Some applications of the Borsuk-Ulam Theorem

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1 Introduction

The Borsuk-Ulam theorem with various generalizations and many proofs is one of the most useful theorems in algebraic topology. This paper will demonstrate this by first exploring the various formulations of the Borsuk-Ulam theorem, then exploring two of its applications. The two major applications under consideration, the ham sandwich theorem and the Kneser conjecture, come from different areas of mathematics and are separately interesting.

Before going further, I would like to credit [1] for inspiring the sequence of the discussion and for introducing me to many of these proofs. I have tried to emulate the clarity and simplicity of its presentation here. First, we will state the theorem in several different ways.

Theorem 1.1 (Borsuk ($B1a$)). *For every continuous function $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

Theorem 1.2 (Borsuk ($B1b$)). *For every continuous antipodal function $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = 0$.*

Theorem 1.3 (Borsuk ($B2a$)). *There does not exist an antipodal mapping $f : S^n \to S^{n-1}$.*

Theorem 1.4 (Borsuk ($B2b$)). *There does not exist a mapping $f : B^n \to S^{n-1}$ that is antipodal on the boundary $\partial B^n = S^n$.*

Theorem 1.5 (Lyusternik and Shnirel'man closed (LSc)). *A covering of S^n by $n+1$ closed sets, $F_1, \ldots, F_{n+1}$, has at least one set that contains a pair of antipodal points.*

Theorem 1.6 (Lyusternik and Shnirel'man open (LSo)). *A covering of S^n by $n+1$ open sets, $U_1, \ldots, U_{n+1}$, has at least one set that contains a pair of antipodal points.*

Points x, y on S^k are said to be antipodal if $y = -x$. Likewise, a function $f : S^k \to X$ is antipodal if f is continuous and, for all $x \in S^k$, $f(-x) = -f(x)$ where X is any space. It also follows immediately from this definition that $F \cap -F = \varnothing$ holds for a set F which does not contain a pair of antipodal points.

A proof of the theorem itself would not be as illustrative as what follows, and given the scope of this paper, will therefore be excluded. However, proving the equivalence of the various forms is not particularly difficult and follows immediately.

Proof. $(B1a) \Rightarrow (B1b)$ Any antipodal continuous function $f : S^n \to \mathbb{R}^n$ has a point $x \in S^n$ that satisfies $f(x) = f(-x) = -f(x)$ by (B1a), which implies $f(x) = 0$.

$(B1b) \Rightarrow (B1a)$ The converse follows by applying (B1b) to $g(x) = f(x) - f(-x)$, since g is antipodal and continuous for any continuous f.

$(B1b) \Rightarrow (B2a)$ An antipodal continuous function $f : S^n \to S^{n-1}$ defines an antipodal function from S^n to $\mathbb{R}^n$ which is nowhere zero, since $0 \notin S^{n-1}$ and $S^{n-1} \subset \mathbb{R}^n$. The existence of such a function is prohibited by (B1b).

$(B2a) \Rightarrow (B1b)$ Similarly, a nonzero antipodal function $f : S^n \to \mathbb{R}^n$ defines a function $g : S^n \to S^{n-1}$ by $g := \frac{f(x)}{\|f(x)\|}$. Such a function contradicts (B2a).

$(B2b) \Rightarrow (B2a)$ Note that the projection $\pi : (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ of the upper hemisphere of S^n to B^n is a homeomorphism. Thus, an antipodal map $f : S^n \to S^{n-1}$ would define a function $g : B^n \to S^{n-1}$ that is antipodal on the boundary ∂B^n according to $g(x) := f(\pi^{-1}(x))$, contradicting (B2b). Here continuity of g is guaranteed since π is a homeomorphism, implying π^{-1} is continuous.

$(B2a) \Rightarrow (B2b)$ A map $g : B^n \to S^{n-1}$ that is antipodal on the boundary ∂B^n defines a function $f : S^n \to S^{n-1}$ by $f(x) := g(\pi(x))$ and $f(-x) := -g(\pi(x))$ for x in the upper hemisphere. Such a function is antipodal by construction, contradicting (B2a).

