0) = \int_0^1 f'(B_t)dB_t + \frac{1}{2}\int_0^1 f''(B_t)dt + 0.$$

We assumed $t = 1$ for simplicity in notation. However, nothing changes from the proof above if we divide partitions of the interval $[0, t]$ instead of $[0, 1]$. Therefore, we conclude that

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

□

Following is an alternative form of Ito's lemma with its derivation. It provides a more intuitive understanding of Ito's lemma and will be used to derive Black-Scholes equation in the later section.

Theorem 3.9 (Ito's Lemma II). *Let $f(t, X_t)$ be an Ito process which satisfies the stochastic differential equation*

$$dX_t = Z_t dt + y_t dB_t.$$

If B_t is a standard Brownian motion and f is a C^2 function, then $f(t, X_t)$ is also an Ito process with its differential given by

$$df(t, X_t) = [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}{y_t}^2]dt + \frac{\partial f}{\partial X_t}y_t dB_t.$$

Proof. Consider a stochastic process $f(t, X_t)$. Note that, since X_t is a standard Brownian motion, $X_0 = 0$. Using a Taylor approximation and taking differentials for both sides, we get

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}(dX_t)^2 + \frac{\partial^2 f}{\partial f \partial Xt}dtdX_t + ... \tag{3.10}$$

Now, note that since the quadratic variation of W_t is t, the term $(dW_t)^2$ contributes an additional dt term. However, all other terms are smaller than dt and thus can be treated like a zero. Such a result is often illustrated as Ito's multiplication table.

Using Ito's multiplication table to simplify the equation 3.10, we get

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}(dX_t)^2.$$

Such a result should be described by the stochastic differential equation for X_t, which is $dX_t = Z_t dt + y_t dB_t$. Therefore, we make a substitution of dX_t to get

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}[Z_t dt + y_t dB_t] + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}(Z_t dt + y_t dB_t)^2. \tag{3.11}$$

Since

$$(Z_t dt + y_t dB_t)^2 = Z_t^2(dt)^2 + 2Z_t y_t dt dB_t + {y_t}^2(dB_t)^2 = {y_t}^2 dt,$$

we make a substitution to equation 3.11 and get

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}[Z_t dt + y_t dB_t] + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}{y_t}^2 dt$$

$$= [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} Z_t + \frac{1}{2} \frac{\partial^2 f}{\partial {X_t}^2} {y_t}^2] dt + \frac{\partial f}{\partial X_t} y_t dB_t.$$

□

4. Black-Scholes Formula

The Black-Scholes formula is often used in finance sector to evaluate option prices. In this paper, we will focus on calculating the value of European call option since put option can be calculated analogously. Although the derivation of Black-Scholes formula does not use stochastic calculus, it is essential to understand significance of Black-Scholes equation which is one of the most famous applications of Ito's lemma. Black-Scholes equation will be discussed in the next section of the paper. To understand Black-Scholes formula and its derivation, we need to introduce some relevant concepts in finance.

Definition 4.1. An **option** is a security that gives the right to buy or sell an asset within a specified period of time.

Definition 4.2. A **call option** is the kind of option that gives the right to buy a single share of common stock.

Definition 4.3. An **exercise price (striking price)** is the price that is paid for the asset when the option is exercised.

Definition 4.4. A **European option** is a type of option that can be exercised only on a specified future date.

Definition 4.5. If random variable Y follows the normal distribution with mean μ and variance σ^2, then $X = e^Y$ follows the **log-normal distribution** with mean and variance

$$\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2} \quad Var[X] = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}.$$

The probability distribution function for X is

$$dF_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{(-\frac{1}{2}(\frac{lnx - \mu}{\sigma})^2)}, \tag{4.6}$$

and the cumulative distribution function for X is

$$F_X(x) = \Phi(\frac{lnx - \mu}{\sigma}), \tag{4.7}$$

where $\Phi(x)$ is the standard normal cumulative distribution function.

