A BRIEF OVERVIEW OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This paper discusses the basic techniques of solving linear ordinary differential equations, as well as some tricks for solving nonlinear systems of ODE's, most notably linearization of nonlinear systems. The paper proceeds to talk more thoroughly about the van der Pol system from Circuit Theory and the FitzHugh-Nagumo system from Neurodynamics, which can be seen as a generalization of the van der Pol system.

Contents

1. General Solution to Autonomous Linear Systems of Differential Equations

Let us begin our foray into systems of differential equations by considering the simple 1-dimensional case

$$x' = ax \tag{1.1}$$

for some constant a. This equation can be solved by separating variables, yielding

$$x = x_0 e^{at} \tag{1.2}$$

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where $x_0 = x(0)$. Before proceeding to examine higher dimension linear, autonomous systems, it seems prudent to define "linear" and "autonomous" in this context. But first, a bit of notation.

Notation 1.3. Let

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, ..., x_n) \\ x_2' &= f_2(t, x_1, x_2, .., x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, ..., x_n) \end{aligned}$$

be a system of differential equations. I will write this as $X' = F(t, X)$ where $X = \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}$.

Unless otherwise specified, we will assume here that $X \in \mathbb{R}^n$ and $F(t, X) : \mathbb{R}^{n+1} \to \mathbb{R}^n$.

Definition 1.4. An n-dimensional system of differential equations $X' = F(t, X)$ is *autonomous* if $F(t, X)$ in fact depends only on X.

Thus in discussion of autonomous systems, we write $X' = F(X)$.

Definition 1.5. An n-dimensional system of differential equations $X' = F(t, X)$ is *linear* if there exists $A \in \mathbb{R}^{n\times n}$ such that $X' = AX$. That is, the system takes the form $\begin{pmatrix} x_1' = a_{11}x_1 + ... + a_{1n}x_1 \\ \vdots \\ x_n' = a_{n1}x_1 + ... + a_{nn}x_n \end{pmatrix}$.

It is worth noting that any linear system of equations must also be autonomous.

Let us now consider the very simple 2-dimensional system

$$\begin{aligned} x' &= ax \\ y' &= by \end{aligned} \tag{1.6}$$

By repeating the 1-dimensional separation of variables and "ignoring" either x or y, we can see that $X(t) = \begin{matrix} x_0e^{at} \\ 0 \end{matrix}$ and $X(t) = \begin{matrix} 0 \\ y_0e^{bt} \end{matrix}$ are solutions to (1.6). In fact, we will show that $X(t) = \begin{matrix} x_0e^{at} \\ y_0e^{bt} \end{matrix}$ is also a solution to (1.6).

Theorem 1.7. *Let $X' = AX$ be a linear system of differential equations with solutions $X(t)$ and $Y(t)$. Then, $(X + Y)(t)$ is also a solution to the system.*

Proof. We know that $(X + Y)'(t) = X'(t) + Y'(t)$ and that $X'(t) + Y'(t) = AX + AY = A(X + Y)$ by linearity. Therefore $(X + Y)'(t) = A(X + Y)$ as required. □

Then, we have that $\begin{matrix} x_0e^{at} \\ y_0e^{bt} \end{matrix}$ is indeed a solution to (1.6). This solution is more interesting than it may at first appear. To clarify, let us rewrite (1.6) as

$$X' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} X \tag{1.8}$$

and the aforementioned solution as

$$X(t) = x_0 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{bt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.9}$$

Now we can clearly observe that, quite interestingly, each of the two terms is of the form $ce^{\lambda t}(V)$, where V is an eigenvector of A and λ its corresponding eigenvalue.

Fortunately, this property is not unique to this system.

Theorem 1.10. *$X(t)$ is a solution to the equation $X' = AX$ if and only if $X(t) = Ve^{\lambda t}$ for some eigenvector V of A, where λ is the corresponding eigenvalue to V.*

Proof. Let $X(t) = Ve^{\lambda t}$. Then, $X' = \lambda V e^{\lambda t}$.
Since V is an eigenvector of A to λ, $X' = AVe^{\lambda t}$. Therefore $X' = AX$ as required. Conversely, if $X(t)$ is a solution to $X' = AX$, $X(t) = Be^{\alpha t}$ for some B and α. Therefore, $\alpha B e^{\alpha t} = ABe^{\alpha t}$. This implies that α is an eigenvalue of A with eigenvector B. □

Therefore, for any linear system of differential equations $X' = AX$, all solutions will be of the form

$$X(t) = \alpha_1 V_1 e^{\lambda_1 t} + \alpha_2 V_2 e^{\lambda_2 t} + ... + \alpha_k V_k e^{\lambda_k t} \tag{1.11}$$

where λ_i are the eigenvalues of A, V_i eigenvectors to λ_i, and α_i constants.

Two other concepts important to define now are the Poincaré map and nullclines.

Definition 1.12. Let $X' = F(t, X)$ be a system of differential equations. Suppose that for any initial condition $X(0)$, we know $X(1)$. Then we can define a function P such that $P(X(0)) = X(1)$. We call this function a *Poincaré map*.

While the Poincaré map may not be particularly useful for linear systems, since we can solve them explicitly, it is a very useful tool for modeling the behavior of messy nonlinear systems.

Definition 1.13. Given the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

the *x-nullcline* is the set of points such that $f(x, y) = 0$. The *y-nullcline* is defined similarly.

Note that nullclines are not a construct used only in 2-dimensional systems. In higher dimensional systems, we will also have z-nullclines, etc.

