BROUWER'S FIXED POINT THEOREM

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ABSTRACT. In this paper, we seek to prove Brouwer's fixed point theorem. We begin by constructing a homeomorphism between the closed n-ball and the standard n-simplex. After proving Sperner's lemma, we use it along with the compactness of the standard n-simplex to prove Brouwer's theorem.

CONTENTS

1. INTRODUCTION

Brouwer's theorem (1911) states that for any continuous function $f : \overline{B^n} \to \overline{B^n}$, there exists $x \in \overline{B^n}$ such that $f(x) = x$. A crucial result in topology, Brouwer's theorem also has applications in various other fields. For example, it is related to existence theorems in differential equations and plays a role in proving the existence of the Nash equilibrium in game theory. It is also worth noting that the one dimensional case of the theorem is a result of the intermediate value theorem. One consequence of the three dimensional case of the theorem is that after stirring a cup of coffee, at least one point of the liquid will return to its original position after coming to a rest.

In the first section, we define the standard n-simplex, Δ^n, and construct a homeomorphism between it and a related set, Δ^n_0. Next, we introduce the notion of a convex body and use the fact that Δ^n_0 is a convex body to construct a homeomorphism between Δ^n_0 and the closed n-ball $\overline{B^n}$. In the third section, we define a proper coloring of a simplicial subdivision, in order to state and prove Sperner's lemma. Finally, we create barycentric subdivisions of the standard n-simplex and use Sperner's lemma to prove Brouwer's theorem.

While Reference [2] was used primarily to clarify the application of simplicial subdivisions at the end of the proof of Brouwer's theorem, Reference [1] was used extensively in both the proof of Sperner's lemma and Brouwer's theorem.

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2. Simplicial Homeomorphism

Definition 2.1. The *standard n-simplex* is

$$\Delta^n = \{\sum_{i=1}^{n+1} \lambda_i \mathbf{e}_i \mid \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\} \subset \mathbb{R}^{n+1}$$

$$= \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_{n+1} \mathbf{e}_{n+1} \mid \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\}.$$

Brouwer's Theorem is stated in terms of a closed n-ball $\overline{B^n}$. We will prove that Δ^n is homeomorphic to $\overline{B^n}$. Then we will show that a proof of Brouwer's Theorem for Δ^n implies that Brouwer's Theorem is satisfied for $\overline{B^n}$. Finally, we will prove the restatement of Brouwer's Theorem for Δ^n, using Sperner's Lemma. We must first define what it means for two spaces to be homeomorphic to each other.

Definition 2.2. A function $f : A \longrightarrow B$ is a *homeomorphism* if it is a continuous bijection such that f^{-1} is also continuous. If there exists a homeomorphism $f : A \longrightarrow B$, we say A is *homeomorphic to* B, denoted $A \cong B$.

In order to prove $\Delta^n \cong \overline{B^n}$, we define a new set Δ_0^n below. We will prove $\Delta^n \cong \Delta_0^n$ and $\Delta_0^n \cong \overline{B^n}$. After proving that the composition of homeomorphisms is a homeomorphism, we will have $\Delta^n \cong \overline{B^n}$.

Definition 2.3. Let

$$\Delta_0^n = \{\sum_{i=1}^{n} \lambda_i \mathbf{e}_i + \lambda_{n+1}(-\mathbf{e}_1 - \cdots - \mathbf{e}_n) \mid \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\} \subset \mathbb{R}^{n+1}$$

$$= \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n + \lambda_{n+1}(-\mathbf{e}_1 - \cdots - \mathbf{e}_n) \mid \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\}$$

$$= \{(\lambda_1 - \lambda_{n+1})\mathbf{e}_1 + \cdots + (\lambda_n - \lambda_{n+1})\mathbf{e}_n \mid \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\}$$

Theorem 2.4. $\Delta^n \cong \Delta_0^n$.

