The representability hierarchy and Hilbert's 13th problem

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Abstract. We propose the *representability hierarchy* of algebraic functions over $\mathbb{C}$, which relates classic questions like the unsolvability of the quintic by radicals to unresolved questions like Hilbert's 13th problem in its original algebraic form. We construct the theory of algebraic functions via both superpositions and algebraic geometry and then justify the known positions of the universal algebraic functions ρ_n in the hierarchy. Relationships to algebraic topology, Galois theory, birational geometry, and prior literature on representability are also discussed.

Contents

1. Introduction

Notation 1.1. Let k denote an algebraically closed topological field. We write $\mathrm{Sym}^n(k) := k^n/S_n$ for the space of n-element multisets of k (it is the quotient of k^n by the non-free permutation action of the n-th symmetric group).

We begin by defining our central object of study:

Definition 1.2. An **algebraic function** of m variables and n values (over k) is a partial map of the form

$$f : k^m \dashrightarrow \mathrm{Sym}^n(k), \quad \vec{a} = (a_1, \dots, a_m) \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ z^n + c_1(\vec{a})z^{n-1} + \cdots + c_n(\vec{a}) \end{matrix} \right\}$$

for some rational functions $c_i(\vec{x}) \in k(x_1, \dots, x_m)$. Note that f is a continuous map on a dense open subset of k^m (namely, the complement of the zeroes of the denominators of c_i).

Algebraic functions are a special type of multi-valued map, a notion we will define later (Definition 2.2). For now, our definition is best motivated by some familiar examples:

Example 1.3. The **radical functions** $\sqrt[n]{-}$ are the algebraic functions of 1 variable and n values given by

$$\sqrt[n]{-} : k \to \mathrm{Sym}^n(k), \quad a \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ z^n - a \end{matrix} \right\}.$$

Example 1.4. The "quadratic solver" ρ_2 is the algebraic function of 2 variables and 2 values given by

$$\rho_2 : k^2 \to \mathrm{Sym}^2(k), \quad (a_1, a_2) \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ z^2 + a_1 z + a_2 \end{matrix} \right\}.$$

Thus, the quadratic solver maps the coefficients of a quadratic to its two solutions.

Remark. Our definition of algebraic function (Definition 1.2) and later treatment (Section 2.1) generalizes that of Arnold [Arn70b], who only worked with entire algebraic functions over $\mathbb{C}$ (where the c_i are polynomials and $k = \mathbb{C}$). Lin might view our definition as that for (partial) **algebroidal functions** [Lin76]. Other authors view algebraic functions as *individual* roots of the polynomial $z^n + c_1(\vec{x})z^{n-1} + \cdots + c_n(\vec{x})$ and hence as formal elements of algebraic closures $\overline{k(x_1, \ldots, x_m)}$ or $\overline{k[x_1, \ldots, x_m]}$ ([Vit04], [Żoł00]). We can reconcile these viewpoints once we derive a Galois-theoretic formulation of representability in Section 2.

1.1. Universal algebraic functions

Example 1.4 is an instance of a special class of algebraic functions:

Definition 1.5. The **universal algebraic functions** ρ_n are the algebraic functions of n variables and n values given by

$$\rho_n : k^n \to \mathrm{Sym}^n(k), \quad (a_1, \ldots, a_n) \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ z^n + a_1 z^{n-1} + \cdots + a_n \end{matrix} \right\}.$$

Thus, ρ_n is the algebraic function that maps the coefficients of an n-th degree monic polynomial to the multiset of its roots.

Other than their natural characterization, universal algebraic functions are special since all n-valued algebraic functions are equivalent to ρ_n up to a rational map (Proposition 2.4). However, unlike rational maps, algebraic functions like ρ_n cannot always be written down in a nicely explicit way:

Example 1.6. Over $\mathbb{C}$, one might write the quadratic solver ρ_2 as the familiar **quadratic formula**:

$$\rho_2(a_1, a_2) = \frac{-a_1 \pm \sqrt[2]{a_1^2 - 4a_2}}{2}.$$

However, ρ_2 is *not* the quadratic formula itself. Instead, ρ_2 is an abstractly defined algebraic function that happens to be "representable" in terms of field operations (addition, division, etc.) and a square root, over the field $\mathbb{C}$. For example, ρ_2 is still well-defined in algebraically closed fields of characteristic 2, while the quadratic formula would not be.