Before we continue, we should be sure that our efforts in solving differential equations is not in vain. We want to be sure that each system with given initial condition has a solution and that the solution is unique. Fortunately, there's a theorem for that.

Theorem 1.14. *Given the initial value problem*

$$X' = F(X), X(t_0) = X_0$$

for $X_0 \in \mathbb{R}^n$. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then, there exists a unique solution to this initial value problem. That is, there exists an $a > 0$ and a solution $X : (t_0 - a, t_0 + a) \to \mathbb{R}^n$ of the differential equation such that $X(t_0) = X_0$.

We will not delve into the totality of the proof of this theorem in this paper. Suffice to say, the proof relies on the technique of Picard iteration. The basic idea of this technique is to construct a sequence of functions which converges to the solution of the differential equation. The sequence of functions $p_k(t)$ is defined by $p_0(t) = x_0$, our initial condition, and

$$p_{k+1} = x_0 + \int_0^t p_k(s)ds$$

This technique is useful not only for proving this theorem, but also for approximating solutions to difficult or impossible to solve equations.

Let us also consider an example which highlights the importance of the C^1 condition in theorem 1.14.

Example 1.15. Consider the differential equation

$$x' = ln(x)$$

Since $f' = \frac{1}{x}$ is not continuous, this equation fails the condition of theorem 1.14. Next, consider the initial condition $x(0) = -1$. But then we have $x'(0) = ln(-1)$, which is nonsense. Therefore, we have no solution with initial condition $x(0) = -1$.

One last important result from theorem 1.14 is that solution curves to a differential equation which satisfies the conditions of theorem 1.14 do not intersect.

Another important concept is the "flow" of an n-dimensional differential equation. The flow is a function

$$\phi \to \mathbb{R} \times \mathbb{R}^n$$

such that $\phi(t, X_0)$ is the solution at time t with $\phi(0, X_0) = X_0$. Then we have the theorem

Theorem 1.16. *Consider the system $X' = F(X)$ where F is C^1. Then $\phi(t, X)$ is C^1, i.e. $\frac{\partial \phi}{\partial X}$ and $\frac{\partial \phi}{\partial t}$ exist and are continuous.*

Again, in the interest of time, we will not delve into the prof of this theorem. Worth noting is that we can calculate $\frac{\partial \phi}{\partial t}$ for any t provided we know the solution through X_0. We have

$$\frac{\partial \phi}{\partial t}(t, X_0) = F(\phi(t, X_0))$$

We also have

$$\frac{\partial \phi}{\partial X}(t, X_0) = D\phi_t(X_0)$$

where $D\phi_t$ is the Jacobian of $X \to \phi_t(X)$ and $\phi_t(X)$ is $\phi(t, X)$ with constant t. Note that $\frac{\partial \phi}{\partial X}$ requires knowledge not only of the solution through X_0, but also through all nearby initial conditions.

2. Sinks, Sources, Saddles, and Spirals: Equilibria in Linear Systems

Definition 2.1. An *equilibrium point* of the n-dimensional autonomous system of differential equations $X' = F(X)$ is a point $Z \in \mathbb{R}^n$ such that $X' = 0$ at $X = Z$.

In particular, 0 is always an equilibrium point of a linear system.

Let us now restrict our discussion to 2-dimensional linear systems $X' = AX$. Specifically, let us look at the eigenvalues of A.

Theorem 2.2. *Let $X' = AX$ be a 2-dimensional linear system. If $det(A) \neq 0$, then $X' = AX$ has a unique equilibrium point (0,0).*

Proof. An equilibrium point $X = (x, y)$ of the system $X' = AX$ is a point that satisfies $AX = 0$. We know from linear algebra that this system has a nontrivial solution if and only if $det(A) = 0$. Therefore if $det(A) \neq 0$, the only solution to $AX = 0$ is $(0, 0)$. □

2.1. **Real Eigenvalues.** If we ignore for now the possibility that $\lambda_i = 0$ and that $\lambda_1 = \lambda_2$, then we are left with three cases:

(1) $0 < \lambda_1 < \lambda_2$
(2) $\lambda_1 < \lambda_2 < 0$
(3) $\lambda_1 < 0 < \lambda_2$

Let us first consider case (1):

Example 2.3. Let A have eigenvalues $0 < \lambda_1 < \lambda_2$ and eigenvectors V_1, V_2 which correspond to λ_1 and λ_2 respectively. Then the general solution is of the form

$$X(t) = \alpha_1 e^{\lambda_1 t} V_1 + \alpha_2 e^{\lambda_2 t} V_2$$

Then, any solution of the system tends to $(0, 0)$ as $t \to -\infty$ and tends to $(\pm\infty, \pm\infty)$ as $t \to \infty$. Thus, we call $(0, 0)$ a "source" in this case.

Next, we consider case (2):

Example 2.4. Let A have eigenvalues $\lambda_1 < \lambda_2 < 0$ and corresponding eigenvectors V_1, V_2. Then the general solution tends to $(0, 0)$ as $t \to \infty$. In this case, we call the equilibrium point a "sink".

And finally, case (3):

Example 2.5. Let A have eigenvalues $\lambda_1 < 0 < \lambda_2$ and corresponding eigenvectors V_1, V_2. Then, the general solution

$$X(t) = \alpha_1 e^{\lambda_1 t} V_1 + \alpha_2 e^{\lambda_2 t} V_2$$

tends to $(0, 0)$ along V_1. That is, for the solution $X(t) = \alpha_1 e^{\lambda_1 t} V_1$, $(0, 0)$ is a sink. We call the line $\{X \in \mathbb{R}^2 | X = \beta V_1, \beta \in \mathbb{R}\}$ the "stable line". Similarly, the solution tends away from $(0, 0)$ as $t \to \infty$. Thus we call the line $\{X \in \mathbb{R}^2 | X = \beta V_2, \beta \in \mathbb{R}\}$ the "unstable line". Overall, we call the equilibrium point of this system a "saddle".

