SCISSORS CONGRUENCE

MAEVE COATES WELSH

Abstract. This expository paper discusses Dehn's answer to Hilbert's third problem, and the stronger Dehn − Sydler Theorem. We start with the Wallace − Bolyai − Gerwien Theorem to introduce Scissors Congruence and follow Jessen's proof of the Dehn − Sydler Theorem. We then present some generalizations of this problem, including open questions.

Contents

1. Introduction

In 1807, Wallace proved the Wallace − Bolyai − Gerwien theorem that stated that if and only if two polygons had the same area, then either one could be cut into finitely many pieces and rearranged through translations and rotations to form the other [1]. These two polygons are then called Scissors Congruent. In 1900, Hilbert extended this to three dimensions and raised the question of whether the same volume was enough to determine that two polyhedra were Scissors Congruent [2]. In 1902, Hilbert's student Dehn showed that this was not true; the two polyhedra must also have the same Dehn Invariant [3]. Sydler proved that these two invariants were in fact enough to determine Scissors Congruence in 1965 [4]. This expository paper explains the Dehn − Sydler Theorem through a combination of Jessen's proof of Sydler's Theorem [5], which uses algebra proven in [6], and Zylev's Theorem [7].

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2. Scissors Congruence in the Plane

Definition 2.1. Two polygons P and Q are called Scissors Congruent in the plane if there exist finite sets of polygons $\{P_1, P_2, \dots, P_m\}$ and $\{Q_1, Q_2, \dots, Q_m\}$ such that the polygons in each respective set intersect with each other only on the boundaries, $\bigcup_{i=1}^{m} P_i = P$ and $\bigcup_{i=1}^{m} Q_i = Q$ and P_i is congruent to Q_i for each $i \in \{1, 2, \dots, m\}$.

Theorem 2.2 (Wallace-Bolyai-Gerwien Theorem, [1])**.** *Two polygons are Scissors Congruent if and only if they have the same area.*

It is easy to observe that if two polygons are Scissors Congruent then they have the same area. The proof follows trivially since congruent polygons have equal area. The harder part is to show that the converse is true.

Claim. *If two polygons have the same area then they are Scissors Congruent.*

Proof of this requires the following three lemmas.

Lemma 2.3. *Given any polygon, it is possible to split it up into a finite number of triangles.*

Proof. This is done by choosing a vertex and drawing lines to all other vertices such that these lines do not exit the polygon. If any non-triangle sub-polygons remain, then apply the same procedure to that sub-polygon using a different vertex than the original. Repeat this process until the initial polygon is entirely decomposed into triangles as in Figure 1. Because every polygon has a finite number of edges,

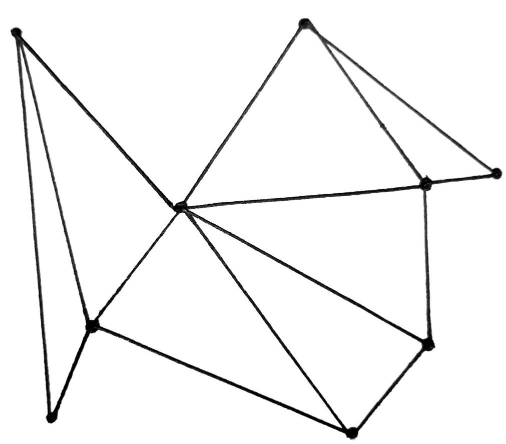

Figure 1.

this takes a finite number of steps and produces a finite number of triangles. □

Lemma 2.4. *A triangle T is Scissors Congruent to a rectangle with the same base.*

Proof. Let b be the base of T and h be the height of T. Let A, B, and C be the vertices of T, as shown in Figure 2. Construct a line l parallel to b at the midpoint G of h, and label the intersections of l with T as D and E.

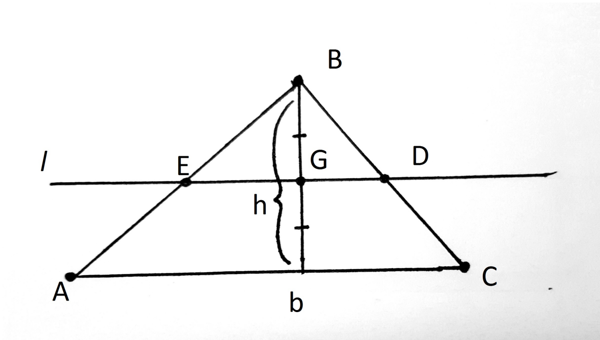

FIGURE 2.