Proof. Let $f : \Delta^n \longrightarrow \Delta_0^n$ such that

$$f((\lambda_1, \lambda_2, ..., \lambda_{n+1})) = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_n \mathbf{e}_n + \lambda_{n+1}(-\mathbf{e}_1 - ... - \mathbf{e}_n)$$

$$= (\lambda_1 - \lambda_{n+1})\mathbf{e}_1 + \cdots + (\lambda_n - \lambda_{n+1})\mathbf{e}_n.$$

Let $f^{-1} : \Delta_0^n \longrightarrow \Delta^n$ such that

$$f^{-1}((\lambda_1 - \lambda_{n+1})\mathbf{e}_1 + \cdots + (\lambda_n - \lambda_{n+1})\mathbf{e}_n) = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_{n+1} \mathbf{e}_{n+1}.$$

By construction, $(f^{-1} \circ f)((\lambda_1, \lambda_2, ..., \lambda_{n+1})) = (\lambda_1, \lambda_2, ..., \lambda_{n+1})$ and $(f \circ f^{-1})((\lambda_1 - \lambda_{n+1})\mathbf{e}_1 + \cdots + (\lambda_n - \lambda_{n+1})\mathbf{e}_n) = (\lambda_1 - \lambda_{n+1})\mathbf{e}_1 + \cdots + (\lambda_n - \lambda_{n+1})\mathbf{e}_n$. Then f is a bijection and $\Delta^n \cong \Delta_0^n$. □

3. Homeomorphism Between n-Simplex and n-Ball

Our next step is to prove $\Delta_0^n \cong \overline{B^n}$. In order to do this, it will be useful to state and prove a proposition about convex bodies, so we will begin with a definition:

Definition 3.1. A *convex body* in $\mathbb{R}^n$ is $X \subset \mathbb{R}^n$ such that
i) X is convex, that is if $x, y \in X$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in X$.
ii) X has a nonempty interior, that is $\exists x \in X$ and $r > 0$ such that $B(x, r) \subset X$.
iii) X is compact.

Proposition 3.2. *Δ_0^n is a convex body in $\mathbb{R}^n$.*

Proof. Δ^n is convex by construction. Since $\Delta_0^n \cong \Delta^n$ by Theorem 2.4, it follows that Δ_0^n is convex.

Next, we will show that Δ_0^n has a nonempty interior. Let $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1} = \frac{1}{n+1}$

Δ^n is closed and bounded in $\mathbb{R}^{n+1}$ and therefore compact, so Δ_0^n is also compact.

Now that we have proven the three conditions in Definition 3.1, we have shown that $\Delta_0^n \subset \mathbb{R}^n$ is a convex body. □

Before we construct a homeomorphism between Δ_0^n and $\overline{B^n}$, we will define some terms.

Definition 3.3. A *ray* from x_0 is $\{x_0 + ty | t \geq 0\}$ for y with $\|y\| = 1$.

Definition 3.4. $\partial X = \overline{X} \cap \overline{X^C}$.

We will now prove a lemma that will lead to a statement about the compactness of ∂X.

Lemma 3.5. *A closed subset Y of a compact set X is compact.*

Proof. Fix a compact set $X \subset \mathbb{R}^n$. Fix $Y \subset X$ such that Y is closed. Let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be an open cover of Y. Let $\mathcal{H} = \{(\mathbb{R}^n \setminus Y), G_\lambda\}_{\lambda \in \Lambda}$. Since Y is closed, $\mathbb{R}^n \setminus Y$ is open. And $\mathcal{G}$ is an open cover of Y, so all G_λ are open. If $c \in Y$, then $c \in G_\lambda$ for some $G_\lambda \in \mathcal{G}$, by the definition of an open cover. If $c \notin Y$, then $c \in \mathbb{R}^n \setminus Y \in \mathcal{H}$. Therefore, $\mathcal{H}$ is an open cover of $\mathbb{R}^n$.