2.2. **Complex Eigenvalues.** Of course, these examples cover only three of the four things I promised to discuss in this section. Next we turn to the possibility of complex eigenvalues and equilibria as spiral sources, spiral sinks, and centers (spiral saddles).

Example 2.6. Let $X' = AX$ be a 2-dimensional liner system of differential equations with

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

with $\beta \neq 0$. (This is the complex analogue of a saddle) Then, A has eigenvalues $\pm i\beta$. So, A has eigenvectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$ to $i\beta$ and $-i\beta$ respectively. Then, $X' = AX$ has the solution

$$X(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + e^{-i\beta t} \begin{pmatrix} -1 \\ i \end{pmatrix} \tag{2.7}$$

Let us briefly focus on only the first term of this equation. By using Euler's formula, this term becomes

$$X(t) = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{pmatrix} \tag{2.8}$$

We can then split this equation into

$$X_R(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, X_I(t) = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix} \tag{2.9}$$

In fact, each of these pieces of a solution is itself a solution, and since

$$X_R(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, X_I(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

therefore any linear combination of X_R and X_I is also a solution. In fact, 2.7 itself is a linear combination of X_R and X_I. Then,

$$X(t) = \alpha_1 X_R(t) + \alpha_2 X_I(t) \tag{2.10}$$

is the general solution to this system of equations.

Then, we can see that any solution is periodic with period $\frac{2\pi}{\beta}$ and that all solutions lie on circles about the origin. Thus, we call the equilibrium point $(0, 0)$ a center.

Example 2.11. Now let us consider $X' = AX$ with $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $\alpha, \beta \neq 0$. Then,A has eigenvalues $\alpha \pm i\beta$. As in the previous example, $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector of A, this time to $\lambda = \alpha + i\beta$. Then, we have

$$X(t) = e^{(\alpha+i\beta)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + ie^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix} = X_R(t) + iX_I(t)$$

and the general solution

$$X(t) = \gamma_1 e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + \gamma_2 e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix} \tag{2.12}$$

Since the solution would form concentric rings about the origin without the $e^{\alpha t}$ term, the solution will either spiral away from the origin (if $\alpha > 0$) or spiral toward the origin (if $\alpha < 0$), making the origin a spiral source or spiral sink respectively.

3. Nonlinear Systems: Linearization

Now, not every system of differential equations can be written in the nice form $X' = AX$.

Definition 3.1. A *nonlinear* system of differential equations is a system that cannot be written as $X' = AX$ for some matrix A.

Then, we will need new methods for discerning the nature of equilibria in nonlinear systems.

Example 3.2. Consider the 2-dimensional nonlinear system of differential equations

$$X' = X + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.3}$$

This system has an equilibrium point at $\begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$. Then, we can introduce a change in coordinate $\bar{x_1} = x_1 + \alpha, \bar{x_2} = x_2 + \beta$ such that the new system

$$\bar{X}' = \bar{X} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = X - \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = X$$

Then, this new system $\bar{X}' = \bar{X}$ is linear with a unique equilibrium point at $(0, 0)$. Moreover, $\bar{X}' = X'$ for any $X \in \mathbb{R}^2$, since $\bar{X}' = (X - \begin{pmatrix} \alpha \\ \beta \end{pmatrix})' = X'$.

Proposition 3.4. *For any system of n-dimensional differential equations of the form $X' = AX + V$ for some $V \in \mathbb{R}^n$ and with unique equilibrium point X_e, then the change of coordinates $\bar{X} = X - X_e$ yields a linear system of differential equations $\bar{X}' = A\bar{X}$ with unique equilibrium point* 0.

Proof. Since X_e is an equilibrium point of the system, $AX_e + V = 0$.
Then, $\bar{X}' = A(\bar{X} + X_e) + V = A\bar{X} + (AX_e + V) = A\bar{X}$.
Then, we have from 3.2 that $\bar{X}' = X'$ for any $X \in \mathbb{R}^n$, as our proof in $\mathbb{R}^2$ did not require any specific characteristics of $\mathbb{R}^2$.

Therefore $0 = X_e' = (X_e - X_e)' = \bar{0}$, i.e. $\bar{X}' = A\bar{X}$ has equilibrium point 0. Since X_e is a unique equilibrium point, 0 is the only element of $\mathbb{R}^n$ with $\bar{X} = 0$. □

More generally, we have the notion of conjugacy.

Definition 3.5. Let $F : X \to X$ and $G : Y \to Y$. F and G are *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ F = G \circ h$

That is, two systems are conjugate if there exists a "change of coordinates" from one system to the other. In studying nonlinear systems, we are notably interested in systems which are conjugate to linear systems. However, most nonlinear systems are not conjugate to linear equations. Most nonlinear systems, in fact, cannot be solved to arrive at a general equation.

Example 3.6. Consider the system

$$\begin{aligned} x' &= x - 3y + x^3 \\ y' &= -x + y - 2y^4 \end{aligned} \tag{3.7}$$

We cannot solve this system explicitly, but we can at least discern the nature of the equilibrium point $(0, 0)$.