Construct a line n parallel to $\overline{AE}$ through C, and label the intersection of l and n to be F, as in Figure 3.

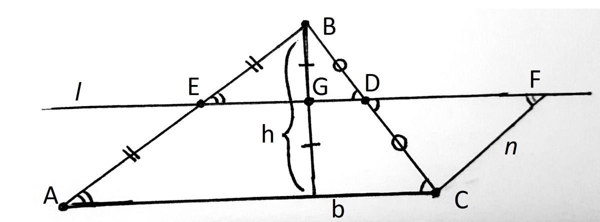

FIGURE 3.

Since G bisects h, it follows from Thales Theorem that D bisects $\overline{BC}$. Using basic geometry, we see that $\angle BDE$ is congruent to $\angle FDC$ and $\angle EAC$ is congruent to both $\angle BED$ and $\angle DFC$. Therefore $\triangle BDE$ is congruent to $\triangle CDF$ by AAS. Then, because $AEDC$ is congruent to $AECD$, it follows that $\triangle ABC$ is Scissors Congruent to $AECF$.

Finally, consider parallelogram $AEFC$. Construct perpendicular lines p and q to the base through points A and C respectively, and label the intersections of p and q with l as K and J respectively, as in Figure 4. Simple geometry shows that $\overline{AK}$

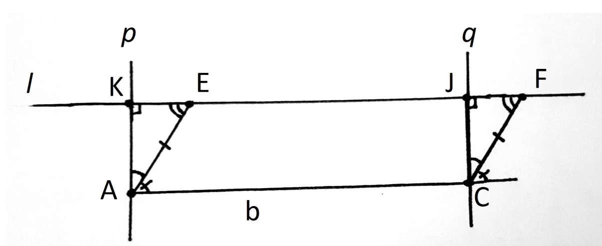

FIGURE 4.

is congruent to $\overline{CJ}$ and $\angle KAE$ is congruent to $\angle JCF$. Thus $\triangle KAE$ is congruent to $\triangle JCF$ by SAS. Because $AEFC$ is congruent to $AEFC$, it follows that $AKJC$

is Scissors Congruent to $AEFC$. Therefore $\triangle ABC$ is Scissors Congruent to the rectangle $AKJC$ with the same base. □

Lemma 2.5. *A rectangle is Scissors Congruent to another rectangle with arbitrary height x, if x is less than the height of the rectangle.*

Proof. Let h be the height of the first rectangle. We will consider the case where $h-x < x$. The case where $h-x = x$ only requires dividing the rectangle in two and stacking the pieces. The case where $h - x > x$ is fairly similar to the case where $h - x < x$, and the set up is demonstrated in Figure 5.

Given a rectangle $ABDC$, extend $\overline{CD}$ by $h - x$ length, as in Figure 5 to a point H. Mark E as the distance x from C on $\overline{AC}$. Construct the rectangle with vertices E, C, and H, and label the 4th point as G. Construct $\overline{AH}$ and label intersections I, J, and F as seen in Figure 5.

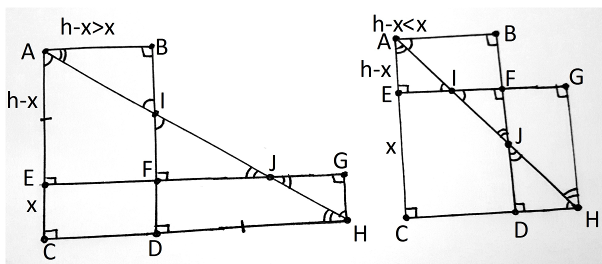

FIGURE 5.