$(LSo) \Rightarrow (LSc)$ For a closed cover, $F_1, \ldots, F_{n+1}$, of S^n define open sets $U_i^\epsilon := \{x \in S^n : \text{dist}(x, F_i) < \epsilon\}$, where $\text{dist}(x, S) := \inf_{y \in S} \|x - y\|$ for a point x and a set S. If no set in the closed cover $F_1, \ldots, F_{n+1}$ contained a pair of antipodal points, then there is some $\epsilon > 0$ such that the diameter of every set is less than $2 - \epsilon$. Thus, $F_i \subset U_i^{\epsilon/2}$, so $U_1^{\epsilon/2}, \ldots, U_{n+1}^{\epsilon/2}$ forms an open cover of S^n which does not contain any antipodal points, violating (LSo).

$(LSc) \Rightarrow (LSo)$ For an open cover, $U_1, \ldots, U_{n+1}$, we will construct $F_1, \ldots, F_{n+1}$, a collection of closed sets with $F_i \subset U_i$ for $i \in [n+1]$, where [n+1] denotes the set $\{1, 2, \ldots, n+1\}$. For each $x \in S^n$, let V_x be an open neighborhood of x

whose closure is completely contained in one of the U_i. Applying compactness gives a cover whose closure, $F_1, \ldots, F_{n+1}$, doesn't contain any pairs of antipodal points, since $F_i \subset U_i$ for each i, contradicting (LSc).

$(B1a) \Rightarrow (LSc)$ For a closed cover, $F_1, \ldots, F_{n+1}$, let $f : S^n \to \mathbb{R}^n$ such that $x \mapsto (\text{dist}(x, F_1), \text{dist}(x, F_1), \ldots, \text{dist}(x, F_n))$. Because the norm is continuous, f is continuous, so there exists a point y such that $y = f(x) = f(-x)$ by $(B1a)$. if some component of y is zero, for example, the ith component, then $\text{dist}(x, F_i) = 0 = \text{dist}(-x, F_i)$. Thus, F_i contains an antipodal pair. On the other hand, if no component of y is zero, then $x, -x \in F_{n+1}$, since $F_1, \ldots, F_{n+1}$ covers S^n and $x, -x \notin F_1, \ldots, F_n$.

$(LSc) \Rightarrow (B2a)$ There exists a covering of S^{n-1} by $n+1$ closed sets such that no set contains a pair of antipodal points. To construct such a covering, $F_1, \ldots, F_{n+1}$, simply project the facets of an n-simplex containing 0 in the interior centrally from 0 onto S^n. Thus, an antipodal mapping $f : S^n \to S^{n-1}$ gives a closed covering $f^{-1}(F_1), \ldots, f^{-1}(F_{n+1})$, no set of which contains a pair of antipodal points, contradicting (LSc). □

Thus, the various theorems above are equivalent statements of the Borsuk-Ulam theorem. As a quick illustration, applying $(B1a)$ guarantees that there is some point on earth which shares a temperature and barometric pressure with its antipode. It is also interesting to observe that Borsuk-Ulam gives a quick proof of the Brouwer fixed point theorem, the important result from algebraic topology which states that, for every continuous function $f : B^n \to B^n$, there exists a point $x \in S^n$ such that $f(x) = x$.

$(B2b) \Rightarrow$ *Brouwer fixed point theorem.* Suppose there exists a continuous function $f : B^n \to B^n$ such that f has no fixed point. Define $g : B^n \to S^{n-1}$ such that $g(x)$ is the point on S^{n-1} that intersects with the ray from $f(x)$ to x. This is a well defined retraction since there is no fixed point at which the function would be ill-defined, and it is a retraction since a ray from anywhere in B^n to a point $x \in \partial B^n = S^{n-1}$ intersects the boundary at x by construction. But, since the identity is clearly antipodal, i.e., $g(-x) = -g(x)$ for $x \in S^{n-1}$, such a function contradicts $(B2b)$. □

Such a simple proof seems reasonable given the similarities between the Borsuk-Ulam theorem and the Brouwer fixed point theorem, and, indeed, there are proofs of each theorem which share many similarities. At this point, it is worth noting that Borsuk-Ulam theorem has many generalizations and a variety of methods of proof. But we will instead focus on proving two interesting theorems, the ham sandwich Theorem and the Kneser conjecture.