We know that as $x, y \to 0$, x^3 and $2y^4$tend to 0 much faster than the linear terms. Therefore, sufficiently close to $(0, 0)$, the system behaves similarly to

$$\begin{aligned} x' &= x - 3y \\ y' &= -x + y \end{aligned}$$

The linear system has eigenvalues $\lambda = 1 \pm i\sqrt{3}$, meaning the equilibrium point is a spiral source. Therefore, in the nonlinear system, we know at least that the equilibrium point is a source.

In fact, we have

Theorem 3.8. *Suppose that the n-dimensional system $X' = F(X)$ has an equilibrium point at X_0 and the eigenvalues of the Jacobian $DF_(X_0)$ have nonzero real part. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of X_0.*

The proof of this theorem is rather advanced and convoluted, so we will not discuss the details here. This style of linearizing nonlinear systems is a useful tool for determining the nature of equilibria and works well. Except when it doesn't.

4. When Linearization Fails

Example 4.1. Let us begin by considering a trivial example.

$$\begin{aligned} x' &= x^2 \\ y' &= -y^3 \end{aligned} \tag{4.2}$$

Then, the linearized system $x' = 0, y' = 0$ is rather boring to look at and, moreover, sheds no light on the behavior of the system even very close to the equilibrium point at the origin.

Example 4.3. Next, consider the less trivial system

$$\begin{aligned} x' &= x^2 \\ y' &= y \end{aligned} \tag{4.4}$$

Then, the system has a unique equilibrium point at the origin, and has all other solutions moving toward the x-axis and to the right.

The linearized system

$$\begin{aligned} x' &= 0 \\ y' &= -y \end{aligned}$$

has equilibrium points all along the x-axis, and all other solutions lie on vertical lines. Thus, however close to the origin we are, we have lost the horizontal movement of the system.

At this point, one might suspect that, logically, we cannot linearize a system with no linear dependence on a variable, but some nonlinear dependence. However, this suspicion is not quite correct.

Example 4.5. To see this, let us consider the system

$$\begin{aligned} x' &= -y + cx(x^2 + y^2) \\ y' &= x + cy(x^2 + y^2) \end{aligned} \tag{4.6}$$

for $c \neq 0$.

Then, the linearized system is

$$\begin{aligned} x' &= -y \\ y' &= x \end{aligned}$$

Then, the linearized system has eigenvalues $\lambda = \pm i$, yielding a center at the origin. The behavior of the nonlinear system, however, changes drastically. To see this clearly, we convert the system to polar coordinates.

We have

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

which yields

$$\begin{aligned} x' &= r'\cos\theta + r\sin\theta \\ y' &= r'\sin\theta - r\cos\theta \end{aligned}$$

Then, we have

$$\begin{aligned} r' &= cr^3 \\ \theta' &= -1 \end{aligned}$$

So, for $c > 0$, the equilibrium point is a spiral source, and a spiral sink for $c < 0$. Thus, as in the previous example, even arbitrarily close to the origin, the linearized system fails to reflect the nature of the nonlinear system.

The commonality between all of these examples is that at least one of the eigenvalues has 0 real part. Then, in any such system we must be wary of linearizing the system.

5. The van der Pol Equation and Oscillating Systems

The van der Pol equation models the oscillation of voltage and current in certain RLC circuits. For us, however, it is foremost a model of another type of "equilibrium".

The system is given by

$$\begin{aligned} x' &= y - x^3 + x \\ y' &= -x \end{aligned} \tag{5.1}$$

First, we note that the linearized system has a spiral source at the origin. Since both eigenvalues have non-zero (in fact, positive) real part, we can be reasonably sure that the nonlinear system has the same behavior near the origin. However, this is not the behavior of the system that interests us.

Theorem 5.2. *There is one nontrivial periodic solution to the van der Pol system, and each other non-equilibrium solution tends to this periodic solution. We say that the system oscillates.*

Proof. We know that the system has a unique equilibrium point at the origin. Moreover, the equilibrium is a source. Next, we must show that all nonequilibrium solutions rotate about the equilibrium. We observe that the x-nullcline is given by $y = x^3 - x$ and the y-nullcline by $x = 0$. Then we divide the x-nullcline into x^+, x^- such that $x > 0$ on x^+ and $x < 0$ on x^-. We divide the y-nullcline into y^+, y^- such that $y > 0$ on $y+$ and $y < 0$ on y^-.

These 4 curves divide the plane into 4 sections and the origin. Let us name the region bounded by y^+ and x^+ A, the region bounded by x^+ and y^- B, the one bounded by y^- and x^- C, and the final region D.

We must now show that a solution starting in one region visits all other regions before returning to its starting region.

Lemma 5.3. *A solution curve starting on y^+ crosses x^+, y^-, and x^+ successively before returning to y^+.*

Proof. Since $x'(0) > 0$, any solution starting on y^+ immediately enters region A. Since $y > 0$ in A, $y' < 0$ in A. Then, since y is decreasing and the solution cannot tend to the source, the solution must intercept x^+. Along x^+, $x' = 0$ and $y' < 0$, so the solution crosses into B. In B, $x', y' < 0$, so the solution either crosses into C or tends to $y = -\infty$.

Now, we must show that the second case is impossible. Let us suppose that it is possible. Then, let (x_0, y_0) be a point in B on this solution and let us examine the function $\phi_t(x_0, y_0) = (x(t), y(t))$. Since $x(t)$ can never be zero, the solution lies in the region R given by $0 < x \leq x_0$ and $y \leq y_0$. We also have $y(t) \to -\infty$ as $t \to t_l$ for some t_l. We observe that

$$y(t) - y_0 = \int_0^t y'(s)ds = \int_0^t -x(s)ds$$

Since we know that $0 < x \leq x_0$, it follows that $y(t) \to -\infty$ only when $t \to \infty$. But then, we know that $x' = y - x^3 + x$ and that $-x^3 + x$ is bounded and $y \to -\infty$ as $t \to \infty$. Then, $x(t) - x_0$ tends to $-\infty$ as $t \to \infty$, which contradicts our assumption that $x > 0$.