Then $\mathcal{H}$ must also be an open cover of X. Since X is compact, there exists a finite subcover $\mathcal{G}'_x \subset \mathcal{H}$. Since $\mathcal{G}'_x$ is an open cover of X and since $Y \subset X$, $\mathcal{G}'_x$ is also an open cover of Y. Since $\mathcal{H} = \{(\mathbb{R}^n \setminus Y), G_\lambda\}_{\lambda \in \Lambda}$, there must exist a finite subcover $\mathcal{G}' \subset \mathcal{G}$ of Y.

We have shown that for all open covers $\mathcal{G}$ of Y, there exists a finite subcover $\mathcal{G}' \subset \mathcal{G}$. By definition, Y is compact. □

Since $\partial X \subset X$ and ∂X is closed, it follows that if X is compact, then ∂X is compact.

Fix a convex body X and an interior point $x_0 \in X$. Let $f : X \to \mathbb{R}^n$ such that $f(x) = x - x_0$ for all $x \in X$. Let $Y = Im(X)$ under this mapping. f is clearly a homeomorphism, so Y is also a convex body. $f(x_0) = 0$. Since f is a homeomorphism and x_0 is an interior point of X, 0 is an interior point of Y.

Lemma 3.6. *Every ray from 0 intersects ∂Y exactly once.*

Proof. Fix $y \in \mathbb{R}^n$ such that $\|y\| = 1$. We want to show there exists a unique t_0 such that $0 + t_0 y \in \partial Y$.

We know the ray $\{ty | t \geq 0\}$ intersects ∂Y at least once, because $0 \in Y$. So there exists $t_0 \geq 0$ such that $t_0 y \in \partial Y$.

Assume for the sake of contradiction that there exists $\epsilon > 0$ such that $(t_0 + \epsilon)y \in Y$. We know that there exists r such that $B(0, r) \subset Y$, because 0 is an interior point of Y. Then $B(t_0 y, \epsilon r)$ is contained in Y, so $t_0 y \notin \partial Y$. We have reached a contradiction. Similarly, if $(t_0 - \epsilon)y \notin Y$, then $B(t_0 y, \epsilon r)$ is contained in Y^C, so $t_0 y \notin \partial Y$ again.

We can conclude that there exists a unique t_0 such that $0 + t_0 y \in \partial Y$, so each ray from 0 intersects ∂Y exactly once. □

Now we will define a new set and a function that we will prove is a homeomorphism.

Definition 3.7. Let $S^{n-1} = \{y \mid \|y\| = 1\} = \partial \overline{B^n}$.

Let $g : S^{n-1} \to \partial Y$ such that $g(y) = \{ty | t \geq 0\} \cap \partial Y$. Let $g^{-1} : \partial Y \to S^{n-1}$ such that $g^{-1}(x) = \frac{x}{\|x\|}$. We want to show that g^{-1} is well-defined.

Fix $x \in \partial Y$. Draw a ray $\{tx\}$ from 0 that passes through x. Since every ray from 0 intersects ∂Y exactly once, x is the only place $\{tx\}$ intersects ∂Y. We want to prove there is a unique point where $\{tx\}$ intersects S^{n-1}. Consider the point $y = \frac{x}{\|x\|}$. This is the vector from 0 in the x direction with magnitude $\|y\| = 1$. By construction, $y \in S^{n-1}$. Therefore, g^{-1} is well-defined because it sends each point in ∂Y to a unique point in S^{n-1}.

Next we will prove that g^{-1} is the inverse of g. We want to show that $(g \circ g^{-1})(x) = x$ for all $x \in \partial Y$ and that $(g^{-1} \circ g)(y) = y$ for al $y \in S^{n-1}$. Fix $x \in \partial Y$. $g^{-1}(x) = y$ as defined in the previous paragraph. $g(g^{-1}(x)) = g(y) = \{ty | t \geq 0\} \cap \partial Y$ by construction. The vector y is parallel to the vector x by the construction of y. Therefore, $\{ty | t \geq 0\} = \{tx | t \geq 0\}$. We know x is the only place $\{tx | t \geq 0\}$ intersects ∂Y, so x is the only place $\{ty | t \geq 0\}$ intersects ∂Y. It follows that $\{ty | t \geq 0\} \cap \partial Y = x$. Then $(g \circ g^{-1})(x) = x$ for all $x \in \partial Y$.