Example 1.7. Cardano's historical solution to the cubic [Tig01] involves two steps. First, he notes that if we are given a polynomial

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0$$

we can take $w = z - \frac{a_1}{3}$ to get

$$w^3 + 3b_1 w - 2b_2 = 0, \quad \text{with} \quad b_1 = \frac{a_2}{3} - \frac{a_1^2}{9}, \quad b_2 = \frac{a_1 a_2}{6} - \frac{a_3}{2} - \frac{a_1^3}{27}$$

Let ν_3 be the algebraic function that takes b_1, b_2 to the roots of $w^3 + 3b_1 w - 2b_2$. Then

$$\rho_3(a_1, a_2, a_3) = \nu_3(b_1(a_1, a_2, a_3), b_2(a_1, a_2, a_3)) + \frac{a_1}{3}.$$

Thus, it suffices to give an explicit expression for ν_3. Cardano gives

$$\nu_3(b_1, b_2) = \sqrt[3]{b_2 + \sqrt[2]{b_1^3 + b_2^2}} + \sqrt[3]{b_2 - \sqrt[2]{b_1^3 + b_2^2}}.$$

Like with ρ_2 and the quadratic formula, it happens that ρ_3 is well-defined in any algebraically closed field, whereas this **cubic formula** for ν_3 and thus for ρ_3 is not valid in characteristics 2 and 3 (due to the divisions in the polynomials for b_1, b_2).

There are also formal considerations that arise with Cardano's solution that cannot easily be glossed over as they were with the quadratic formula. For example, if $\sqrt[2]{-}$ and $\sqrt[3]{-}$ are algebraic functions and thus multi-valued, what does it mean to take specific roots as values for $\sqrt[2]{b_1^3 + b_2^2}$ and $\sqrt[3]{\cdots \pm \sqrt[2]{\cdots}}$? The interpretation of Cardano's expression involves fixing a square root of $b_3^3 + b_2^2$ and then running through the cube roots of each of

$$b_2 + \sqrt[2]{b_1^3 + b_2^2}, \quad b_2 - \sqrt[2]{b_1^3 + b_2^2}.$$

This gives the desired 3 solutions for z, although with the repetition inherent in making $2 \cdot 3 \cdot 3 = 18$ choices. We will later say that even over $\mathbb{C}$, the quadratic formula faithfully represents ρ_2, while the cubic formula only (classically) represents ρ_3. This terminology is due to Lin [Lin76] and is formalized in Definition 2.7.

1.2. Hilbert's 13th problem

From our examples, it appears that both ρ_2 and ρ_3 can be easily written using only field operations and the radical functions, as long as we work in $\mathbb{C}$. To capture this intuitive notion of "being able to write down", we say that:

Definition 1.8 (**informal**)**.** An algebraic function is **representable** by a set of algebraic functions $\mathcal{S}$ if it can be written as the composition of elements of $\mathcal{S}$ and field operations, up to superfluous roots.

In our case, our algebraic functions are ρ_2 and ρ_3, which are representable over $\mathbb{C}$ by the radical functions $\mathcal{S} = \{\sqrt[n]{-}\}$. This definition is currently informal since characterizing both compositions of multi-valued maps like algebraic functions and the existence of superfluous roots requires some consideration (see Definition 2.7). However, for now this suffices in allowing us to formulate two questions of historical interest:

Question 1.9 (Gauss)**.** Can the general quintic be solved in radicals? That is, can the algebraic function ρ_5 be represented by the radical functions $\{\sqrt[n]{-}\}$? This is known as the **unsolvability of the quintic**.

Question 1.10 (Hilbert)**.** Can the solutions z to

$$z^7 + a_1 z^3 + a_2 z^2 + a_3 z + 1 = 0,$$

viewed as a function of the three variables a_1, a_2, a_3, be written as the composition of two-variable functions? That is, can the algebraic function ρ_7 be represented by algebraic functions of 2 variables or fewer? This is known as **(the algebraic) Hilbert's 13th problem**.