Therefore, the solution must cross y^-. Then, we can exploit the skew symmetry of the system about the origin. That is, if $V(x, y)$ is the van der Pol vector field, then $-V(x, y) = V(-x, -y)$. Thus, the solution must pass x^- and return to y^+. This proves the lemma. □

Then, we can define a Poincaré map P on y^+. For $(0, y_0) \in y^+$, we define $P(y_0)$ as the first return of $\phi_t(0, y_0)$ to y^+ with $t > 0$. P is injective and surjective. Then, let $P^n(x, y) = P(P^{n-1}(x, y))$ represent the n^{th} return of ϕ_t to y^+.

Now, we must prove

Theorem 5.4. *The Poincaré map has a unique fixed point in y^+. The sequence $P^n(y_0)$ tends to this fixed point as $n \to \infty$ for any $y_0 \in y^+$.*

Proof. We observe that any fixed point of P must lie on a periodic solution. Alternatively, if $P(y_0) \neq y_0$, then y_0 must not lie on a periodic solution. We note, then, that if $P(y_0) > y_0$, then $P^n(y_0) > P^{n-1}(y_0)$ for any n. Thus, the solution spirals upward to infinity and, importantly, never returns to y_0. Similarly, if $P(y_0) < y_0$, the solution cannot return to y_0.

We then define a new map $\alpha : y^+ \to y^-$ by letting $\alpha(y)$ be the first intersection of ϕ_t with y^- for $t > 0$. Then, define

$$\delta(y) = \frac{1}{2}(\alpha(y)^2 - y^2)$$

Observe for future use that there exists a unique point $(0, y^*) \in y^+$ and time t^* such that

(1) $\phi_t(0, y^*) \in A$ for $0 < t < t^*$
(2) $\phi_{t^*}(0, y^*) = (1, 0) \in x^+$

That is, there is a unique point such that the solution beginning at the point, after its first quarter rotation, hits $(1, 0)$.

The end draws nearer to sight, as the theorem will follow from the following proposition:

Proposition 5.5. *$\delta(y)$ satisfies*

(1) $\delta(y) > 0$ if $0 < y < y^*$
(2) $\delta(y)$ decreases monotonically to $-\infty$ as $y \to \infty$

□

In a bid to confound and irritate the reader, we will prove the proposition shortly. First, we show that Theorem 5.4 does indeed follow from the proposition.

By using the skew symmetry of the vector field, we see that if $(x(t), y(t))$ is a solution curve, so too is $(-x(t), -y(t))$.

By the intermediate value theorem and Proposition 5.5, there is a unique $y_0 \in y^+$ such that $\delta(y_0) = 0$. Then, $\alpha(y_0) = -y_0$ and by symmetry, the solution through $(0, y_0)$ is periodic. Since $\delta(y) \neq 0$ for $y \neq y_0$, $\phi_t(0, y_0)$ is the unique periodic solution.

Next, we show that all other non-equilibrium solutions tend to this periodic solution. So, we define a map $\beta : y^- \to y^+$, which sends a point in y^- to its solution's first intersection with y^+. By symmetry, we know that $\beta(y) = -\alpha(-y)$. Also, $P(y) = \beta(\alpha(y))$.

We order $y^- \cup y^+$ such that $y_1 > y_2$ if y_1 is above y_2 on the y-axis. This ordering corresponds with our usual ordering of numbers. Note that α and β reverse ordering, while P preserves ordering.

Now, we choose $y \in y^+$ with $y > y_0$. Then, we have $\alpha(y) < -y_0 = \alpha(y_0)$. Also, $\delta(y) < 0$, which implies that $\alpha(y) > -y$. Therefore $P(y) < y$. We have already shown that $y > y_0$ implies $y > P(y) > y_0$. Then, $P^n(y) > P^{n+1}(y) > y_0$ for all $n > 0$.

Then, $P^n(y)$ has a limit $y_1 \geq y_0$ in y^+. Note that, by continuity of P, y_1 is a fixed point of P.

Then, since P has only one fixed point, we have $y_1 = y_0$. Thus, each solution that starts above y_0 spirals toward the periodic solution. We can see by similar methods that non-equilibrium solutions starting below y_0 spiral toward the periodic solution. This proves the theorem. $\square$

Now, all that remains is to prove Proposition 5.5.

Proof. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth curve in the plane and $F : \mathbb{R}^2 \to \mathbb{R}$. We write $\gamma(t) = (x(t), y(t))$ and define

$$\int_\gamma F(x, y) = \int_a^b F(x(t), y(t))dt$$

If we have $x'(t) \neq 0$ for all $t \in [a, b]$, then along γ, y is a function of x. Then, we can write

$$\int_a^b F(x(t), y(t))dt = \int_{x(a)}^{x(b)} F(x, y(x))\frac{dt}{dx}dx$$

Then,

$$\int_\gamma F(x, y) = \int_{x(a)}^{x(b)} \frac{F(x, y(x))}{dx/dt}dx$$

We have a similar expression if $y'(t) \neq 0$. Now we introduce even another function

$$W(x, y) = \frac{1}{2}(x^2 + y^2)$$

Let $p \in y^+$. Suppose $\alpha(p) = \phi_\tau(p)$ for some $\tau > 0$. Let $\gamma(t) = (x(t), y(t))$ for $0 \leq t \leq \tau = \tau(p)$ be the solution curve joining p to $\alpha(p)$. By definition,

$$\delta(p) = \frac{1}{2}(y(\tau)^2 - y(0)^2) = W(x(\tau), y(\tau)) - = W(x(0), y(0))$$

Then,

$$\delta(p) = \int_0^\tau \frac{d}{dt}W(x(t), y(t))dt$$

Then, we have

$$W' = (xx' + yy') = -x(x^3 - x)$$

Then,

$$\delta(p) = \int_0^{\tau} x(t)^2(1 - x(t)^2)$$

Since the integrand is positive for $0 < x(t) < 1$, we have proven part (1) of Proposition 5.5.