2 Ham Sandwich Theorem

We'll begin by defining some concepts and proceed by stating a version of the ham sandwich theorem. We then prove the theorem as well as some generalizations. Finally, we apply the theorem to a simple problem.

Definition 2.1. A *hyperplane* in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace, or equivalently, the set $\{x \in \mathbb{R}^n : \langle a, x\rangle = b\}$ for some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$. A hyperplane defines two closed *half-spaces* of the form $\{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$ for some $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

The concept of a measure, somewhat informally described as a volume function defined from sets to $\mathbb{R}^+ \cup \infty$, is also important, but we will not concern ourselves with a construction here (for such a construction, see [2]). As such, we will assume familiarity with measures.

Definition 2.2. A *finite Borel measure* μ on $\mathbb{R}^d$ is a measure on $\mathbb{R}^d$ such that all open subsets of $\mathbb{R}^d$ are measureable and $0 < \mu(\mathbb{R}^d) < \infty$.

An illustrative example is that of the restriction of the Lebesgue measure to a compact subset. If $A \subset \mathbb{R}^d$ is a compact subset and λ^d is the d-dimensional Lebesgue measure, then $\mu(X) := \lambda^d(X \cap A)$ for Lebesgue measurable sets $X \subseteq \mathbb{R}^d$.

Theorem 2.1 (Ham sandwich theorem)**.** *Let $\mu_1, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the μ_i. Then there exists a hyperplane h such that*

$$\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d) \text{ for } i = 1, \ldots, d,$$

where h^+ is one of the half-spaces defined by h.

Informally, this theorem states that any arrangement of ham, bread, and cheese in space can be bisected by a single cut. The fact that any ham sandwich can be bisected with only one well placed cut is quite surprising.

Proof. Let $u = (u_0, u_1, \ldots, u_d) \in S^n$. If one of the components $u_1, \ldots, u_d$ is nonzero, assign the half-space

$$h^+(u) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d \leq u_0\}$$

to u. Antipodal points correspond to opposite half-spaces:

$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : -u_1x_1 + \cdots + -u_dx_d \leq -u_0\} = h^+(-x) =$$
$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d \geq u_0\} = -h^+(u).$$

We also have

$$h^+((1, 0, \ldots, 0)) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d = 0 \leq 1\} = \mathbb{R}^d,$$

$$h^+((-1,0,\dots,0)) = \{(x_1,\dots,x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d = 0 \le -1\} = \varnothing.$$

Then, define $f : S^d \to \mathbb{R}^d$ by $f_i = \mu_i(h^+(u))$.

The hyperplane which defines a half-space $h^+(u_0)$ such that $f(u_0) = f(-u_0)$ satisfies the theorem. Additionally, by sub-additivity of a measure and since $\mu(\mathbb{R}^d) > 0$,

$$f((-1,0,\dots,0)) = (\mu_1(\varnothing),\dots,\mu_d(\varnothing)) \neq (\mu_1(\mathbb{R}^d),\dots,\mu_d(\mathbb{R}^d)) = f(1,0,\dots,0),$$

which guarantees a well-defined hyperplane. Thus, it suffices to prove that f is continuous in order to apply Borsuk-Ulam.

We will show that $\mu_i(h^+(u_n)) \to \mu_i(h^+(u))$ for a sequence $(u_n)_{n=1}^\infty$ which converges to $u \in S^d$. For $x \notin \partial h^+(u)$ and sufficiently large n, $x \in h^+(u_n)$ if and only if $x \in h^+(u)$. Therefore, the characteristic function $g_n := \chi_{h^+(u_n)}$ approaches the characteristic function $g = \chi_{h^+(u)}$, except on $\partial h^+(u)$. By assumption, $\mu_i(\partial h^+(u)) = 0$, so g_n converges almost everywhere to g in the μ_i measure. Lebesgue's dominated convergence theorem gives $\mu_i(h^+(u_n)) = \int g_n d\mu_i \to \int g \, d\mu_i = \mu_i(h^+(u))$. This follows since 1 dominates g_n and further since the assumption of a finite measure guarantees the integrability of 1. Therefore, f is continuous and Borsuk-Ulam applies, completing the proof. □

