A SURVEY ON THE MONODROMY GROUPS OF ALGEBRAIC FUNCTIONS

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Abstract. The study of polynomials is one of the most ancient subjects in mathematics, dating back to the Babylonian's search for solving the quadratic and even further. In this paper we shall prove theorems that have been central to the study of polynomials, such as the Abel-Ruffini Theorem, by studying their monodromy. Monodromy is the study of how objects "run round" a singularity, and so the viewpoint of this paper shall be geometric.

Contents

1. Introduction

In this paper we shall study the "functions" that return the roots of polynomials. Take for example the polynomial $p(z) = z^2 - a$, where $z, a \in \mathbb{C}$. We define *f(a)* as the "function" defined by the equation $p(z) = 0$. We can simply see $f(a) = \sqrt{a}$ (with $\sqrt{a}$ we refer to $\pm\sqrt{a}$).

However, *f(a)* is not a well-defined function, for it returns two values $\{+\sqrt{a}, -\sqrt{a}\}$, we call it a *multivalued function.* We shall call a special case of multivalued "functions" *algebraic functions*; their definition follows.

Definition 1.1. *An* algebraic function *is a relation*

$$f : \mathbb{C}^n \longrightarrow \mathbb{C}$$
$$(a_0, ...a_n) \longmapsto \{z|\ p(a_0, ...a_n, z) = a_0z^n + a_1z^{n-1} + ... + a_n = 0\}$$

Let's look closely at how the two single valued functions $+\sqrt{a}, -\sqrt{a}$ of $f(a)$ behave. Take a point $a_0 \in \mathbb{C} \setminus \{0\}$ and look at its two images under $+\sqrt{a}$ and $-\sqrt{a}$. Now draw a loop $\mathcal{L}$ given by $a(t)$, $t \in [0,1]$, that wraps around 0 such that $a(0) = a_0$. Define by continuity the two images of the loop. One notices that $+\sqrt{a(1)} = -\sqrt{a(0)}$ and $-\sqrt{a(1)} = +\sqrt{a(0)}$. In other words, when going around 0 the two single valued functions switched. This can be visualized Figure 1. This phenomenon is the origin of the monodromy groups we will discuss in the following sections.

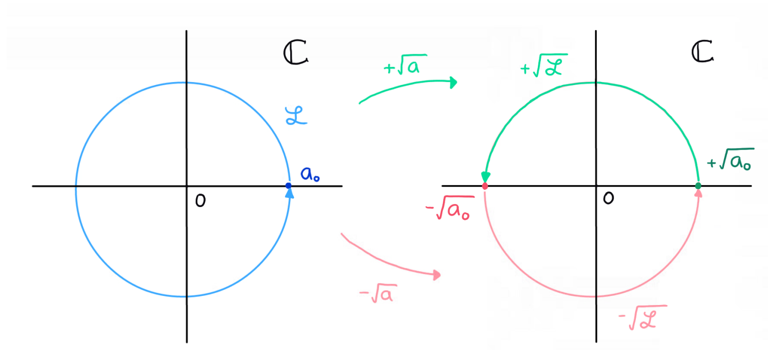

FIGURE 1. Loop that goes around 0 and its two images under $\sqrt{a}$

In the first section we will study multivalued functions and how to define them as continuous single valued functions on what we will call Riemann surfaces. Riemann surfaces are complicated surfaces and we will give a detailed construction of them in this section. In the second section we will define two different monodromy groups which will help us study the invariants of algebraic functions. Finally in the last section we will sketch two applications of the permutation monodromy group and talk a little about applications of the braid monodromy group.

2. RIEMANN SURFACES

2.1. Riemann surface of $\sqrt{a}$.

In this section we will go over the construction of the Riemann surface of $\sqrt{a}$. Observe

$$f(a) = \sqrt{a} = \begin{cases} f_1(a) = +\sqrt{r}\exp\{i\frac{\varphi}{2}\} \\ f_2(a) = +\sqrt{r}\exp\{i\frac{\varphi}{2} + \pi\} \end{cases} \tag{2.1}$$

We call $f_1(a) = +\sqrt{a}, f_2(a) = -\sqrt{a}$ the *single valued functions* or *branches* of $f(a)$.

Before building the Riemann surface of f(a) we must understand the phenomenon described in the introduction.

Let C be a curve given by $a(t) : [0,1] \to \mathbb{C} \setminus \{0\}$. Define $\varphi(t) : t \mapsto \arg(a(t))$ so that it is a continuous function. We say the variation of the argument of the curve C is equal to $\text{var}(C) := \varphi(1) - \varphi(0)$. For example in the following picture we have, $\text{var}(C_1) = 2\pi - 0 = 2\pi$, $\text{var}(C_2) = \varphi(1) - \varphi(0) = 0$ and $\text{var}(C_3) = 4\pi - 0 = 4\pi$.