Now fix $y \in S^{n-1}$. By construction, $g(y) = \{ty | t \geq 0\} \cap \partial Y$. Let $\{ty | t \geq 0\} \cap \partial Y = x$. Then $x \in \{ty | t \geq 0\}$, so $\{tx | t \geq 0\}$ passes through y. $\{ty | t \geq 0\} = \{tx | t \geq 0\}$ because both rays start from 0 and pass through y. So $g^{-1}(x) = \frac{x}{\|x\|} = \frac{y}{\|y\|} = y$ because y has a magnitude of 1 by construction. Then $(g^{-1} \circ g)(y) = y$ for al $y \in S^{n-1}$.

We have now shown that g^{-1} is the inverse of g, so g is a bijection. Now we will show g^{-1} is continuous. Since $f(x) = x$ and $h(x) = \frac{1}{\|x\|}$ are both continuous, $g^{-1}(x) = \frac{x}{\|x\|}$ is continuous because the product of continuous functions is continuous. In order to prove g is a homeomorphism, we will prove a quick lemma.

Lemma 3.8. *A continuous bijection from a compact space is a homeomorphism.*

Proof. Let B be a compact space. Let A be closed in B. Since B is compact, A is compact. Let $j : B \to C$ be a continuous bijection. $j(A)$ is compact because j is continuous. Then $j(A) \subset C$ is closed. Let $k^{-1} = j$. So for all closed spaces $A \subset B$, $k^{-1}(A) \subset C$ is closed. Then k is continuous. Since $j = k^{-1}$ is a continuous bijection and k is continuous, $k = j^{-1}$ is a homeomorphism. Then j is a homeomorphism. Therefore, a continuous bijection from a compact space is a homeomorphism. □

Theorem 3.9. $\Delta_0^n \cong \overline{B^n}$.

Proof. First, we will prove that any convex body X is homeomorphic to $\overline{B^n}$. Fix a convex body X. Let $f : X \to Y$ such that $f(x) = x - x_0$. f is clearly a homeomorphism. Using g and g^{-1} as defined after Definition 3.7, and the result from Lemma 3.8, we have a homeomorphism between ∂Y and S^{n-1}. This homeomorphism and the result from Lemma 3.6 give us a homeomorphism between Y and $\overline{B^n}$. Therefore, we have a homeomorphism between X and $\overline{B^n}$.

From Proposition 3.2, we know that Δ_0^n is a convex body. We can conclude that $\Delta_0^n \cong \overline{B^n}$. □

Corollary 3.10. $\Delta^n \cong \overline{B^n}$.

Proof. We have that $\Delta^n \cong \Delta_0^n$ and that $\Delta_0^n \cong \overline{B^n}$. Since there exist homeomorphisms $f : \Delta^n \to \Delta_0^n$ and $g : \Delta_0^n \to \overline{B^n}$ such that f, g, f^{-1}, and g^{-1} are continuous, and f and g are bijections, we know $g \circ f$ and $(g \circ f)^{-1}$ are both continuous and $g \circ f$ is a bijection. So $g \circ f : \Delta^n \to \overline{B^n}$ is a homeomorphism, and $\Delta^n \cong \overline{B^n}$. □

4. Sperner's Lemma

The next two sections detail a proof by Jacob Fox of MIT, cited as the first reference at the end of this paper.

Before we state Sperner's Lemma, we will define a few terms.

Definition 4.1. An *n-dimension simplex* is

$$C = \{\sum_{i=1}^{n+1} \alpha_i v_i \mid \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1\}.$$

Note that the standard n-simplex is an n-dimensional simplex.