Simple geometry shows that $\angle BJI$ is congruent to $\angle GHA$, $\angle HJD$, $\angle FJI$, and $\angle IAE$. Additionally, $\angle GIH$ is congruent to $\angle JHD$, $\angle BAI$, and $\angle AIE$. By SAS, $\triangle AIE$ is congruent to $\triangle JHD$. Thus $\overline{AI}$ is congruent to $\overline{JH}$. It follows that $\triangle ABJ$ is congruent to $\triangle IGH$. Thus we have

$$ABCD = CEIJD + EIA + AJB$$
$$\Leftrightarrow ABCD = CEIJD + DHJ + IHG.$$

Therefore $ABCD$ is Scissors Congruent to $CEGH$ which has base x. □

Thus we can see that given two polygons P and Q, we can decompose them into rectangles of some same base x. We know $x > 0$, but x can be arbitrarily small to accommodate arbitrarily small triangles. By stacking these rectangles as in Figure 6, we can see that because P and Q have the same area, the rectangles will reach the same height. It is evident that additional divisions of the rectangles can be made so the rectangles are clearly Scissors Congruent to each other. Thus we can see that P and Q are Scissors Congruent because they can be decomposed into exactly the same pieces.

3. Scissors Congruence in 3-Space

3.1. Some Definitions and Introduction to Group Theory.

Definition 3.1. Two polyhedra P and Q are Scissors Congruent in 3-space if there exist finite sets of polyhedra $\{P_1, P_2, \dots, P_m\}$ and $\{Q_1, Q_2, \dots, Q_m\}$ such that the

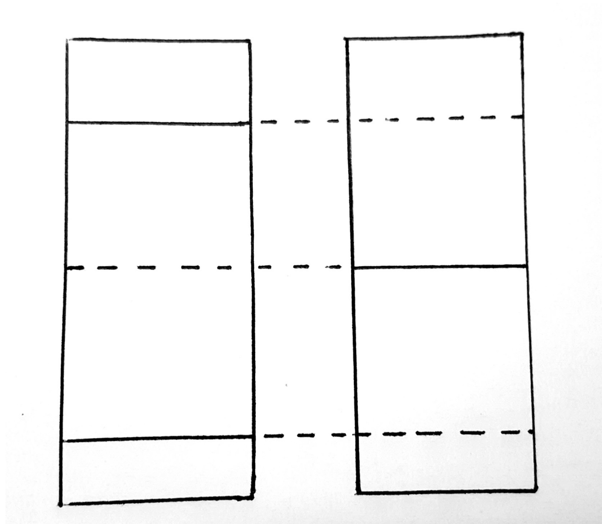

FIGURE 6.

polyhedra in each respective set intersect with each other only on the edges or faces, $\bigcup_{i=1}^{m} P_i = P$ and $\bigcup_{i=1}^{m} Q_i = Q$ and P_i is congruent to Q_i for each $i \in \{1, 2, \ldots, m\}$.

We can easily observe that

Proposition 3.2. *If two polyhedra are Scissors Congruent, then they have the same volume.*

In 2 dimensions, it was possible to tell if two polygons were Scissors Congruent by determining if they had the same area. In 3 dimensions, it is not the case that two polyhedra having the same volume implies Scissors Congruence, as will be shown explicitly in a later example. There is another criterion that must be met; namely, the Dehn Invariants of the two polyhedra must be the same. However, in order to introduce the Dehn Invariant, we must first cover some basic group theory.

Definition 3.3. A group is a set of elements G with a binary operation $\cdot$ and an identity element e such that the following statements hold true:

(1) For any $g, h \in G$, we have $g \cdot h \in G$.
(2) For $f, g, h \in G$, we have $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.
(3) For any $g \in G$, we have $g \cdot e = e \cdot g = g$.
(4) For any $g \in G$, there exists an element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

Definition 3.4. A group G is abelian if the binary operation $\cdot$ is commutative, i.e. for all $f, g \in G$, $f \cdot g = g \cdot f$.

Definition 3.5. Let G be a group with identity e and operation $\cdot$. Let H be a group with identity e' and operation $\star$. A homomorphism $\psi : (G, \cdot, e) \to (H, \star, e')$ is a function such that for all $f, g \in G$,

$$\psi(f \cdot g) = \psi(f) \star \psi(g).$$

Definition 3.6. A subset H of a group G is a subgroup of G if H is itself a group under the same group operation restricted to H and with the same identity element as G.

Definition 3.7. Let G be a group and H be a subgroup of G. Let x be an element of G. A coset of H is a set of the form $\{xh \mid h \in H\}$, written xH, or of the form $\{hx \mid \in H\}$, written Hx. A coset of the form xH is a left coset of H, and a coset of the form Hx is a right coset.

Definition 3.8. Let N be a subgroup of a group G. If $xNx^{-1} = N$ for every $x \in G$, then N is a normal subgroup of G.