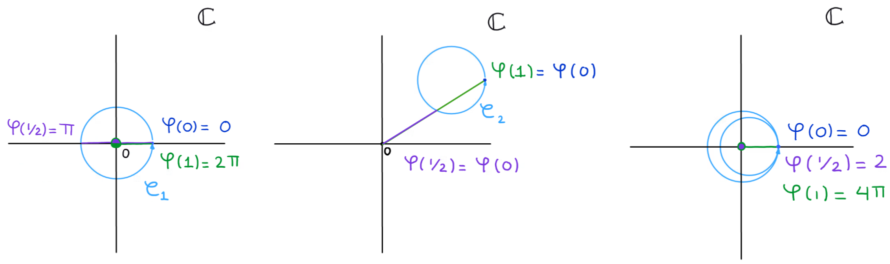

FIGURE 2. Example of 3 curves and how their argument varies

Let $n = \frac{1}{2\pi - 1}\int_C \frac{dz}{z}$ be the winding number of C around 0. It should be obvious that the variation of the argument of the C is equal to $2\pi \cdot n$. In particular, a loop $\mathcal{L}$ must have $\text{var}(\mathcal{L}) = 2\pi \cdot k$. Now let's look at how the argument of a curve's image under $f(a)$ varies. From equation (2.1) we know $\arg(f(a(t))) = \arg(a(t))/2$. Consequently $\text{var}(f(C)) = \text{var}(C)/2$. So for the image of a curve under f(a) to be closed, we must have $\text{var}(C) = 2(2\pi \cdot k)$. We have proved the following Lemma.

Lemma 2.2. *Let C be curve given by $a(t)$ such that $a(0) = a(1)$.*
Then $f(a(0)) = f(a(1))$ iff C has an even winding number around 0.

Lemma 2.3. *Let C be curve given by $a(t)$, then if:*
(1) C is in $\mathbb{C} \setminus (0, \infty)$, its image under $f(a)$ satisfies $f(a(t)) = f_i(a(t))$ for all $t \in [0, 1]$ for a fixed i=1,2
(2) C crosses the line $(0, \infty)$, its image is not given by only one single valued function.

The proof of the first part of the lemma follows from Lemma 2.2. Suppose $f(a(0)) \neq f(a(t))$ for some t, then $f(a(0)) = f_1(a(0))$ and $f(a(t)) = f_2(a(t))$ or $f(a(0)) = f_2(a(0))$ and $f(a(t)) = f_1(a(t))$. Because of the previous Lemma, we know that this doesn't happen unless we go around 0 at least once. Consequently, for every loop $\mathcal{L}$ given by $a(t)$ in $\mathbb{C} \setminus (0, \infty)$, we can fix $f(a(0)) = f_i(a(0))$ and have $f(a(t)) = f_i(a(t))$ for all $t \in [0, 1]$ and a fixed $i = 1, 2$.

To prove the second part of the lemma, take $a_0, a_1 \in \mathbb{C} \setminus (0, \infty)$ and a curve $C_1 \subset \mathbb{C} \setminus \{0\}$ that joins them and cuts the line $(0, \infty)$. Fix $w_0 = f(a_0), w_1 = f(a_1)$. Now draw another curve C_2 joining a_0 and a_1 that doesn't cross the line $(0, \infty)$. Because of the first part of the lemma, we know the image of C_2 is given by only one single valued function. Say we fix $f(a_0) = w_0$, then defining $w_1' = f(a_1)$ by continuity along C_2 we know w_1' belongs to the same branch as w_0. Now follow the loop $C_2^{-1}C_1$, because this goes once around 0 we know its image is not a closed curve. This means $w_1 \neq w_1'$, and because w_1' and w_0 belong to the same branch, w_1 and w_0 must belong to different branches. □

See Figure 3 for a visual representation.

Definition 2.4. *We call a point a at which the set $\{z|z = f(a)\}$ does not have maximal possible cardinality, a* branchpoint *of $f(a)$.*

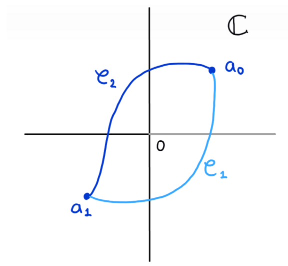

FIGURE 3. Representation of the curves used in the proof above

Consider two copies of the complex plane minus the cut from zero to infinity, we shall call these planes *sheets*. Take the function $f_1(a)$ on one sheet and $f_2(a)$ on the other. In this way, we can now view $f(a)$ as a single valued function defined not on the complex plane but on a surface consisting of two sheets.