Next, we rewrite the previous equation as

$$\delta(p) = \int_{\gamma} x^2(1 - x^2)$$

We will focus in on points $p \in y^+$ with $p > y^*$. We divide the curve γ into $\gamma_1, \gamma_2, \gamma_3$ with γ_1 defined for $0 \leq x \leq 1$ and $y > y_2$, γ_2 for $y_1 \leq y \leq y_2$, and γ_3 for $0 \leq x \leq 1$ and $y < y_1$. Then, $\delta(p) = \delta_1(p) + \delta_2(p) + \delta_3(p)$ where

$$\delta_i(p) = \int_{\gamma_i} x^2(1 - x^2)$$

We notice that along γ_1, $y(t)$ is a function of x. Then,

$$\delta_1(p) = \int_0^1 \frac{x^2(1 - x^2)}{dx/dt} dx = \int_0^1 \frac{x^2(1 - x^2)}{y - x^3 + x} dx$$

Then, as p moves up the y-axis, $y - x^3 + x$ increases (on γ_1). Hence, $\delta_1(p)$ decreases as p increases. $\delta_3(p)$ mimics this behavior.

On γ_2, $x(t)$ is a function of y defined for $y \in [y_1, y_2]$ and $x \geq 1$. Therefore, since $\frac{dy}{dt} = -x$, we have

$$\delta_2(p) = \int_{y_2}^{y_1} -x(y)(1 - x(y)^2) dy = \int_{y_1}^{y_2} x(y)(1 - x(y)^2) dy$$

Then, we can see that $\delta_2(p)$ is negative.

As p increases, the domain $[y_1, y_2]$ becomes larger. Since the function $y \to x(y)$ depends on p, we write it $x_p(y)$. As p increases, γ_2 moves to the right, so $x_p(y)$ increases, meaning that $x_p(y)(1 - x_p(y)^2)$ decreases. Therefore, $\delta_2(p)$ decreases as p increases, so $lim_{p \to \infty} \delta_2(p) = -\infty$. Therefore, $\delta(p)$ as a whole tends to $-\infty$ as $p \to \infty$, proving the proposition and ending this whole affair. □

While this may seem an inordinate amount of work messing with a system just introduced, this proof can serve as a guide for determining if other systems have a similar periodic solution.

6. Hopf Bifurcations

Let us now turn to a generalization of the van der Pol system, given by

$$\begin{aligned} \tfrac{dx}{dt} &= y - f_\mu(x) \\ \tfrac{dy}{dt} &= -x \end{aligned} \tag{6.1}$$

where μ is a parameter. Let us examine the case where

$$f_\mu(x) = x^3 - \mu x$$

and $\mu \in [-1, 1]$. Note that when $\mu = 1$ we have the van der Pol system exactly. For any μ, as with the van der Pol system, the only equilibrium point is the origin and the linearized system is

$$X' = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} X$$

with eigenvalues

$$\lambda = \frac{1}{2}(\mu \pm \sqrt{\mu^2 - 4})$$

Then, for $\mu < 0$, the origin is a spiral sink, and a spiral source for $\mu > 0$. In fact, for $\mu < 0$, the graph of f_μ lies entirely in the first and third quadrant, meaning that the sign of x' is always the same as in the linearized system. Therefore, as in the linearized case, all solutions tend to the origin. In particular, no solutions tend to a periodic solution as they do in the van der Pol system. Note also that the system retains this spiral sink even when $\mu = 0$ and the linearized system has a center at the origin.

However, once μ becomes positive, the system oscillates as with the van der Pol system, since none of the proof of theorem 5.2 required that $\mu = 1$. Just as with the van der Pol system, all non-equilibrium solutions tend to this new periodic solution. This vast change in behavior as we alter μ is called a Hopf Bifurcation.

7. Example: Neurodynamics

Let us now take an in-depth look at the FitzHugh-Nagumo system used to model neurodynamical behavior. The system is given by

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} + I \\ y' &= -x + a - by \end{aligned} \tag{7.1}$$

where I is a parameter and

$$0 < \frac{3}{2}(1 - a) < b < 1$$

This system is a simplification of neurodynamical systems and is reminiscent of differential equations used to model circuits. Specifically, the system shares many similarities with the van der Pol system. Here, x is analogous to voltage in circuits and y represents an amalgam of forces which restore the system to rest. I represents the "excitability" of the system, for reasons we will see later; a and b are constants.

We can then immediately see that $\frac{1}{3} < a < 1$. Let us begin by examining the system when $I = 0$.

7.1. Ignoring I.

First, let us find all equilibria points of the system (7.1) when $I = 0$. We know that any equilibria will occur at the intersection of the x- and y-nullclines.

The x-nullcline is given by

$$y_x = \frac{x^3}{3} - x \tag{7.2}$$

and the y-nullcline by

$$y_y = \frac{a}{b} - \frac{x}{b} \tag{7.3}$$

First, we observe that, because $y_x \to \infty$ as $x \to \infty$ and $\to -\infty$ as $x \to -\infty$, and $y_y \to -\infty$ as $x \to \infty$ and vice versa, the nullclines must intersect at some point.