Definition 4.2. A *simplicial subdivision* of an n-dimensional simplex is a partition of the simplex into smaller simplicies, called cells, such that any two cells either are disjoint or share an entire face of a certain dimension.

Definition 4.3. Let S denote an n-dimensional simplex. A *proper coloring of a simplicial subdivision of S* is an assignment of $n + 1$ colors to the vertices of the subdivision, such that each vertex of S receives a different color, and points on each face of S receive only the colors of the vertices defining that respective face of S.

Lemma 4.4. *(Sperner's Lemma) Every properly colored simplicial subdivision contains a cell whose vertices have all different colors.*

Proof. We will prove a statement slightly stronger than Sperner's Lemma: Every properly colored simplicial subdivision contains an odd number of cells whose vertices have all different colors. We will let Δ^n be our n-dimensional simplex and call cells with all different colored vertices *rainbow cells.* Before proving the general case, we will consider cases $n = 1$ and $n = 2$.

<u>Case $n = 1$</u>. Consider a properly colored simplicial subdivision of Δ^1. Δ^1 is a line segment, so a simplicial subdivision of Δ^1 is a division of one line segment (a, b) into smaller segments. By Definition 4.2, a and b have different colors t_1 and t_2 respectively, and the vertices in between a and b are each colored either t_1 or t_2.

This means that we can construct a sequence $\{t_1, \cdots, t_2\}$ which has the colors of the vertices in order from a to b. In this sequence, if we start at t_1, the color of a, then no matter what the intermediate elements of the sequence are, the color t_i will change an odd number of times before it changes to the final element of the sequence, t_2. For example, in the sequence $\{t_1, t_2, t_2, t_1, t_2\}$, the color changes three times. Each time the color changes, a segment's second vertex is a different color than its first. Therefore, the number of color changes is equal to the number of rainbow segments, or rainbow cells in Δ^1. We can conclude that there are an odd number of rainbow cells in any coloring of a simplicial subdivision of Δ^1.

Case $n = 2$. Take any properly colored simplicial subdivision of Δ^2, which is a triangle T. Let Q be the number of cells in T that are colored with the vertices (t_1, t_1, t_2) or (t_1, t_2, t_2). Let R be the number of rainbow cells in T, colored (t_1, t_2, t_3). Let X equal the number of edges that are colored (t_1, t_2) that are completely on the boundary of the original triangle T. Let Y equal the number of edges that are colored (t_1, t_2) that are not completely on the boundary of T and are instead in the interior of T. So if an edge has only one endpoint on the boundary, it is counted in Y.

Note than in every cell counted in Q, there are two edges that are counted in either X or Y, while in every cell counted in R, there is one edge that is counted in either X or Y. Therefore, we can count the total edges in the subdivision by writing $2Q + R$. However, this method of counting counts internal edges twice, because each internal edge borders two cells, but counts boundary edges once, because each boundary edge borders only one cell. Then we have

$$2Q + R = 2Y + X.$$

Looking at the boundary of T, edges counted in X can only be on the edge between vertices a and b, by the definition of a proper coloring. By the reasoning used in the $n = 1$ case, there are an odd number of edges colored (t_1, t_2) between a and b. It follows that X is odd. Since $2Q + R = 2Y + X$, R is odd too. We can conclude that there are an odd number of rainbow cells in any coloring of a simplicial subdivision of Δ^2.

General case for dimension n. We will prove the general case by induction on n. We have already proven the base case, $n = 1$. For the inductive hypothesis, assume that any properly colored simplicial subdivision of Δ^{n-1}, which is colored with n colors $t_1, \cdots, t_n$, contains an odd number of rainbow cells. We want to show that any properly colored simplicial subdivision of Δ^n, which is colored with $n+1$ colors $t_1, \cdots, t_{n+1}$, contains an odd number of rainbow cells.