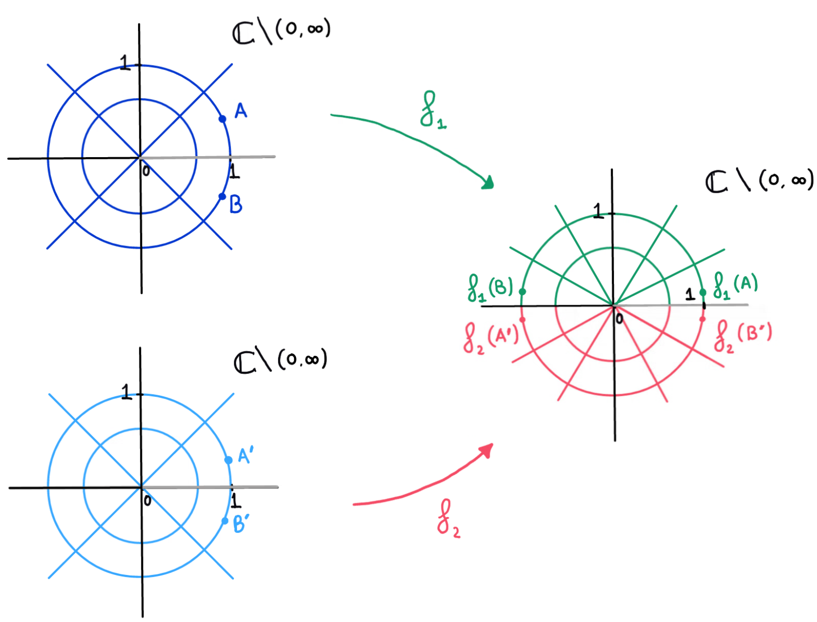

FIGURE 4. Mapping of both sheets under $f_1(a)$ and $f_2(a)$

As we can see in the previous image, although two close points on either side of the cut (eg: A, B) are mapped to points far from each other under each single valued function (eg: $f_1(A), f_1(B)$). They are mapped to points near each other when one maps one under f_1 and the other under f_2 (eg: $f_1(A), f_2(B')$). Consequently, if while traversing the cut point a moves from one sheet to the other, the single valued function defined on the surface varies continuously. To guarantee the point moves as requested, we "glue" the two sheets as follows. Along the cut from zero to infinity, we glue the side of the first sheet where $Im(a) > 0$ to the side of the second sheet where $Im(a) < 0$. Analogously, we glue the side of the first sheet where $Im(a) < 0$

to the side of the second sheet where $Im(a) > 0$, along the same cut. The resulting surface is what we shall call the Riemann surface of the function. It is represented in the following picture.

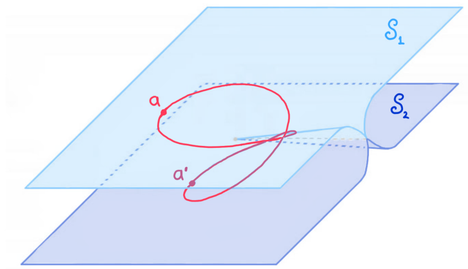

FIGURE 5. Riemann surface of $f(a) = \sqrt{a}$

Call the two sheets of the Riemann surface $\mathcal{S}_1$ and $\mathcal{S}_2$. We wish to follow a loop γ around 0 starting at $a \in \mathcal{S}_1$. To avoid confusion, call a' the point on $\mathcal{S}_2$ with same coordinates as a (we can find such a point for all a in $\mathcal{S}_1$ because both sheets are copies of $\mathbb{C} \setminus (0, \infty)$). γ must start at a, go halfway around 0 on $\mathcal{S}_1$ and meet the cut, which is glued to $\mathcal{S}_2$. γ then continues on $\mathcal{S}_2$, passing through a' and looping around 0, to come back to where it came onto $\mathcal{S}_2$. γ then gets back onto $\mathcal{S}_1$, goes halfway around zero (now on the other side) and comes back to a. γ is drawn in the previous figure. By construction, if we take the image of γ under $f(a)$, we obtain a loop (unlike in Figure 1). Notice that if we collapse the Riemann surface to obtain the complex plane, we obtain a loop that wraps twice around 0, as lemma 2.2 tells us.

In general, to build the Riemann surface of a function we must find its branchpoints, separate the single valued functions, and determine how their sheets are connected. However, the Riemann surfaces of more complicated functions are difficult to visualize. Because of this we will represent them schematically, with diagrams. For $f(a)$ for example, the diagram represented in Figure 6 tells us the Riemann surface has two single valued functions, zero as its only branchpoint, and that the two sheets are glued together at a cut going from 0 to infinity. Moreover, the arrows indicate the direction of passage from one sheet to the other, which in this case is simply passage in both directions. In this way, a diagram determines the Riemann surface.

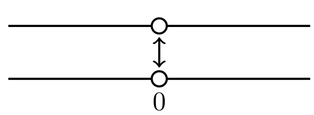

FIGURE 6. Diagram of $f(a) = \sqrt{a}$

2.2. General Method for building the Riemann surface of a function representable by radicals.

Before giving the general method to build the Riemann surfaces of a "function representable by radicals", we will introduce one last concept which is important in building the Riemann surface.

Example 2.5. *Riemann surface of* $\sqrt{a^2}$.

The single valued functions of $\sqrt{a^2}$ are $-a, +a$, the diagram sought has thus two sheets.

Notice that $\arg(a)$ varies by 2π when going around 0, so $\arg(a^2)$ varies by 4π when going around 0 and consequently, $\arg(\sqrt{a^2})$ varies again by 2π. This tells us $\sqrt{a^2}$ doesn't have any branchpoints. However, because at 0 both single valued functions return the same value, when passing through 0 we may remain on the same sheet or move onto the other one.