We next consider point sets, but the meaning of a hyperplane bisecting a point set with an odd number of points carries some ambiguity. We say that a hyperplane h bisects a point set A if each of the open half-spaces defined by h contains at most $\lfloor \frac{1}{2} |A| \rfloor$ points in A. This definition allows at most k points from a set containing $2k+1$ points in each open half-space, thereby requiring that at least one point in an odd magnitude point set lie on the hyperplane. With this clarification, we may prove a variation of the ham sandwich theorem.

Theorem 2.2 (Point set ham sandwich theorem)**.** *Let $A_1,\dots,A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane h that simultaneously bisects $A_1,\dots,A_d$.*

Proof. We would like to apply the ham sandwich theorem for measures to the point sets by imagining them as tiny balls. Suppose each A_i has odd cardinality and the disjoint union is in general position, i.e., $A_i \cap A_j = \varnothing$ for $i \neq j$ and no hyperplane intersects more than d points. Then, let A_i^ϵ be the set A_i with each point replaced by a ball of radius ϵ centered at that point. Thus, there exists an $\epsilon > 0$ such that no hyperplane intersects more than d balls of the union $A_1^\epsilon \cup \cdots \cup A_d^\epsilon$. The ham sandwich theorem gives a hyperplane that bisects the sets A_i^ϵ. This bisection must intersect at least one of point from each A_i^ϵ since they each have an odd number of balls, and, since h intersects a maximum of d balls, each A_i^ϵ is intersected exactly once. The bisection guarantees that the hyperplane passes through the center of a ball which it intersects. Therefore, h bisects each of the original A_i.

We will relax the assumptions on position. Let $A_{i,\eta}$ for $\eta > 0$ denote a perturbation of A_i where each point is moved by at most η such that the disjoint union of $A_{i,\eta}$, in general position as before. Then, the previous argument gives h_η a hyperplane which bisects the $A_{i,\eta}$. For each h_η, we may write $h_\eta = \{x \in \mathbb{R}^d : \langle a_\eta, x\rangle = b_\eta\}$ for some a_η, a unit vector in $\mathbb{R}^d$ and some scalar b_η. The b_η lie in a bounded interval and the a_η lie on a $d-1$-sphere. Thus, we may apply compactness to guarantee a cluster point $(a, b) \in R^{d+1}$ of pairs (a_η, b_η) as $\eta \to 0$. We claim that the hyperplane $h = \{x \in S^n : \langle a, x\rangle = b\}$ bisects A_i. Take a sequence $\eta_1 > \eta_2 > \cdots$ which converges to 0 such that $(a_{\eta_j}, b_{\eta_j}) \to (a, b)$. Then, for x a distance $\delta > 0$ away from h, x is at least $\frac{1}{2}\delta$ away from h_{η_j} for sufficiently large j. Thus, if one of the open half-spaces of h contains k points in A_i, then the corresponding open half space defined by h_{η_j} must also contain at least k points of A_{i,η_j} for large enough j. Since these open half spaces cannot contain more than $\lfloor \frac{1}{2} | A | \rfloor$ points from A_i by construction, there can be no more than $\lfloor \frac{1}{2} | A_i | \rfloor$ points from A_i in the open half space defined by h, so h must bisect each A_i as required.

Now, we relax the condition on the parity of the A_i. Suppose some of the A_i have an even number of points. For each such A_i, remove an arbitrary point and bisect the resulting collection of point sets as before. Then, replacing the deleted points maintains the bisection because each open half space from the half-space already contained fewer than half of the points in the original sets. □

These two theorems give surprising results regarding the ability of hyperplanes to bisect arbitrary sets. We would like to apply these theorems to a different problem, but we need to modify the previous theorem slightly.