Take any properly colored simplicial subdivision of Δ^n. Let R again be the number of rainbow cells. Let Q be the number of cells that are colored with all colors except t_{n+1}. Each cell has $n + 1$ vertices, so for each cell counted in Q, exactly two vertices are colored the same color and the remaining $n - 1$ vertices are each colored a different color.

As in case $n = 2$, consider all the $(n - 1)$-dimensional faces that use exactly the colors $t_1, \cdots, t_n$. Let X be the number of these faces on the boundary of Δ^n, and let Y equal the number of these faces that are in the interior of Δ^n.

Note that in every cell counted in Q, there are two $(n-1)$-dimensional faces that are counted in either X or Y. In every cell counted in R, there is one $(n-1)$-dimensional faces that is counted in either X or Y. Therefore, we can count the total $(n-1)$-dimensional faces in the subdivision by writing $2Q+R$. However, this method of counting counts internal $(n-1)$-dimensional faces twice, because each internal $(n-1)$-dimensional face borders two cells. The method of counting counts boundary $(n-1)$-dimensional faces once, because each boundary $(n-1)$-dimensional faces borders only one cell. Then we have again

$$2Q + R = 2Y + X.$$

Looking at the boundary of Δ^n, $(n-1)$-dimensional faces counted in X can only be on the face $F \subset \Delta^n$ where F has vertices colored $t_1, \cdots, t_n$, by the definition of a proper coloring. F is a properly colored $(n-1)$-dimensional simplicial subdivision. By the inductive hypothesis, F has an odd number of rainbow cells. It follows that X is odd. Since $2Q+R = 2Y+X$, R is odd too. We can conclude that there are an odd number of rainbow cells in any coloring of a simplicial subdivision of Δ^n. □

5. Brouwer's Theorem

First we will prove a lemma that will allow us to rephrase Brouwer's Theorem.

Lemma 5.1. *Suppose any continuous function $g : X \to X$ has a fixed point. Let $f : X \to Y$ be a homeomorphism and let $h : Y \to Y$ be continuous. Then h has a fixed point.*

Proof. Let $g : f^{-1} \circ h \circ f$. Since f is a homeomorphism and h is continuous, g is the composition of continuous functions, so g is continuous. By construction, g maps from X to X. Then by hypothesis, g has a fixed point. This means there exists $x_0 \in X$ such that $g(x_0) = x_0$. By construction,

$$\begin{aligned} g(x_0) &= f^{-1}(h(f(x_0))) \\ x_0 &= f^{-1}(h(f(x_0))) \\ f(x_0) &= f(f^{-1}(h(f(x_0)))) \\ f(x_0) &= h(f(x_0)). \end{aligned}$$

This means h has the fixed point $f(x_0)$, which is what we wanted to show. □

Definition 5.2. A *barycentric subdivision* of Δ^n is a simplicial subdivision where the faces are formed by the following construction:

1. Draw the barycenters (centroids) of all faces, from dimensions 0 to n, of Δ^n.
2. For all n, consider each $(n+1)-$tuple of barycenters such that each barycenter in the $(n+1)$-tuple is the barycenter of a different dimensional face. Connect this $(n+1)-$tuple of barycenters by an n-dimensional simplex.

Theorem 5.3. *(Brouwer's Theorem) Let $\overline{B^n}$ be a closed n-dimensional ball. For any continuous function $f : \overline{B^n} \to \overline{B^n}$, there exists $x \in \overline{B^n}$ such that $f(x) = x$.*

Proof. Lemma 5.2 implies that since $\Delta^n \cong \overline{B^n}$, if we prove Brouwer's Theorem for Δ^n, then Brouwer's Theorem will be true for $\overline{B^n}$. So we wish to show that for any continuous function $f : \Delta^n \to \Delta^n$, there exists $x \in \Delta^n$ such that $f(x) = x$. We will proceed by contradiction.