Now, let's calculate the expected value of X conditional on $X > x$ denoted as $L_X(K) = \mathbb{E}[X|X > x]$.

$$L_X(K) = \int_K^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{lnx - \mu}{\sigma})^2} dx.$$

Changing variables as $y = lnx$, $x = e^y$, $dx = e^y dy$, and Jacobian is e^y. Therefore, we can rewrite the equation 4.6 as

$$L_X(K) = \int_{lnK}^{\infty} \frac{e^y}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2} dy \tag{4.8}$$

$$= exp(\mu + \frac{1}{2}\sigma^2)\frac{1}{\sigma}\int_{lnK}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}(\frac{y-(\mu+\sigma^2)}{\sigma})^2)dy.$$

Notice that the integral in equation 4.7 has the form of standard normal distribution. Therefore, we can express it as

$$L_X(K) = exp(\mu + \frac{\sigma^2}{2})\Phi(\frac{-lnK + \mu + \sigma^2}{\sigma}). \tag{4.9}$$

Theorem 4.10 (Black-Scholes Formula)**.** *The value of an European call option (C_0) can be calculated given its stock price (S_0), exercise price (X), time to expiration (T), standard deviation of log returns (σ), and risk-free interest rate (r). Assume that the option satisfies the following conditions:*

a) The short-term interest rate is known and is constant through time.

b) The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant.

c) The stock pays no dividends or other distributions.

d) The option is "European," that is, it can only be exercised at maturity.

e) There are no transaction costs in buying or selling the stock or the option.

f) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

g) There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

Then, the price can be calculated by

$$C_0 = S_0 N(d_1) - Xe^{-rT}N(d_2),$$

where

$$d_1 = \frac{ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, d_2 = \frac{ln(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

and $N(x)$ represents a cumulative distribution function for normally distributed random variable x.

Proof. Calculating for the present value of the expected return of the option, we get

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_0 - X)^+|F_0]$$

Now, calculating the expected value using integration,

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_0-X)^+|F_t] = e^{rT}\int_X^\infty (S_0-X)dF(S_0)$$

$$= e^{-rT}\int_X^\infty S_0 dF(S_0) - e^{-rT}X\int_X^\infty dF(S_0). \tag{4.11}$$

Now, note that the distribution of possible stock prices at the end of any finite interval is log-normal. Therefore, recall equation 4.9 to evaluate the first integral of the equation 4.11:

$$e^{-rT}\int_X^\infty S_0 dF(S_0) = e^{rT}L_{S_T}(X) \tag{4.12}$$

$$= e^{-rT}exp(lnS_0 + (r-\frac{\sigma^2}{2})T + \frac{\sigma^2 T}{2}) * \Phi(\frac{-lnX + lnS_0 + (r-\frac{\sigma^2}{2})T + \sigma^2 T}{\sigma\sqrt{T}})$$

$$= e^{-rT}S_0 e^{rT}\Phi(d_1) = S_0\Phi(d_1).$$

Now let's calculate the second integral of 4.11 using the equation 4.6.

$$r^{-rT}X\int_X^\infty dF(S_0) = e^{rT}X[1-F(X)] \tag{4.13}$$

$$= e^{-rT}X[1-\Phi(\frac{lnX - lnS_0 - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})]$$

$$= e^{-rT}X[1-\Phi(-d_2)] = e^{rT}X\Phi(d_2).$$

Combining the results of equations 4.11, 4.12 and 4.13, we get

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_0-X)^+|F_0] = S_0N(d_1) - Xe^{rT}N(d_2).$$

□

Example 1. *Let's try Finding the price of an European call option whose stock price is \$90, months to expiration is 6 months, risk-free interest rate is 8%, standard deviation of stock is 23%, exercise price is \$80.*

Since $S_0 = 90, T = 0.5, r = 0.08, \sigma = 0.23, and X = 80$, *plug in those values into the Black-Scholes formula to get*

$$C_0 = 90 * N(d_1) - 80 * e^{-0.08*0.5}N(d_2),$$

where

$$d_1 = \frac{ln(\frac{90}{80}) + (0.08 + \frac{0.23^2}{2})0.5}{0.23 * \sqrt{0.5}} = 1.0515$$

and

$$d_2 = \frac{ln(\frac{90}{80}) + (0.08 - \frac{0.23^2}{2})0.5}{0.23 * \sqrt{0.5}} = 0.8889.$$

Now, use the normal distribution table to find the values of $N(1.0515)$ *and* $N(0.8889)$ *to get*

$$N(1.0515) = 0.8535, \quad N(0.8889) = 0.813.$$

Therefore, the value of the option is

$$C_0 = 90 * 0.8535 - 80 * e^{-0.08*0.5} * 0.813 = 14.33.$$

Black and Scholes have done empirical tests of Black-Scholes formula on a large body of call-option data. Although the formula gave a good approximation, they found that the option buyers pay prices consistently higher than those predicted by the formula.