Corollary 2.2.1. *Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$ be disjoint, finite point sets in general position, i.e., no hyperplane contains more than d points of the disjoint union of A_i.*
Then there exists a hyperplane h that bisects each A_i such that there are exactly $\lfloor \frac{1}{2} | A_i | \rfloor$ points from A_i in each open half-space defined by h, and at most one point of A_i on the hyperplane h.

Proof. Applying the previous theorem gives a hyperplane h where each of the corresponding half-spaces has at most half of the points in each A_i. However, the hyperplane could contain up to d points of a single A_i if some of the A_i have even cardinality.

Fix a coordinate system so h has the equation $x_d = 0$. Let $B := h \cap (A_1 \cup \cdots \cup A_d)$; then B consists of at most d affinely independent points. We claim that it is possible to move h slightly so that any point in B is above or below or remains on h, whichever is required to achieve the desired bisection.

To demonstrate this, add $d - |B|$ points to B to obtain a d-point affinely independent set $C \subset h$. For each $a \in C$, choose a point a' such that either $a' = a$, in the case where a is one of the new points and for points of B which should remain on h in order to achieve the required bisection, or $a' = a + \epsilon e_d$, or $a' = a - \epsilon e_d$, where e_d is the standard basis vector in the d dimension. Let $h' = h'(\epsilon)$ be the hyperplane determined by the d points $a', a \in C$. For all sufficiently small $\epsilon > 0$, the a' remain affinely independent, so $h'(\epsilon)$ is well defined, and the motion of $h'(\epsilon)$ is continuous in ϵ. Thus, for all sufficiently small ϵ, h' bisects the A_i as required by the corollary. □

This version of the ham sandwich theorem will allow us to prove a theorem related to a necklace. We motivate this theorem with an anecdote.

Two thieves steal a necklace made up of d different types of jewels of unknown value. In order to equally divide the value of the necklace, the thieves agree to cut up the necklace so that they both get the same number of jewels. This is made easier since the necklace is open and there are an even number of gemstones of each type. However, the jewels are set in platinum, a precious metal which the thieves don't want to waste. So, the thieves decide to make as few cuts as possible.

For a necklace with all of the stones of one type followed by all the stones of a second type, and so on, through all d different types, it will be necessary to make d cuts to divide each type of jewel evenly. Then next theorem shows this is the maximum possible number of cuts.

Theorem 2.3 (Necklace)**.** *Every open necklace with d kinds of stones can be divided between two thieves using at most d cuts.*

Prior to giving the proof, we introduce one more concept, the moment curve. The moment curve in $\mathbb{R}^d$ is the curve $\{\gamma(t) : t \in \mathbb{R}\}$ given by $\gamma(t) := (t, t^2, \ldots, t^d)$. We will need a lemma regarding this curve.

Lemma 2.4. *No hyperplane intersects the moment curve γ in $\mathbb{R}^d$ in more than d points. Therefore, any set of $d+1$ distinct points on the moment curve is affinely independent.*

Proof. A hyperplane has equation $a_1x_1 + \cdots + a_dx_d = b$ with $(a_1, a_2, \ldots, a_d) \neq 0 \in \mathbb{R}^d$. A point in $\gamma(t)$ which intersects h satisfies $a_1t + a_2t^2 + \cdots + a_dt^d = b$. These intersections correspond to real roots of the polynomial $p(t) = (\sum_{i=1}^{d} a_it^i) - b$ whose degree is at most d. A polynomial of degree $\leq d$ has at most d roots, so there are a maximum of d intersections. □

This will allow us to prove the necklace theorem quite easily.

Proof of Necklace theorem. We embed the necklace into $\mathbb{R}^d$ along the moment curve. For a necklace of n stones, let

$$A_i = \{\gamma(k) = (k, k^2, \ldots, k^d) : \text{ the } k\text{th stone is of the } i\text{th kind, } k = 1, 2, \ldots, n\}.$$

We will refer to the points of A_i as the stones of the ith kind. By the lemma, h cuts the moment curve in at most d places, so we may apply 2.2.1. Therefore, there exists a hyperplane h simultaneously bisecting each A_i. Furthermore, since every set is even, h contains no stones, and so cutting the necklace where h intersects the moment curve divides the stones as required by the theorem. □

This theorem is surprisingly difficult to prove without topological methods, with combinatorial proofs typically using some version of Tucker's lemma, the combinatorial version of the Borsuk-Ulam Theorem, as with [3].