Question 1.9 was first formulated by Gauss in his *Disquisitiones Arithmeticae* [Gau01] and answered in the negative by the celebrated **Abel-Ruffini theorem** proved in 1824 by classical means, and later reproved by Galois using techniques of what we now call Galois theory [Tig01].

Meanwhile, Question 1.10 was first described in Hilbert's address to the ICM in the 1900s, then published as the 13th problem in his famous list of twenty-three problems [Hil02]. Its intended formulation as an algebraic problem was clarified in his later writings [Hil27]. At the time of this paper's writing, this problem remains open.

Remark. We include the prefix *algebraic* as some versions of Question 1.10 are solved, namely the continuous and faithful variants (see the end of Section 1.3). Furthermore, the equivalence of representing the algebraic function for $z^7 + a_1 z^3 + a_2 z^2 + a_3 z + 1$ to that of representing the algebraic function ρ_7 is not obvious, and is due to Tschirnhaus transformations (Section 3.1). These generalize Example 1.7's passage from ρ_3 to ν_3, where ν_3's formula in radicals must also give one for ρ_3, and vice versa.

1.3. The representability hierarchy

Our discussion encourages us to depict what we shall call the **representability hierarchy** of algebraic functions over $\mathbb{C}$. In a sense, the objective of this paper is to justify Figure 1.

The middle area depicts the universe of algebraic functions over $\mathbb{C}$. It is partitioned into a number of levels. The left-hand side of Figure 1 indicates which algebraic functions suffice to represent elements of that level. The right-hand side indicates the classifying invariant associated to that level; we will discuss monodromy in Section 2.

We have already seen that ρ_2, ρ_3 are representable using $\sqrt[2]{-}$ and $\{\sqrt[2]{-}, \sqrt[3]{-}\}$ respectively. We will see that the same holds for ρ_4 (since intuitively, $\sqrt[4]{-} = \sqrt[2]{\sqrt[2]{-}}$). Abel-Ruffini's theorem is equivalent to the statement that ρ_5 is not representable by the radicals $\{\sqrt[n]{-}\}$. In fact, we will see that ρ_5 is representable by the radicals, as long as one includes the Bring radical BR in the representing set $\mathcal{S}$. All this will occur in Section 3.

The radicals and the Bring radical are all algebraic functions of 1 variable, and in general, Tschirnhaus transformations show that for $n \geq 5$, ρ_n is representable by algebraic functions of at most $n-4$ variables. Furthermore, Hilbert and Wiman showed that for $n \geq 9$, ρ_n is representable in at most $(n-5)$-variable algebraic functions (hence the identical ranges for ρ_8 and ρ_9). These two results give the