Let's think about the reason behind such a discrepancy. In the real market, real interest rates are not constant as assumed in Black-Scholes model. Most stocks pay some form of distributions including dividends. Due to such factors, volatility (σ) in Black-Scholes formula may be underestimated. Since the price of an option (C_0) is a monotonically increasing function of the volatility (σ), such a difference in volatility could be one of the reasons for underestimation of option prices.

5. Black-Scholes Equation

Now we are able to find the price of an option. However, investors are often interested in predicting the future price of an option to build a profitable portfolio. Black-Scholes partial differential equation does the work by describing the price of option over time.

Theorem 5.1 (Black-Scholes Equation)**.** *Let the value of an option be $f(t, S_t)$, standard deviation of stock be stock's returns be σ, and risk-free interest rate be r. Then the price of an option over time can be expressed by the following partial differential equation:*

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2} = rf.$$

Proof. Let's first create a portfolio that consists of ϕ units of stock share and φ units of cash. Denote the amount of share and cash at time t as ϕ_t and φ_t, respectively. Then, the value of the portfolio at time t (V_t) will be the sum of the value of stock share ($\phi_t S_t$) and the amount of real interest that can be earned by possessing the cash for dt amount of time ($rPdt$) so that

$$V_t = \phi_t S_t + \varphi_t rPdt.$$

Now, to apply Ito's lemma, let's calculate the partial derivatives of V_t.

$$\frac{\partial V_t}{\partial t} = \varphi_t rPdt, \quad \frac{\partial V_t}{\partial s} = \phi_t, \quad \frac{\partial^2 V_t}{\partial s^2} = 0.$$

Now, recall Ito's lemma II from the previous section and modify it with slightly different notations to write

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t\frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2}\right)dt + \sigma S_t\frac{\partial f}{\partial S_t}dZ_t. \tag{5.2}$$

Substitute V_t in place of f and plug in the values of derivatives to the equation to get

$$dV_t = (\varphi_t rPdt + \mu S_t \phi_t + \frac{1}{2}\sigma^2 * 0)dt + \sigma S_t \phi_t dZ_t \tag{5.3}$$

$$= (\varphi_t rP + \mu S_t \phi_t)dt + \sigma S_t \phi_t dZ_t.$$

Now, we need to come up with the formula for ϕ and φ by equating coefficients of equations 5.2 and 5.3. Since we do not know the expressions for ϕ and φ, first compare the coefficients for dZ_t to get

$$\sigma S_t f_{S_t} = \sigma S_t \phi_t, \quad \frac{\partial f}{\partial S_t} = f_{S_t} = \phi_t.$$

Therefore,

$$V_t = f = \frac{\partial f}{\partial S_t} S_t + \varphi_t P, \quad \varphi_t = \frac{1}{P}[f - \frac{\partial f}{\partial S_t} S_t].$$

Plug in the values of ϕ and φ into the equation 5.3 and compare coefficients of 5.2 and 5.3 for dt to get

$$\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = \mu \frac{\partial f}{\partial S_t} S_t + \frac{1}{P}[f - \frac{\partial f}{\partial S_t} S_t] rP.$$

Simplifying,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = rf - \frac{\partial f}{\partial S_t} S_t r$$

and therefore

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t} S_t + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = rf.$$

□

Having Black-Scholes formula and equation, the natural question is to ask if solving Black-Scholes partial differential equation gives Black-Scholes formula. Indeed, using Feynman-Kac Theorem and the boundary condition $f(t, S_t) = max(S_t - X)$, we can derive Black-Scholes formula from Black-Scholes equation.

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