Assume for the sake of contradiction that there exists a continuous function $f : \Delta^n \to \Delta^n$ with no fixed points. Construct a series of barycentric subdivisions of Δ^n, denoted $\Delta^n_1, \Delta^n_2, \Delta^n_3, \cdots$ such that each Δ^n_j is a barycentric subdivision of Δ^n_{j-1}. This means that as $j \to \infty$, the size of each cell in Δ^n_j approaches zero.

We will now use f to get a coloring of the jth barycentric subdivision, Δ^n_j, for each $j \geq 0$. Note that for each point $x \in \Delta^n$, $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n (f(x))_i = 1$ by the definition of Δ^n. Since $f(x) \neq x$, there are coordinates x_i and $x_{i'}$ such that $(f(x))_i < x_i$ and $(f(x))_{i'} > x_{i'}$. If there are multiple coordinates x_i such that $(f(x))_i < x_i$, take the least i. Assign each element of $[n+1]$ a different color. Then for each vertex $x \in \Delta^n_j$, we can assign a color $c(x) \in [n+1]$ such that $(f(x))_{c(x)} < x_{c(x)}$.

Now we will check that this coloring satisfies the conditions of Sperner's Lemma, namely that this coloring is a properly colored simplicial subdivision. The vertices of Δ^n can be written as $v_1, \cdots, v_{n+1}$, where $v_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ and 1 is the value of the ith coordinate. i is the only coordinate where $(f(x))_i < x_i$ is possible, because all other coordinates are zero, and there is nothing less than zero that is still a possible coordinate of $f(x) \in \Delta^n$. So $c(x) = i$ for vertices of Δ^n because $(f(x))_{c(x)} < x_{c(x)}$. Let $A \subset \{1, \cdots, n+1\}$ and let F be the face defined by the vertices in $\{v_i \mid i \in A\}$. Fix $x \in F$. For the coordinates of x that are zero, we cannot have $(f(x))_i < x_i$, by the same reasoning as above. So $(f(x))_i < x_i$ only for $i \in A$. Then $c(x) \in A$. Since $c(x) = i$ for vertices of Δ^n and $c(x) \in A$ for vertices on a particular face of Δ^n, this coloring is a properly colored simplicial subdivision, and the conditions of Sperner's Lemma are satisfied.

We can then apply Sperner's Lemma to say there exists a rainbow cell with vertices

$$x^{(j,1)}, \cdots, x^{(j,n+1)} \in \Delta^n_j.$$

Because of the coloring we just defined, $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for each $i \in [n+1]$, where $x^{(j,i)}$ is the vertex with color i.

What we just did holds for each jth barycentric subdivision for all $j \geq 0$. This means we have a sequence $a_j = \{x^{(j,1)}\}$ inside Δ^n, which we know to be compact. It follows that a_j has a convergent subsequence $\{x^{(n_j,1)}\}$. We will ignore the points outside this subsequence, and relabel $\{x^{(n_j,1)}\}$ to be $\{x^{(j,1)}\}$. Now $\{x^{(j,1)}\}$ is convergent. We can do this relabeling for all i, from $\{x^{(j,2)}\}$ to $\{x^{(j,n+1)}\}$. Since the size of each cell in Δ^n_j approaches zero as $j \to \infty$, $\lim_{j\to\infty} x^{(j,i)}$ is the same for all $i \in \{1, \cdots, n+1\}$. Let this limit be $x^* = \lim_{j\to\infty} x^{(j,i)}$.

By our assumption, $f(x^*) \neq x^*$. Then $(f(x^*))_i > x^*_i$ for some i. But $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for each $i \in [n+1]$ for all $j \geq 0$. f is continuous, so since $x^* = \lim_{j\to\infty} x^{(j,i)}$, $(f(x^*))_i \leq x^*_i$. This is a contradiction, so we can conclude that for any continuous function $f : \Delta^n \to \Delta^n$, there exists $x \in \Delta^n$ such that $f(x) = x$. □

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References

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