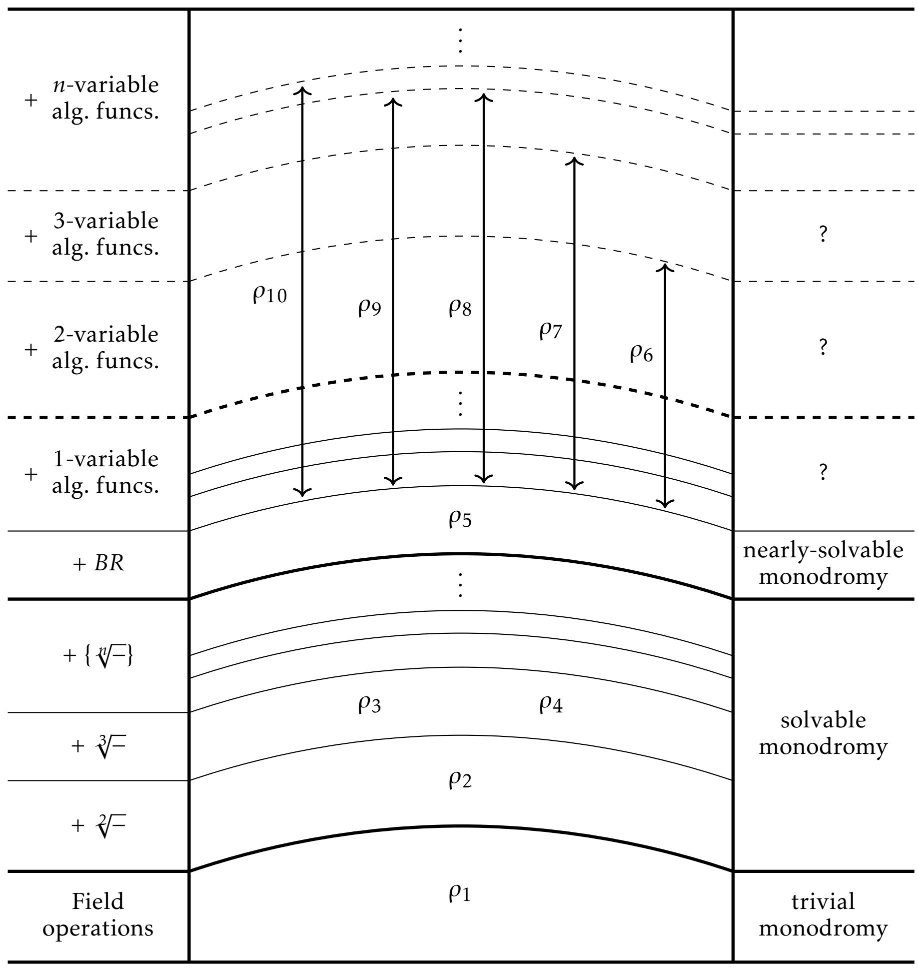


FIGURE 1. The representability hierarchy of algebraic functions over $\mathbb{C}$, with known placements of ρ_n and associated invariants.

visible arrows in Figure 1. There are further results of this type, but they are always upper bounds (on the number of variables used by the algebraic functions required to represent ρ_n), not lower bounds [Bra75].

Although the representability of ρ_7 by algebraic functions of 2 variables is the classic Hilbert's 13th problem, the reality is that Hilbert's question is only a glimpse into our general lack of understanding about the hierarchy in Figure 1. Hilbert realized this, leading him to phrase the analogous conjecture for ρ_6 in [Hil27], and others to phrase the general problem for ρ_n in a Galois-theoretic manner (see Section 2.4).

Our puzzlement regarding the hierarchy is compounded by the following observations over $\mathbb{C}$:

- The dashed lines allow for the possibility that *all* n-variable algebraic functions are representable by 1-variable algebraic functions.

- This is not implausible; the **Kolmogorov-Arnold representation theorem** is the statement that n-variable *continuous* functions are representable by 1-variable continuous functions (and the field operation of addition) [Vit04].
- The hierarchy involves classical representability (which allows for superfluous roots like in the cubic formula), but what about faithful representability? Arnold and Lin showed that here, the dashed lines are in fact solid, with ρ_n faithfully representable by $(n-1)$-variable algebraic functions (and no better). This distinction may seem superficial, but at its core is the vast divide between birational and isomorphic geometry. See Section 2.4 for further discussion.

Ultimately, we are wondering *how rich the hierarchy of algebraic functions is*. To this end, in Section 2 this paper formalizes the notion of representability and establishes several viewpoints regarding the study of algebraic functions. We use these in Section 3 to establish the positions of ρ_2 through ρ_5 as depicted in Figure 1, in a way that takes a modern perspective on Tschirnhaus transformations and proofs of the Abel-Ruffini theorem.

2. Theory of algebraic functions

2.1. Multi-valued maps and graphs

When working with continuous functions between topological spaces, we are bound by the definitional constraint that a function $f : X \to Y$ is a map giving an output of one element $f(x) \in Y$ for each $x \in X$. Is there a natural generalization where f gives n elements in Y for each $x \in X$?

Example 2.1. Consider the cube root function which takes

$$\sqrt[3]{-} : x \mapsto \{r, \omega r, \omega^2 r\}$$

where $\omega = e^{2\pi i/3}$ and r is a fixed choice of cube root. We want to interpret this as a continuous "3-valued function" (intuitively, small changes in x produce small changes in its cube roots). Note that $\sqrt[3]{-}$'s outputs are all the same when $x = 0$.