3 Kneser's Conjecture

We will now explore a somewhat different topic in Kneser's conjecture. Consider the n-element subsets of a set with $2n+k$ elements. Then, Kneser conjectured that the minimum number of classes required to partition the subsets so that the pairwise intersection of subsets in each class is non-empty is $k+2$. We can exhibit such a partition by considering the set $[2n+k]$. Then if K_i denotes the collection of all n-subsets whose least element is i, then $K_1, K_2, \ldots, K_{k+1}$, and $K_{k+2} \cup \cdots \cup K_{n+k+1}$ generates a suitable partition with $k+2$ classes. Thus, Kneser's conjecture claims that any partition with fewer classes results in at least one class with a pair of disjoint subsets.

The conjecture was first proved by Lovász nearly 20 years after Kneser proposed it. Lovász introduced a perhaps surprising recoding in combinatorial terms which we shall introduce below. But for now, we will follow the logic of the simple proof first given by [4]. Thus, we desire a lemma generalizing the Lyusternik and Shnirel'man theorems introduced above.

Lemma 3.1 (Generalized Lyusternik and Shnirel'man)**.** *If S^d is covered with $d+1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points.*

Proof. Induct on the number t of closed sets in the cover of S^d. The base case $t=0$ is a cover composed entirely of open sets. This is simply (LSo) proved above. Now assume $0<t<d+1$ and that the conclusion holds for fewer than t closed sets as an induction hypothesis. Let C be a cover of S^d with $d+1$ sets, with t closed sets and $d+1-t$ open sets. Fix a closed set F in C. Suppose F did not contain a pair of antipodal points. It follows that F has diameter $2-\epsilon$ for some $\epsilon>0$. We let U be the open set of points in S^n with distance from F less than $\epsilon/2$. Thus, $F \subset U$ and $(C \backslash \{F\}) \cup U$ forms a cover of S^d consisting of $t-1$ closed sets and $d+2-t$ open sets. The induction hypothesis guarantees that at least one set in this cover contains a pair of antipodal points. However, U does not contain a pair of antipodal points by construction. So some other set in C must contain such a pair, as required. □

We now have a slightly stronger form of the Lyusternik and Shnirel'man theorem which gives us everything required to prove Kneser's conjecture.

Theorem 3.2 (Lovász-Kneser)**.** *If the n-element subsets of a $2n+k$ element set are partitioned into $k+1$ classes, then one of the classes must contain a pair of disjoint subsets.*

Proof. Distribute $2n+k$ points on S^{k+1} in general position, i.e., such that no $k+2$ points lie on a great k-sphere. Partition the n-element subsets of these points into $k+1$ classes, call them $A_1, \dots, A_{k+1}$. For $i \in [k+1]$, let U_i denote the set of points $a \in S^k$ such that the open hemisphere centered at a, which has the form $\{x \in S^{k+1} : \langle a, x\rangle > 0\}$, contains an n-subset in the class A_i. Each U_i is clearly open, so $F := S^{k+1} \backslash (U_1 \cup \cdots \cup U_{k+1})$ is closed. The collection of all of these sets covers $S^k + 1$ with $k+2$ sets. The lemma asserts that at least one set contains a pair of antipodal points $\pm a \in S^{k+1}$. F cannot contain $\pm a$, since then both open hemispheres centered at a and $-a$ would contain fewer than n points, which is impossible since then at least $k+2$ points must lie on the great k sphere given by $\{x \in S^{k+1} : \langle a, x\rangle = 0\}$. This contradicts the fact that the points are in general position. Thus, a and $-a$ must lie in U_i for some i. Therefore, the open hemispheres centered at a and $-a$ both contain an n-element subset in A_i, and these subsets must be disjoint. □

Kneser's conjecture thus receives a simple proof by embedding the sets on the sphere and applying Borsuk-Ulam. However, we can encode the information differently, by imagining the sets as graphs and asking about the chromatic number of the graph.