Then, we observe that the x-nullcline is monotonically increasing except on the region $[-1,1]$, on which it is monotonically decreasing. The y-nullcline is monotonically decreasing. Then, if the nullclines have multiple intersections, one intersection must occur on the region $[-1,1]$ with respect to x. In fact, we must also have that the y-nullcline passes through both vertical edges of the rectangle $[-1,1]\times[-\frac{2}{3},\frac{2}{3}]$, since the x-nullcline has $y_x(1)=-\frac{2}{3}$ and $y_x(-1)=\frac{2}{3}$.

But, we see that, since $\frac{1}{b}>1$, if $y_y(-1)=\frac{2}{3}$, then $y_y(1)<\frac{2}{3}-2=-\frac{4}{3}$.

Therefore, the y-nullcline is too steep to intersect the x-nullcline twice. So we have

Proposition 7.4. *For $I=0$, the system has a unique equilibrium point (x_0,y_0).*

Next, we examine the nature of this equilibrium point. Now, we consider the vector field at points slightly displaced from equilibrium along the x- and y-nullclines. Along the x-nullcline, we have

$$y' = -x + a - b(\frac{x^3}{3} - x) = a - \frac{bx^3}{3} + (b-1)x \tag{7.5}$$

Then, we see that for ϵ small, $y'(x_0+\epsilon) < y'(x_0)$ and $y'(x_0-\epsilon) > y'(x_0)$.

Along the y-nullcline, we have

$$x' = \frac{a}{b} - \frac{x}{b} + x - \frac{x^3}{3} = \frac{a}{b} + (1-\frac{1}{b})x - \frac{x^3}{3} \tag{7.6}$$

Then, we have that $x'(x_0+\epsilon) < x'(x_0)$ and $x'(x_0-\epsilon) > x'(x_0)$.

So, minor displacement along the nullclines pushses solutions back toward the equilibrium. Then we have

Proposition 7.7. *The equilibrium point (x_0,y_0) is always a sink.*

7.2. **Acknowledging I.** Our first order of business now that we allow $I\neq 0$ is to discern whether or not this changes the number of equilibrium points.

We now have the x-nullcline

$$y_x = \frac{x^3}{3} - x - I \tag{7.8}$$

Thus, the addition of I shifts the graph of the x-nullcline up or down. However, because the end behavior of the nullclines is unchanged, we must still have at least one equilibrium point. Then, if we are to have multiple equilibria, we must have the y-nullcline pass through the vertical edges of $[-1,1]\times[-\frac{2}{3}-I,\frac{2}{3}-I]$. However, our argument from the previous subsection still applies here, so we still have a unique equilibrium point x_I, y_I.

In fact, we also see that, if $I>0$, the graph of the x-nullcline shifts down, shifting the equilibrium point down. Then, since the equilibrium must lie along the y-nullcline, the equilibrium point is shifted right. Similarly, if $I<0$, the equilibrium point is shifted left. Therefore, we have

Proposition 7.9. *The system has a unique equilibrium point (x_I,y_I) and x_I varies monotonically with I.*

For points sufficiently close to the equilibrium point (the origin under the new coordinates), the system behaves similarly to

$$\begin{aligned} x' &= y + (1-x_I^2)x + I \\ y' &= -x + a - by \end{aligned} \tag{7.10}$$

Then, we see that the along the y-nullcline,

$$x' \approx \frac{a}{b} - (\frac{1}{b} + 1 - x_I^2)x + I \tag{7.11}$$

Then, for $x_I \in (-\infty, -\frac{1}{b} - 1) \cup (1 + \frac{1}{b}, \infty)$, solutions starting near the equilibrium along the y-nullcline are pushed away from the equilibrium.

Along the x-nullcline, we have

$$y' \approx (-\frac{1}{c} - b)y - \frac{I}{c} + a \tag{7.12}$$

for $c = (x_I^2 - 1)$. Then, for $x_I \in (-\infty, 1 - \frac{1}{b}) \cup (\frac{1}{b} - 1, \infty)$. Therefore, the equilibrium point is a source for $x_I \in (-\infty, -\frac{1}{b} - 1) \cup (1 + \frac{1}{b}, \infty)$.

Next, we return to the non-approximated system to examine the behavior of the system for points far from the equilibrium. We observe that the $\frac{x^3}{3}$ dominates x', so solutions tend to the right for $x < 0$ and vice versa. For y', we notice that solutions tend up for $y < 0$ and vice versa, except in the case that $|x|$ very large. However, we know that if $|x|$ is very large, x' restores x toward the origin. Thus, for large x, y y' also restores y toward the origin.

However, we then have solutions tending to the origin, and more importantly, the equilibrium for x, y large, yet the equilibrium is a source. Then, we have

Proposition 7.13. *For $x_I \in (-\infty, -\frac{1}{b} - 1) \cup (1 + \frac{1}{b}, \infty)$, the equilibrium point is a source and there exists a stable limit cycle. For any other value of x_I, the equilibrium point is a sink.*

Next, let us examine what happens at the old equilibrium point (x_0, y_0) with the addition of I. First, we observe that (x_0, y_0) still has $y' = 0$, since neither a nor b has changed. Then, since we know that $y_0 + x_0 - \frac{x_0^3}{3} = 0$, (x_0, y_0) has $x' = I$. I is referred to as the "excitability" of the system, a name which makes sense, given that x_0' increases as $|I|$ increases.