Definition 2.6. *Points where two distinct single valued functions return the same value, but are not branchpoints, are called* non-uniqueness points *of the given multivalued function.*

Points that are either branchpoints or non-uniqueness points are called singular points.

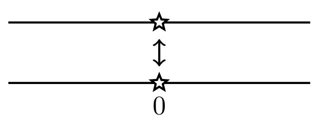

FIGURE 7. Diagram of $f(a) = \sqrt{a^2}$

The star on the diagram tells us 0 is a non-uniqueness point and that the two sheets of the Riemann surface connect at that point (and that point only).

Definition 2.7. *Let f(a) be an Algebraic Function associated to a polynomial p(a,z).*
We define the Riemann surface *M of f(a) along with the map*

$$\pi : \{(a,z)|p(a,z) = 0\} \longrightarrow \{a\},$$

as a covering space of $\mathbb{C} \setminus \{$*singular points of f(a)*$\}$.

An important family of Algebraic functions are those that are representable by radicals.

Definition 2.8. *We say a function h(a) is* representable by radicals *if it can be written in terms of the function id(a)=a and of constant functions by means of the the field operations and extraction of a root of integer order.*

Although we will not do it in this paper, it can be proven that for every function representable by radicals one can build a Riemann surface, along which the function is continuous and single valued.

Knowing this, we will now give a general method to build the Riemann surface of any function representable by radicals.

Construction 2.9. *Riemann surface of* $h(a) = \sqrt[n]{(a-a_0)^{i_0}...(a-a_m)^{i_m}}$ *for given* $n, m \in \mathbb{N}$ *and* $i_0, ..., i_m \in \mathbb{Z}$.

First notice that $h(a)$ is n-valued. Now notice that the $a_j's$ will all be either branchpoints or non-uniqueness points, because at those points the n distinct single valued functions return the same value. To distinguish the branchpoints look at $\mathrm{var}(h(\mathcal{L}))$, where $\mathcal{L}$ is a small loop given by $a(t)$ that wraps once around the point in question. If $\mathrm{var}(h(\mathcal{L})) \neq 2\pi \cdot k$, we have $h(a(0)) \neq h(a(1))$, and so the point is a branchpoint; if not it is a non-uniqueness point. Finally, label each single valued function h_i as $\epsilon_n^{i-1} \cdot h_1$, where h_1 is an arbitrarily chosen single valued function and $\epsilon_n = \exp\{\frac{2\pi}{n}\}$. Use this notation to determine how the sheets are connected.

Construction 2.10. *Riemann surface of h(a)=f(a)* $\bullet$ *g (a), where f(a), g(a) are functions with known Riemann surfaces and the operation* $\bullet$ *is a field operation.*

Let $\{f_1, ..., f_n\}$ be the branches of $f(a)$ and $\{g_1, ..., g_m\}$ be the branches of $g(a)$. The branches of h(a) are then: $\{f_1 \bullet g_1, ..., f_1 \bullet g_m, ..., f_n \bullet g_1, ..., f_n \bullet g_m\}$. Observe that on the sheet corresponding to $f_1 \bullet g_1$, one must go around a branchpoint or non-uniqueness point of $f(a)$ that connects f_1 with f_2 to go onto the sheet $f_2 \bullet g_1$; and around a branchpoint or non-uniqueness point of $g(a)$ to go onto the sheet $f_1 \bullet g_2$. Because of this, the branchpoints and non-uniqueness points of $h(a)$ are those of $f(a)$ and those of $g(a)$. Additionally, to connect the sheets of $h(a)$ it suffices to look at how the sheets of $f(a)$ and $g(a)$ are connected.

An important observation is that it is sometimes necessary to "merge" two sheets, for sometimes not all $n \cdot m$ single valued functions are distinct. For example $h(a) = \sqrt{a} + \sqrt[4]{a^2}$ has 7 single valued functions not 8.

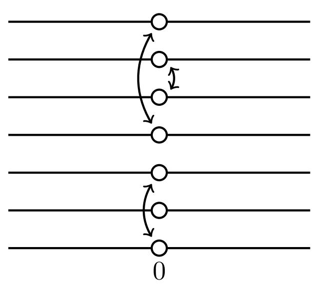

FIGURE 8. Diagram of $h(a) = \sqrt{a} + \sqrt[4]{a^2}$

Remark 2.11. Algebraic functions can be defined continuously as single valued functions on more than one Riemann surface. So the reader should not be discouraged if when drawing the diagram of a function, it does not correspond to the one represented in the paper.

Construction 2.12. *Riemann surface of* $h(a) = f(a)^n$ *where* $f(a)$ *is a function whose Riemann surface we know.*

It follows from the previous construction that the branchpoints and non-uniqueness points of $h(a)$ are those of $f(a)$. Say f is an m-valued function, with single valued functions $\{f_1, ..., f_m\}$. Then $h(a)$ has at most m single valued functions, $\{f_1^n, ..., f_m^n\}$. And their sheets connect just as the sheets of the multivalued function $f(a)$ connect.