Definition 3.1. A (simple, undirected) *graph* G is a pair (V, E) where V is a set, known as the vertex set, and $E \subseteq \binom{V}{2}$ is the edge set, where $\binom{S}{2}$ is the collection of subsets of S with 2 elements. We say that the vertices $v, v' \in V$ are *adjacent* if $\{v, v'\} \in E$.

Furthermore, a legal *k-coloring* of G is a mapping $c : V \to [k]$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$. The *chromatic number* of G, denoted $\chi(G)$ is the smallest k such that G has a k-coloring.

We say that edges connect vertices, giving a simple picture for graphs. Additionally, a coloring is legal if no two adjacent vertices share a color. With these concepts in mind we may recode the conjecture as follows. For a set X and $\mathcal{F} \subset 2^X$, where 2^X is the power set of X, the Kneser graph of $\mathcal{F}$, denoted $\mathrm{KG}(\mathcal{F})$, has a vertex set $\mathcal{F}$ and two sets $F_1, F_2 \in \mathcal{F}$ are adjacent if and only if $F_1 \cap F_2 = \varnothing$. Symbolically,

$$\mathrm{KG}(\mathcal{F}) = (\mathcal{F}, \{\{F_1, F_2\} : F_1, F_2 \in \mathcal{F}, F_1 \cap F_2 = \varnothing\}).$$

If we denote the Kneser graph of $\mathcal{F} = \binom{[2k+n]}{n}$ by $\mathrm{KG}_{n+2k,n}$, where $\binom{[2k+n]}{n}$ is the n-element subsets of $[2k+n]$, then Kneser's conjecture asserts $\chi(\mathrm{KG}_{n+2k,n}) = n+2$. To be clear, the upper bound follows from the discussion at the beginning of the section which exhibits a coloring by $k+2$ colors, while the lower bound is given by the Lovász-Kneser theorem.

This interpretation is interesting for several reasons. For example, if $k < n$, then there is no triplet of mutually disjoint n-element subsets, which means that $\mathrm{KG}_{n+2k,n}$ has no triangles. Yet the chromatic number is arbitrarily large, growing with k. This is perhaps surprising since, when first introduced to the concept of a coloring, one might expect that a high degree of interconnectedness, for example in the form of triangles, is necessary to force a high chromatic number by reducing the size of independent sets. The Kneser graphs deny this expectation by allowing high chromatic number without triangles.

Definition 3.2. *Fractional chromatic number*, denoted $\chi_f(G)$, is the infimum of the fraction $\frac{a}{b}$ such that the vertex set is covered by a independent sets in such a way that every vertex is covered at least b times. An *independent set* is a subset of the vertices, $V' \subseteq V$, such that, for all v, $v' \in V'$, $v \not\sim v'$.

This notion of coloring is slightly different. Conceptually, it allows for multiple colors of different weights which each represent a legal coloring and whose total weight for each vertex is at least one. This reduces to the previous definition when each color has weight one, which gives $\chi_f(G) \leq \chi(G)$. Returning to the Kneser graph, we find that $\chi_f(\mathrm{KG}_{n+2k,n}) \leq \frac{n+2k}{n}$, since the collection of $n + 2k$ independent sets of the form $A_i = \{S \in \binom{[n+2k]}{n} : i \in S\}$ covers each point n times. Therefore, there is a large gap between the fractional chromatic number of the Kneser graph and its chromatic number. In general, giving lower bounds for $\chi_f(G)$ is easier than doing the same for $\chi(G)$, making the gap between the two types of chromatic number quite unusual.

This discussion links a theorem whose proof is highly dependent upon the topology of the sphere with a class of combinatorial objects. We thus see an unexpected connection between algebraic topology and graph theory.

4 Conclusion

This paper will hopefully provide an introduction to phrasings and applications of the Borsuk-Ulam theorem for those who are interested. Additional discussion as well as a variety of additional applications and several proofs of the theorem can be found in [1].

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