Definition 3.9. Let G be a group, and N be a normal subgroup of G. The quotient group G/N, or "G modulo N," is the set of left cosets of N (which is the same as the set of right cosets) in G.

Definition 3.10. Let G and H be abelian groups. We define the tensor product of G and H, denoted $G \otimes H$ as

$$G \otimes H := \left\{ \sum_{finite} a_i(g_i \otimes h_i) \mid g_i \in G, h_i \in H, a_i \in \mathbb{Z} \right\}$$

modulo the following equivalence relations:

(1) $a_1(g \otimes h) + a_2(g \otimes h) = (a_1 + a_2)(g \otimes h)$.
(2) $(g_1 \cdot g_2) \otimes h = (g_1 \otimes h) + (g_2 \otimes h)$.
(3) $g \otimes (h_1 \cdot h_2) = (g \otimes h_1) + (g \otimes h_2)$.

Consider the group $\mathbb{R}/\pi\mathbb{Q}$ with operation $+$ and identity 0. We want to focus on $\mathscr{V} = \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$. The Dehn invariant of a polyhedron P is defined as

$$D(P) = \sum \text{length}(e) \otimes [\theta(e)] \in \mathscr{V}$$

where $\theta(e)$ is the interior dihedral angle (the angle formed by two faces of a polyhedron that is "inside" the polyhedron) at the edge e and the sum is over all edges e of P.

Observe that

Theorem 3.11. *If P and Q are scissors congruent, then* $\text{vol}(P) = \text{vol}(Q)$ *and* $D(P) = D(Q)$.

Proof. It is easy to observe that if two polyhedra are Scissors Congruent, then they have the same volume.

Because P and Q are Scissors Congruent, there exist decompositions of P and Q into $\{P_1, P_2, \ldots, P_n\}$ and $\{Q_1, A_2, \ldots, Q_n\}$ respectively such that for each $i \in \{1, 2, \ldots, n\}$, P_i is congruent to Q_i. Consider the cuts made in P and Q to get to these decompositions, and their effect on the Dehn Invariant. We want to show that if a polyhedron A is cut into two polyhedra A_1 and A_2, then $D(A) = D(A_1) + D(A_2)$.

A cut made transverse to an edge is one that goes through that edge by crossing it at one point on the edge. If a cut is made transverse to an edge E, then the dihedral angle at the two new edges E_1 and E_2 remains the same, and $\text{length}(E) = \text{length}(E_1) + \text{length}(E_2)$. Using the equivalence relations of the tensor product, we can see that

$$\text{length}(E) \otimes [\theta(E)] = [\text{length}(E_1) + \text{length}(E_2)] \otimes [\theta(E)]$$
$$\Rightarrow \text{length}(E) \otimes [\theta(E)] = \text{length}(E_1) \otimes [\theta(E)] + \text{length}(E_2) \otimes [\theta(E)]$$

and we can see that this does not affect the Dehn Invariant.

If a cut is made through an edge E, then the combined length of the edges formed, E_1 and E_2, remains the same, and the dihedral angle $\theta(E)$ is split up into two angles such that $\theta(E_1) + \theta(E_2) = \theta(E)$. By the same properties of tensors as before, this does not affect the Dehn Invariant.

A cut made transverse to a face is one that makes a line through the middle of a face. If a cut is made transverse to a face, then two new edges E_1 and E_2 are created. E_1 and E_2 have the same length, and $\theta(E_1) + \theta(E_2) = \pi$. Thus we can see that

$$\begin{aligned}
&\text{length}(E_1) \otimes [\theta(E_1)] + \text{length}(E_2) \otimes [\theta(E_2)] \\
=&\text{length}(E_1) \otimes [\theta(E_1) + \theta(E_2)] \\
=&\text{length}(E_1) \otimes [\pi] \\
=&\text{length}(E_1) \otimes [0] \\
=&0
\end{aligned}$$

i.e. the total Dehn invariant doesn't change.

Thus we can see that when we decompose P and Q, we have that

$$\begin{aligned}
D(P) &= D(P_1) + D(P_2) + \cdots + D(P_n) \\
&= D(Q_1) + D(Q_2) + \cdots + D(Q_n) \\
&= D(Q).
\end{aligned}$$

□

This theorem allows us to answer Hilbert's 3rd problem in the negative as is apparent from the following example, which was originally proven by Dehn [3], although we follow the proof of Conant [11].