We say at most because as in the previous constructions, some of these functions might not be distinct, and so to build the correct Riemann surface, we must "merge" their sheets.

Construction 2.13. *Riemann surface of* $h(a) = \sqrt[n]{f(a)}$ *where f(a)is a function whose Riemann surface we know.*

Observe that the possible branchpoints of $h(a)$ are the branchpoints and non-uniqueness points of $f(a)$. Similarly, the non-uniqueness points of $h(a)$ are the remaining non-uniqueness points of $f(a)$. Let $\{f_1, ..., f_m\}$ be the branches of $f(a)$, $g(a)$ a single valued continuous branch of $\sqrt[n]{a}$ and set $\epsilon_n = \exp\{\frac{2\pi}{n}\}$. Consequently, the single valued functions of $\sqrt[n]{a}$ are $\{g, \epsilon_n \cdot g(f_1), ..., \epsilon_n^{n-1} \cdot g(f_1)\}$; and the single valued functions of $h(a)$ are

$$\{g(f_1), ..., g(f_m), \epsilon_n \cdot g(f_1), ..., \epsilon_n \cdot g(f_m), ..., \epsilon_n^{n-1} \cdot g(f_1), ..., \epsilon_n^{n-1} \cdot g(f_m)\}.$$

We can thus say that to every branch of $f(a)$ there corresponds a "bunch" of n branches of the function $h(a)$. Let a_0 be a branchpoint of $f(a)$ and suppose that going once around it, one moves from the branch $f_i(a) \to f_j(a)$. Thus for the function $h(a)$, when going around a_0 one moves from all the branches of the "bunch" corresponding to $f_i(a)$ to all branches of the bunch which corresponds to $f_j(a)$. On the other hand, when moving around a non- uniqueness point of $f(a)$ that is a branchpoint for $h(a)$, we move from a certain branch $\epsilon_n^k \cdot g(f_i) \to \epsilon_n^{k+1} \cdot g(f_i)$ for every $i = 1, ..., m$. An example is given in Figure 9.

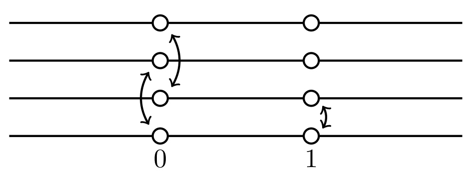

Figure 9. Diagram of $h(a) = \sqrt{\sqrt{a} - 1}$

3. Monodromy groups

3.1. The permutation group as a monodromy group.

In this section we will associate a permutation group to the Riemann surfaces. On the diagram of the Riemann surface, the arrows can be viewed as a permutation of the sheets when going around a branchpoint. Let $\{\sigma_1, ..., \sigma_n\}$ be these permutations, we call the subgroup generated by them the *permutation group of the diagram.*

Example 3.1. The permutation group of the following diagram is $\langle(12), (13)(24)\rangle$.

There is also another way we can associate a permutation group to the Riemann surface of an n-valued function $f(a)$. Take $a_0 \in \mathbb{C} \setminus \{b |\ b \text{ is a singular point }\}$. And let $\mathcal{L} \subset \mathbb{C} \setminus \{b |\ b \text{ is a singular point }\}$ be a loop given by $a(t)$ with $a(0) = a_0$. Fix $f(a(0)) = f_i$ and define by continuity along $\mathcal{L}$ the value $f_j = f(a(1))$. Observe that if we start with different values f_i we obtain different values f_j, hence there corresponds a certain permutation σ of the values $f_1, ..., f_n$ to the curve C.

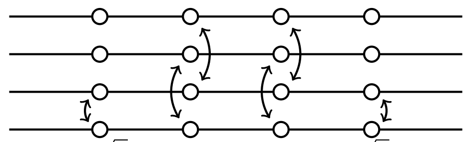

FIGURE 10. Diagram of $h(a) = \sqrt{\sqrt{a)^2+1}-1}$

We define the *group of permutations of the values of* $f(a_0)$ as the group generated by the permutations corresponding to the loops with basepoint a_0.

Lemma 3.2. *Let* G_1 *be the group of permutations of the values of* $f(a_0)$ *and* G_2 *the permutation group of a diagram of the n-valued function f(a).* G_1 *and* G_2 *are isomorphic.*

Let $f_1, ...f_n$ be the values of $f(a_0)$, and $b_1, ..., b_m$ the branchpoints. Number the sheets in the diagram so that the $i'th$ sheet corresponding to the the branch $f_i(a)$, satisfies $f_i(a_0) = f_i$. Two loops correspond to the same permutation, if and only if they wrap around the same branchpoints. On the other hand, if to the loop $\mathcal{L}$ there corresponds the permutation σ, to the loop $\mathcal{L}^{-1}$ there corresponds the permutation σ^{-1}. And if to the loops (with same basepoint) $\mathcal{L}_1, \mathcal{L}_2$ there correspond the permutations σ_1, σ_2, then to the loop $\mathcal{L}_1\mathcal{L}_2$ there corresponds the permutation $\sigma_2 \circ \sigma_1$. Hence, G_1 is generated by the permutations corresponding to the each element in $\pi_1(\mathbb{C} \setminus \{b|\ b \text{ is a singular point } \})$.