7.3. **Altering Parameters and Bifurcations.** We have observed that allowing $I \neq 0$ introduces a bifurcation of the system. Now, let us examine what bifurcations are caused by altering restrictions on a and b.

First let us consider the case that $a = I = 0$ and $b > 0$ (with no other restrictions on b). Then, the system becomes

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} \\ y' &= -x - by \end{aligned} \tag{7.14}$$

The linearized version of the system has characteristic polynomial

$$\lambda^2 + c\lambda - c = 0$$

where $c = b - 1$. Then, we have

$$\lambda = \frac{1}{2}(-c \pm \sqrt{c^2 + 4c}) \tag{7.15}$$

So, for $b > 1$, we have $c > 0$. Then, we have one positive and one negative eigenvalue, so the equilibrium is a saddle. For $b < 1$, we have $c < 0$. For $-3 < b < 1$, both eigenvalues have positive real part, so the equilibrium is a (spiral) source. For $b = -3$, both eigenvalues are negative real numbers, yielding a sink. Then, for $b < -3$, we again have a real saddle. Lastly, for $b = 1$, both eigenvalues are 0.

Thus, we must in this case be cautious about trusting the result of the linearized system. The full system in this case is

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} \\ y' &= -x - y \end{aligned} \tag{7.16}$$

Then, along the x-nullcline of the system, we have

$$y' = -\frac{x^3}{3}$$

Then, displacing slightly from the equilibrium to the right along the x-nullcline yields $y' < 0$. Since the x-nullcline is decreasing around the equilibrium point, this displacement pushes solutions farther away from the equilibrium point. This is also true of displacement to the left.

Along the y-nullcline, we have

$$x' = -\frac{x^3}{3}$$

Then, we see that slight displacement from the origin along the y-nullcline restores solutions to equilibrium. Therefore, we have a real saddle at the equilibrium for $b = 1$. To summarize, we have

Proposition 7.17. *The system has a saddle for $b \geq 1$ and $b < -3$, a spiral source for $-3 < b < 1$, and a sink for $b = -3$.*

Next, we allow the same restriction on b as above and also allow I to vary again. Then we have the x-nullcline

$$y = \frac{x^3}{3} - x - I$$

and the y-nullcline

$$y = -\frac{x}{b}$$

Along the x-nullcline, we have

$$y' = -\frac{bx^3}{3} + (b-1)x$$

and along the y-nullcline

$$x' = (1 - \frac{1}{b})x - \frac{x^3}{3} + I$$

First, we consider the case that $I < 0$. In this case, the equilibrium point has $x_e > 0$ and $y_e < 0$. Then, for $b < 1$, we have a (spiral) sink. For $b > 1$, we have a saddle if $x_e \in (-1, 0)$ and a sink otherwise. For $b = 1$, we have a real sink.

Next, if $I > 0$, we have an equilibrium point with $x_e > 0$ and $y_e < 0$, but the system's behavior as b changes mimics the previous case.

Now, we extend this inquiry to the case where $b \leq 0$. Firstly, we observe that for $b = 0$, we have

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} + I \\ y' &= -x \end{aligned} \tag{7.18}$$

which is the van der Pol system with the addition of I. This system has an equilibrium point at $(0, -I)$ and the equilibrium is a sink for any I.

Then, we examine the system when $b < 0$. To clarify the sign of terms, let $d = -b$. Then the system is

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} + I \\ y' &= -x + dy \end{aligned} \tag{7.19}$$

For $I = 0$, the system has an equilibrium at the origin and linearizes with eigenvalues

$$\lambda = \frac{1}{2}(-g \pm \sqrt{g^2 - 4g})$$

where $g = d + 1$. Then, the equilibrium is always a sink, and is a spiral sink for $b > -3$.

However, in this case the origin is not a unique equilibrium. Examining the intersection of nullclines, we find that the system has three equilibrium points, the other two at $(\pm 3\sqrt{1+h}, \pm 3h\sqrt{1+h})$, where $h = \frac{1}{d}$. Examining small displacement along nullclines from these equilibria, we see that both points are saddles, with solutions tending toward the equilibrium along the y-nullcline and away from the equilibrium along the x-nullcline.

Next, we consider $I \neq 0$. Then, depending on $|I|$, we may have 1 or 3 equilibria. Similarly to our argument concerning equilibrium uniqueness in §7.2, we have multiple equilibria when the y-nullcline passes through the region $[-1, 1] \times [-\frac{2}{3} - I, \frac{2}{3} - I]$ though we no longer require passage through the vertical edges of the region. Then, we have a single equilibrium as long as

$$\frac{1}{b} < -(\frac{2}{3} - I)$$

Then we have

Proposition 7.20. *The system has one equilibrium point, a saddle, when* $\frac{1}{b} < -(\frac{2}{3} - I)$*, and three equilibria, two saddles and a sink, otherwise.*

Lastly, we turn our attention to the case that $b = 0$ and $I, a \neq 0$. Then, the system is

$$\begin{aligned} x' &= y + x - \tfrac{x^3}{3} + I \\ y' &= -x + a \end{aligned} \tag{7.21}$$

Firstly, we observe that now, since the y-nullcline is a vertical line, the system will always have a single equilibrium (x_e, y_e). Then, for points sufficiently close to the equilibrium, the system behaves similarly to

$$\begin{aligned} x' &= y + (1 - x_e^2)x + I \\ y' &= -x + a \end{aligned} \tag{7.22}$$

Then, by examining displacement along the nullclines, we see that the equilibrium is a spiral sink for $|x| < 1$, a center for $|x| = 1$, and a spiral source otherwise. This concludes our analysis of this system.

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