Example 3.12. A cube C and a tetrahedron T of unit volume are not Scissors Congruent.

Proof. Every interior dihedral angle of the cube is equal to $\pi/2$, Therefore

$$\begin{aligned}
D(C) =& \sum_{i=1}^{12} |E_i| \otimes [\pi/2] \\
=& \sum_{i=1}^{12} |E_i| \otimes [0] \\
=&0.
\end{aligned}$$

Now consider the tetrahedron. Geometry shows that for a tetrahedron of volume 1, the length of each edge is $|72^{1/3}|$, and the measure of each angle is $\arccos(1/3)$. Thus

$$D(T) = \sum_{i=1}^{6} |72^{1/3}| \otimes [\arccos(1/3)].$$

We now want to show that $\arccos(1/3) \notin \pi\mathbb{Q}$. Suppose for contradiction that a rational p/q exists such that $\arccos(1/3) = p\pi/q$. Thus because $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we have that

$$e^{(p/q)\pi i} = \frac{1}{2} + (\sqrt{8}/3)i$$
$$\Rightarrow 1 = (1/3 + (\sqrt{8}/3)i)^{2q}$$

Claim. $(1/3 + (\sqrt{8}/3)i)^n = (a_n/3^{n+1} + i\sqrt{2}b_n/3^{n+1})$.

This holds for $n = 1$, with $a_1 = 3$ and $b_1 = 6$. Consider, then, if it holds for $n-1$. We want to use induction to show that it must hold for n. Thus we have

$$(1/3 + \sqrt{8}/3)i)^n = (1/3 + (\sqrt{8}/3)i)(a_{n-1}/3^n + i\sqrt{2}b_{n-1}/3^n).$$

By multiplying the right hand side, we have that $a_n = a_{n-1} - 4b_{n-1}$, and that $b_n = b_{n-1} + 2a_{n-1}$. This means that a_n and b_n are integers.
We now want to show that b_n can never be equal to 0. Thus $(1/3 + (\sqrt{8}/3)i)^n$ will always have some imaginary component and cannot equal 1. Using mod3 and our a_1, b_1, we realize that we have the following:

$$a_1 = 1, b_1 = 0$$
$$a_2 = 1, b_2 = 2$$
$$a_3 = 2, b_3 = 1$$
$$a_4 = 1, b_4 = 2.$$

Thus we can see that this cycles, because $a_2 = a_4$, $b_2 = b_4$. Thus $b_n \neq 0 \pmod 3$ for all $n > 1$, and we know that for $n = 1$, $b = 6$. Thus $[\arccos(1/3)] \neq 0$. This implies $D(T) \neq 0$, so $D(T) \neq D(P)$, and hence by theorem 3.11, T and P are not Scissors Congruent. □

Of course, this answer to Hilbert's problem is really just a start as it immediately raises other questions:

- Are volume and Dehn invariant sufficient to classify polytopes up to scissors congruence?
- What about other dimensions?
- What about other geometries, $\mathbb{H}^3$, $\mathbb{S}^3$ etc.?

We show in the rest of this section that the answer to the first question is 'Yes', as was shown by Sydler in 1965.

3.2. **The Scissors Congruence Group and Zylev's theorem.** Let $\mathscr{P}$ be the set of formal sums of all polyhedra. We can give $\mathscr{P}$ a group structure, with the empty polyhedron as the identity and the operation as formal sum modulo the following equivalence relations:

- $nP + mP = (n+m)P$
- $P = P_1 + P_2$ if P_1 and P_2 intersect only on edges or faces, and $P = P_1 \cup P_2$.
- $P = Q$ if P is congruent to Q.

The group defined as above is called the scissors congruence group of $\mathbb{E}^3$. Note that in the previous section, we have shown that the corresponding group in $\mathbb{E}^2$ is isomorphic to $\mathbb{R}$.

If we have two polyhedra P and Q that are Scissors Congruent, we have that $[P] \in \mathscr{P}$ is equal to $[P_1]+[P_2]+\cdots+[P_n]=[Q_1]+[Q_2]+\cdots+[Q_n]=[Q]$. Thus if two polyhedra are Scissors Congruent, then they are in the same equivalence class in $\mathscr{P}$. This, along with above theorem, enables us to define a map

$$(\mathrm{vol}, D) : \mathscr{P} \to \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q})$$

The main theorem of this paper, theorem 3.22 shows that this map is injective.