Observe that the permutation σ_i corresponding to the branchpoint b_i is the same (up to inversion) as the permutation θ_i corresponding to a loop that goes around the branchpoint b_i. Consequently, $G_1 = \langle\theta_1, ..., \theta_m\rangle = \langle\sigma_1, ..., \sigma_m\rangle = G_2$ □

It follows from the lemma that the permutation group of the values of $f(a_0)$ for all points a_0 and the permutation group of all diagrams of an algebraic function are isomorphic.

Definition 3.3. *The* permutation monodromy group of the algebraic function $f(a)$ *is the group previously defined as the permutation group of the values of* $f(a_0)$*, for some point* a_0*, or the permutation group of the diagram of f(a), for some diagram of f(a).*

3.2. **Braid groups as a monodromy group.**

For this section we will assume knowledge of the braid group. If the reader is not acquainted with this wonderful group, we recommend he refer to the tutorial on braids in *Berrick* [3].

Braids can be viewed in many ways, in this paper we will choose to view them as dances of particles through time. To look at how the braid group can act as a monodromy groups to an algebraic function, let's look again at the phenomenon we described in the introduction, but through time.

Looking at the picture above, the association of a braid group to the algebraic function $f(a) = \sqrt{a}$ becomes quite natural.

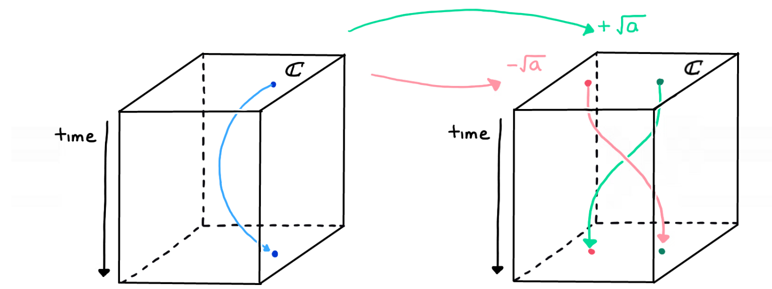

FIGURE 11. Loop that goes around 0 and its two images under $\sqrt{a}$ viewed through time

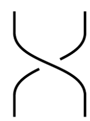

FIGURE 12. The braid associated to a turn around the branchpoint of $f(a) = \sqrt{a}$ is σ_1

Definition 3.4. *The* braid monodromy group *of the algebraic function $f(a)$ is the group generated by the images under $f(a)$ of each element in*

$$\pi_1(\mathbb{C} \setminus \{b|\ b\ is\ a\ branchpoint\ of\ \ f(a)\})$$

viewed through time.

Recall that there exists a natural surjective homorphism between the Braid Group and Permutation group. We shall cal this homorphism Φ. By restricting this homorphism to the braid monodromy group, we can thus define a surjective homomorphism onto the permutation monodromy group.
Let$f(a)$ be an algebraic function, with single valued functions $f_1(a), ..., f_n(a)$. We define the surjective homomorphism ϕ which sends braids σ, from the braid monodromy group MBn, into permutations $\phi(\sigma)$ of the values of $f(a_0)$, in the permutation monodromy group MSn.

$$\phi : MB_n \longrightarrow MS_n$$

$$\sigma \longmapsto \phi(\sigma) : f_i(a_0) \mapsto f_{\Phi(\sigma(i))}(a_0)$$

This homomorphism gives us insight into how the braid monodromy group gives more information than the permutation monodromy group. Let's look at an example of two functions with same permutation monodromy group but different braid monodromy groups.

From the previous chapters, we know the permutation monodromy group of $\sqrt{a}$ is $\langle(12)\rangle = S_2$. Additionally, we just computed its braid monodromy group $\langle\sigma_1\rangle = B_2$.

Let's now look at the monodromy groups of $\sqrt{a}^3$. As is apparent in Figure 13, a turn around the branchpoint of the 2-valued function $\sqrt{a}^3$, switches the two single valued functions. Consequently the permutation monodromy group associated to

$\sqrt{a}^3$ is also $\langle(12)\rangle = S_2$.

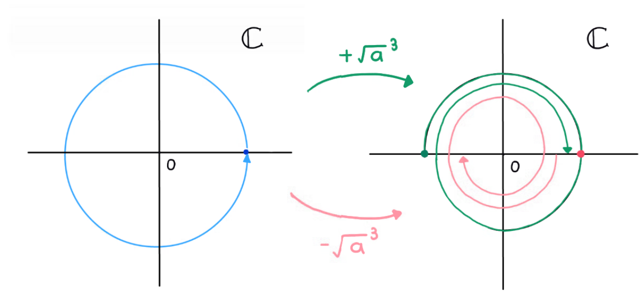

FIGURE 13. Loop around 0 and its two images under $\sqrt{a}^3$ (the two images of the loop were deformed so as not to overlap)

However looking at the images of the loop around zero through time (Figure 14), we notice the braid associated to the branchpoint is not σ_1 but σ_1^3, consequently the braid monodromy group is $\langle\sigma_1^3\rangle \leq B_2$.

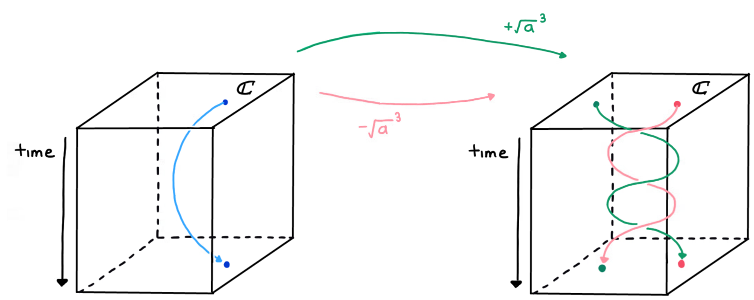

FIGURE 14. Loop that goes around 0 and its two images under $\sqrt{a}^3$ viewed through time

4. Applications of monodromy groups

4.1. **The formulas to solve cubic and quartic equations.**

By looking at the Riemann surface of the associated algebraic functions of a cubic or quartic polynomial, we can get a better understanding of how the roots of these polynomials look.

Set the cubic equation

$$p(z) = z^3 + \alpha z^2 + \beta z + a = 0 \tag{4.1}$$

and $f(a)$ its associated algebraic function. The branchpoints of $f(a)$, are a subset of the points a where $f(a)$ admits less than three values, ie: the points a such that $p(z)$ has repeated roots. $p(z)$ admits repeated roots if it and $p'(z)$ have roots in common. $p'(z)$ admits two distinct roots z_1, z_2 if $\alpha^2 - 3\beta \neq 0$ and a repeated root z_0 if $\alpha^2 - 3\beta = 0$. The case for $\alpha^2 - 3\beta = 0$ $f(a)$, is the degenerate case and we shall not study it.

In the case of $\alpha^2 - 3\beta \neq 0$, $f(a)$ admits two values at

$$a = a_1 := -(z_1^3 + \alpha z_1^2 + \beta z_1) \ and \ a = a_2 := -(z_2^3 + \alpha z_2^2 + \beta z_2)$$

We will now prove that we can construct a Riemann surface with branchpoints a_1 and a_2 where f(a) is continuously and singlevaluedly defined.

Observe first that for any curve $C \in \mathbb{C} \setminus \{a_1, a_2\}$ f(a) is uniquely defined by continuity along C. Hence $f(a)$ will be continuously and singlevaluedly defined on its Riemann surface. Suppose a_1, a_2 are both branchpoints, then both at a_1 and a_2 two sheets must meet (this can be proven by taking small disks around a_1 and a_2 and looking at the roots given by $f(a)$ on the disks). Observe finally that all sheets of the Riemann surface must be connected, hence because a_1, a_2 can connect only two sheets, they must both be branchpoints. We can thus represent the diagram as pictured in Figure 15.

Figure 15. Diagram of $f(a)$

The permutation monodromy group associated to p(z) is thus $\langle (12), (23) \rangle = S_3$.

Following a similar procedure we find that

$$p(z) = z^4 + \alpha z^3 + \beta z^2 + \theta z + a = 0 \tag{4.2}$$

admits the Riemann surfaces represented in Figures 16 and 17(ignoring the degenerate case).

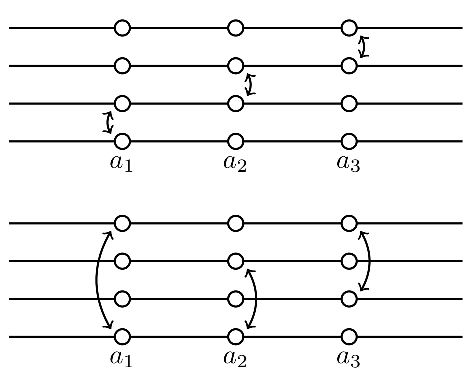

Figure 16. Diagrams associated to $p(z)$ if $\Delta_3(p') \neq 0$

In the three cases, the permutation monodromy group associated to the diagram is S_4.

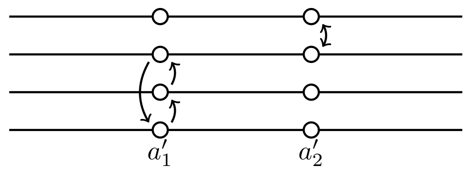

FIGURE 17. Diagram associated to $p(z)$ if $\Delta_3(p') = 0$ and $3\alpha^2 - 8\beta \neq 0$

Notice that both S_3 and S_4 are not abelian. We will now prove that any algebraic function that does not require nesting of roots is abelian. Hence the monodromy groups of the cubic and quartic polynomial give us insight into the form of the roots.