Note that from definitions, having $[P] = [Q]$ in $\mathscr{P}$ means that there exists a polyhedron A such that $P \cup A$ is scissors congruent to $Q \cup A$ and P and A (resp. Q and A) only intersect on their faces. Two such polyhedra are said to be *stably scissors congruent* which doesn't immediately imply that they are scissors congruent. However the following theorems states that in $\mathbb{E}^2$ and $\mathbb{E}^3$, these are equivalent ideas.

Theorem 3.13. *Two polygons P and Q in the Euclidean Plane are Scissors Congruent if and only if they are Stably Scissors Congruent.*

Proof.

Claim. *If P and Q are Scissors Congruent, then they are Stably Scissors Congruent.*

By theorem 2.2, P and Q have the same area. Let A be some polygon such that P and A intersect only on their boundaries, and Q and A intersect only on their boundaries. Because A is congruent to A, it follows that $P \cup A$ has the same area as $Q \cup A$. Therefore P and Q are Stably Scissors Congruent.

Claim. *If P and Q are Stably Scissors Congruent, then they are Scissors Congruent.*

Because P and Q are Stably Scissors Congruent, there exists some polygon A such that A intersects P and Q only on their edges, and $P \cup A$ is Scissors Congruent to $Q \cup A$. By theorem 2.2, $P \cup A$ and $Q \cup A$ have the same area. Therefore, P and Q must have the same area. Thus by theorem 2.2, P and Q are Scissors Congruent.

Thus we can see that in the Euclidean Plane, Scissors Congruence is equivalent to Stable Scissors Congruence. □

Theorem 3.14 (Zylev, [7]). *For two polyhedra P and Q in $\mathbb{E}^3$, P is Scissors Congruent to Q if and only if P is stably Scissors Congruent to Q.*

Sketch of proof. We will follow the proof by Calegari [12]. If P is Scissors Congruent to Q, then clearly so are $P \cup A$ and $Q \cup A$. Conversely, if $P \cup A$ is Scissors Congruent to $Q \cup A$, then there is some division of A into pieces. We can further subdivide A into sufficiently small pieces A_i, $i \in \{1, 2, \ldots, n\}$. Let us define $f : P \cup A \to Q \cup A$ to be the function that maps a polyhedron in the decomposition of $P \cup A$ to the polyhedron it is congruent to in the decomposition of $Q \cup A$. In the same manner, let any subdivisions of polyhedra map to the corresponding subdivisions of the polyhedra that they map to. So, if some polyhedron X maps to some polyhedron Y, then $X_1 \subset X$ maps to $Y_1 \subset Y$ where Y_1 is congruent to X_1. Consider the image $f(A_1)$ in $Q \cup A$. If A_1 is small enough, then it must be equidecomposable with some subset $B_1 \in Q/f(A_1)$. We can then exchange B_1 with $f(A_1)$, decomposing any pieces that intersect B_1, and then we exchange A_1 with B_1, decomposing any pieces that intersect A_1. Therefore, A_1 is now in the correct place and we have

further decomposed pieces that intersect $A_1 \subset Q \cup A$ and $B_1 \subset Q \cup A$.
Now consider the image of A_2. We can do the same thing with A_2, although A_2 may have been further subdivided at this point, and f may be slightly different because of these further subdivisions. $f'(A_2) \subset Q \cup A$ is equidecomposable with A_2, which is equidecomposable with some $B_2 \subset Q \setminus f'(A_2)$, so we may exchange them as before, decomposing first pieces which intersect B_2 and then pieces which intersect A_2. This process repeats inductively with A_i, B_i, and $f'(A_i)$, and at the ith stage, we have that $A_1 \cup A_2 \cup \cdots \cup A_i$ do not move. Thus after n steps, we come to a decomposition of $P \cup A$ and $Q \cup A$ such that the pieces in $A_1 \cup A_2 \cup \cdots \cup A_n$ do not move with f. Thus P and Q are Scissors Congruent.

□

3.3. **Prisms and Jessen's proof of Sydler's Theorem.**

Definition 3.15. A prism is a polyhedron with two identical, parallel polygonal faces for ends and flat parallelogram sides that connect corresponding edges of the polygonal faces. An orthogonal prism is one where the sides are all rectangles.