Lemma 4.3. *Let $f(a), g(a)$ be two algebraic functions with abelian permutation monodromy groups F and G. Then $h(a) = f(a) \bullet g(a)$, where the operation $\bullet$ is a field operation, has an abelian permutation monodromy group.*

To prove this lemma, observe first that the monodromy group H_1, associated to the Riemann surface of $h(a)$ before merging equal sheets, is isomorphic to a subgroup of $F \times G$. Observe now that there exists a surjective homomorphism from H_1 to H_2 where H_2 is the monodromy group associated to the Riemann surface with merged sheets. Hence because F and G are abelian, $F \times G$ is abelian and so are H_1 and H_2. □

Notice now that the permutation monodromy group of an algebraic function of the form $h(a) = \sqrt[n]{(a - a_0)^{i_0} ... (a - a_m)^{i_m}}$ is always abelian; for to each branchpoint, there corresponds a permutation of all the sheets of the form

$$h_1 \to h_2 \to ... \to h_n \to h_1$$

ie: all the branchpoints are associated to the same permutation and the monodromy group is cyclic.

With this, we conclude that a non-abelian permutation monodromy group, implies the Algebraic function involves nesting of roots. Hence the roots of equations (4.1) and (4.2) involve nesting of roots.

4.2. **The Abel-Ruffini Theorem.**

In this section we will outline a proof of the Abel-Ruffini theorem given by Arnold in the sixties to his High School students in Moscow.

Lemma 4.4. *If a multivalued function h(x) is representable by radicals, its permutation monodromy group is solvable.*

The proof of this lemma is done in several steps.

(1) Observe the identity and constant functions have trivial permutation monodromy groups, and so solvable monodromy groups.

(2) Prove that given two functions $f(a), g(a)$ with solvable permutation monodromy groups, the monodromy groups of $f(a) + g(a), f(a) * g(a), f(a)^n$ are also solvable. The proof of this is very similar to the one given for lemma 4.1, so we will skip it.
(3) Prove that given $f(a)$ with solvable permutation monodromy groups, the monodromy group of $\sqrt[n]{f(a)}$ is also solvable.

Let F and H be the respective monodromy groups of $f(a)$ and $\sqrt[n]{f(a)}$. Recall that to every sheet of the Riemann surface of $f(a)$ there corresponds a pack of n sheets in the Riemann surface of $h(a)$. Moreover when going around a branchpoint of $h(a)$ one moves from all sheets of one pack to all sheets of another pack. Hence the packs are preserved under the permutations of H, and to each permutation $\sigma \in H$ there corresponds a permutation σ' of the packs. We define

$$\Gamma : H \longrightarrow \text{group generated by the permutations of the packs.}$$

Observe Γ is a surjective homomorphism, and that $\text{Im}(\Gamma) \cong F$. $\text{Ker}(\Gamma)$ corresponds to those permutations in H which transform each pack into itself, that is they displace all the sheets in the pack by a fixed number i. Hence for any two permutations $\sigma_1, \sigma_2 \in \text{Ker}(\Gamma)$, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, ie: $\text{Ker}(\Gamma)$ is abelian.

Using the fundamental theorem of group homomorphisms we have

$$H/\text{Ker}(\Gamma) \cong F$$

And because $\text{Ker}(\Gamma)$ is abelian and F is solvable by hypothesis, we prove that the group H is solvable. □

We can now tackle the Abel-Ruffini theorem.

Theorem 4.5. ***(Abel-Ruffinni)*** *For $n \geq 5$ the general algebraic equation of degree n*

$$a_0 z^n + a_1 z^{n-1} + ... + a_n = 0$$

is not solvable by radicals.

To prove the theorem, it remains only to show that there exists a polynomial of degree five whose associated algebraic function has a non-solvable monodromy group.

Using the method outlined in the previous section, one can show that the function expressing the roots of

$$3z^5 - 25z^3 + 60z - a = 0$$

has S_5 as its monodromy group, and is thus the function we were looking for.

For, because of the previous lemma, we know it can't be representable by radicals and thus we cannot find a general method to solve the equation of degree 5 or higher by radicals. □

4.3. **Braid group applications.**

Because the Braid group is more complicated to study than the permutation group, the applications of the braid monodromy group are not as simple as the applications previously mentioned. Because of this we will not go into detail about these applications in this paper, although some good references to check might be the papers by *Arnold* [4] and *McMullen*[5]. Additionally, the reader should keep in mind, that the potential of the braid monodromy group is unlikely to have been reached yet, and that many interesting results are yet to come from it.

Acknowledgments

First I would like to thank Professor Peter May for organizing the Chicago Math REU, which has been a very enlightening experience for me. I would also like to thank Professor Benson Farb and Jesse Wolfson for organizing weekly meetings during the REU, and for guiding me with my project. Finally I would like to thank the graduate students at UChicago that helped me out, especially Claudio Gonzalez and my mentor Ronno Das for helping me understand the subjects that I couldn't grasp.

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