Lemma 3.16. *Two prisms P and Q are Scissors Congruent if and only if they have the same volume.*

Proof. Let P and Q be two prisms with the same volume. Because the end faces of prisms are parallel, we can slice them with planes and rearrange the pieces to make orthogonal prisms, as shown in Figure 7. Now, consider P and Q from above, so they look like 2 dimensional polygons. We can use the same techniques that we did for polygons in Theorem 2.2 and turn the faces of the prisms into rectangles with length 1, by making scissor cuts through the top faces that are perpendicular to these faces. We can pick 1 as our length because we can pick any arbitrary height. Turn the prisms on their sides, so now the height of both prisms is 1. Because the prisms have the same volume, the faces that are now on top of the prisms must have the same area. We can use the methods of Theorem 2.2 again, to turn these faces of P and Q into congruent rectangles. Thus we can see that P and Q are Scissors Congruent. □

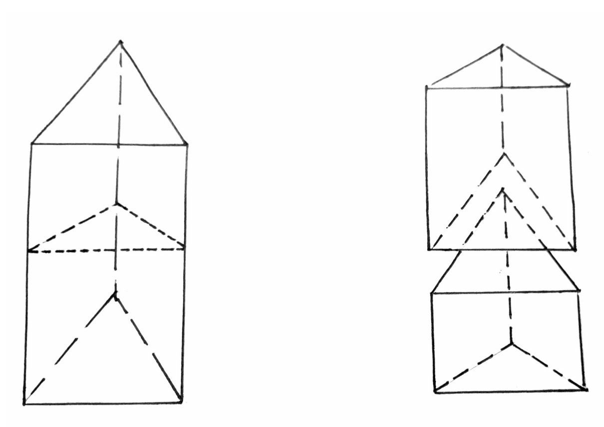

FIGURE 7.

Theorem 3.17. *All prisms have zero Dehn Invariant.*

Proof. Let P be an orthogonal prism. We can assume it is orthogonal by the same logic as lemma 3.16. Using the same techniques as in lemma 3.16, we can create a rectangular prism P' with edges $e_1,\ e_2, \ldots, e_{12}$ that is Scissors Congruent to P. Therefore, we can see that all dihedral angles are $\pi/2$. Because D maps to $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$, we have the following:

$$\begin{aligned} D(P') &= \sum_{i=1}^{12} \text{length}(e_i) \otimes [\pi/2] \\ &= \sum_{i=1}^{12} \text{length}(e_i) \otimes [0] \\ &= 0. \end{aligned}$$

Recall that Scissors Congruence implies the same Dehn Invariant. Therefore, because $D(P') = 0$, it follows that $D(P) = 0$. □

Now we consider another group. Let $\mathscr{P}/\mathscr{C}$ be the group of formal sums of all polyhedra modulo formal sums of prisms. This means that if you have a polyhedron P and a formal sum of prisms Q, such that P and Q do not intersect except on edges or faces, then P is equivalent to $P \cup Q$ in $\mathscr{P}/\mathscr{C}$. Let $j : \mathscr{P} \to \mathscr{P}/\mathscr{C}$ be the function that maps polyhedra to their equivalence class in $\mathscr{P}/\mathscr{C}$. Since the Dehn invariant of Prisms are zero, the map $D : \mathscr{P} \to \mathscr{V}$ factors through $\mathscr{P}/\mathscr{C}$ and the following proposition holds.

Proposition 3.18. *There exists a function $\delta : \mathscr{P}/\mathscr{C} \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$ such that $\delta \circ j(P) = D(P)$. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\ D\ } & \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q} \\ \big\downarrow j & \nearrow \delta & \\ \mathscr{P}/\mathscr{C} & & \end{array}$$

We will next prove Sydler's Theorem which shows that volume and the Dehn Invariant are the only two invariants necessary to show Scissors Congruence. However, in order to do so, we must first show that δ is injective.

Definition 3.19. An orthoscheme is a tetrahedron isometric to the convex hull of the points

$$(0, 0, 0); (x, 0, 0); (x, y, 0); (x, y, z)$$

Note that an orthoscheme has right triangles for faces, and 3 of the 6 dihedral angles are $\pi/2$.

We cite the existence of the following function, as it is beyond the scope of this paper.

Theorem 3.20. *We want to create a homomorphism $\phi : \mathbb{R} \to \mathscr{P}/\mathscr{C}$ such that*

(1) $\phi(a + b) = \phi(a) + \phi(b)$
(2) $\phi(na) = n\phi(a)$ *for* $n \in \mathbb{Z}$
(3) $\phi(\pi) = 0$

(4) $[T] = \sum_{i=1}^{6} \text{length}(e_i)\phi(\theta_i) \in \mathscr{P}/\mathscr{C}$

We follow the construction of Zakharevich [13] *for* ϕ*. Suppose there exists a function* $h : (0,1) \to \mathscr{P}/\mathscr{C}$ *such that* $[T(a,b)\bar{h}(a)+h(b)-h(a,b)$ *and* $ah(a)+bh(b) = 0$ *if* $a+b=1$*. We can then define* $\phi(\alpha) = tan(\alpha) \cdot h(sin^2(\alpha))$ *where* $(n\pi/2) = 0$*.* ϕ *is then our good function. By* [6]*, such a function exists.*

Proposition 3.21. *If such a function* ϕ *exists, then* δ *is injective.*

Proof. ϕ is an abelian homomorphism, and is $\mathbb{Q}$ -linear. It follows that $\phi(\alpha) = 0$ for all $\alpha \in \pi\mathbb{Q}$. Hence, ϕ induces a well-defined $\mathbb{Q}$-linear map from $\mathbb{R}/\pi\mathbb{Q} \to \mathscr{P}/\mathscr{C}$. Let us define a function $\Phi : \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q} \to \mathscr{P}/\mathscr{E}$ such that $\Phi(x \otimes y) = x\phi(y)$. We want to prove that $\Phi \circ \delta =$ identity, to show that δ is injective.

We can see that

$$\Phi \circ \delta([T]) = \Phi \circ D(T) = [T]$$

for any orthoscheme T. Any tetrahedron can be split up into finitely many orthoschemes, and every polyhedron can be split up into finitely many tetrahedrons, so we can see that $\mathscr{P}/\mathscr{C}$ is generated by orthoschemes. From this, we can realize that $\Phi \circ \delta$ is the identity, so δ must be injective. Because δ is injective, Sydler's Theorem can be proven. □

Theorem 3.22 (Sydler, [4])**.** *If two polyhedra* P *and* Q *have the same volume and the same Dehn Invariant, then* P *and* Q *are scissors congruent.*

Proof. We have shown that δ is injective. Now suppose we have two polyhedra P and Q with $\text{vol}(P) = \text{vol}(Q)$ and $D(P) = D(Q)$. Because δ is injective, $[P] = [Q] \in \mathscr{P}/\mathscr{C}$. Then there exist prisms R and S such that R only intersects P on faces, S only intersects Q on faces, and $[P \cup R] = [Q \cup S] \in \mathscr{P}$. By Theorem 3.11, $P \cup R$ and $Q \cup S$ have the same volume. Because $\text{vol}(P) = \text{vol}(Q)$, and the respective intersections of P and R, and of Q and S have volume 0, it follows that $\text{vol}(R) = \text{vol}(S)$. Because R and S are prisms, this means that they are scissors Congruent. Thus $[R] = [S]$ and consequently $[P] = [Q] \in \mathscr{P}$. Finally, therefore by Zylev's theorem 3.14, P is Scissors Congruent to Q. □

4. Generalizations and other Open Questions

We have discussed Scissors Congruence in two and three dimensional Euclidean Space, but there is no reason to limit study to these areas.

4.1. **Scissors Congruence in Higher Dimensions.** In [10], Zakharevich discusses how Jessen showed that Sydler's Theorem can be extended directly to four dimensions in [5]. However, it is an open question as to whether the Dehn Invariatnt can be generalized to higher dimensions than four.

4.2. **Scissors Congruence in Mixed Dimensions.** Scissors congruence also need not be limited to single dimensions. In [10], Zakharevich showed that when we keep track of the effects of cuts on multiple dimensions instead of treating polytopes like physical objects where lesser-dimensional errors can be ignored, we obtain a new notion of scissors congruence: mixed-dimensional scissors congruence. Here, length and the Euler characteristic are invariants for $\mathscr{P}(\mathbb{E}^1)$. We can also construct Goodwillie's bending invariant as an invariant for higher dimensions, although it is an open question as to whether volume and Goodwillie's bending invariant are enough to determine equivalence classes in $\mathscr{P}(\mathbb{E}^3)$.

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