We could simply consider functions of the form $X \to Y^n$. However, this has two caveats: first, the n elements might not be distinct; second, such a map imposes an order on the n elements. The first is not necessarily a problem, as it might be useful to encode the notion of *multiplicity* in our codomain. The second can be resolved by instead mapping to $\mathrm{Sym}^n(Y) = Y^n/S_n$, which makes equivalent the ordered tuples that are permutations of each other.

Definition 2.2. An **(entire)** n**-valued map** $f : X \to_n Y$ is a continuous map $f : X \to \mathrm{Sym}^n(Y)$.

Definition 2.3. A **dominant** n**-valued map** $f : X \dashrightarrow_n Y$ is an n-valued map $f : U \to_n \mathrm{Sym}^n(Y)$ where U is a dense open subset of X.

Remark. There are further generalizations: for example, the complex logarithm could be viewed as a dominant "∞-valued map"

$$\log : \mathbb{C} \dashrightarrow_\infty \mathbb{C}, \quad \text{(i.e., } \mathbb{C}\backslash\{0\} \to \mathrm{Sym}^\infty(\mathbb{C})).$$

The construction of complex manifolds that encode continuous multi-valued maps from $\mathbb{C}$ to $\mathbb{C}$ is explored by the study of **Riemann surfaces**.

We see that an algebraic function of n values is a special type of dominant n-valued map that can be defined algebraically. In fact, we can reconcile our original definition (Definition 1.2) with multi-valued maps in an natural way:

Proposition 2.4. *The algebraic functions of m variables and n values (over k) are exactly the dominant n-valued maps*

$$f : k^m \dashrightarrow_n k$$

that factor as $f = \rho_n \circ c$ where ρ_n is the n-th universal algebraic function and where $c : k^m \dashrightarrow k^n$ is a rational map defined over k. That is, c is of the form

$$c(x_1,\dots,x_m) = (c_1(x_1,\dots,x_m),\dots,c_n(x_1,\dots,x_m))$$

where c_i are rational functions with coefficients in k, i.e., $c_i \in k(x_1,\dots,x_m)$. As before, f is defined and continuous on the complement of the zeros of the denominators of the c_i.

The main sticking point in formally defining representability (recall that Definition 1.8 was informal, due to ambiguities highlighted in Cardano's solution in Example 1.7) is defining composition when the codomains are $\mathrm{Sym}^n(k)$ instead of k or k^n. Happily, things work out nicely by replacing composition with the notion of superposition for multi-valued maps:

Definition 2.5. The **superposition** of two dominant multi-valued maps (or informally, their **composition**)

$$g_1 : X \dashrightarrow_{n_1} Y, \quad (\text{i.e., } X \to \mathrm{Sym}^{n_1}(Y))$$
$$g_2 : Y \dashrightarrow_{n_2} Z, \quad (\text{i.e., } Y \to \mathrm{Sym}^{n_2}(Z))$$

is given by

$$g_2 \circ g_1 : X \dashrightarrow_{n_1 n_2} Z, \quad g_2 \circ g_1 := \pi_{n_1,n_2} \circ \tilde{g}_2 \circ g_1$$

where

$$\tilde{g}_2 : \mathrm{Sym}^{n_1}(Y) \dashrightarrow \mathrm{Sym}^{n_2}(\mathrm{Sym}^{n_1}(Z)), \quad \{y_1,\dots,y_{n_1}\} \mapsto \{g_2(y_1),\dots,g_2(y_{n_1})\}$$
$$\pi_{n_1,n_2} : \mathrm{Sym}^{n_2}(\mathrm{Sym}^{n_1}(Z)) \to \mathrm{Sym}^{n_1 n_2}(Z), \quad \{S_1,\dots,S_{n_2}\} \mapsto \bigcup_{i=1}^{n_2} S_i.$$

In the definition of π_{n_1,n_2}, note that the S_i are multisets of size n_1, and the union over the n_2 S_i's is in the sense of multisets. Since $\tilde{g}_2$ and π_{n_1,n_2} are continuous, and since finite intersections of open dense sets are dense, then $g_2 \circ g_1$ is also a dominant $n_1 n_2$-valued map.

Intuitively, superposition takes the n_1 outputs of the first multi-valued map g_1, applies g_2 elementwise to get n_2 outputs for each n_1, then normalizes so that the $n_1 n_2$ outputs live in the correct space ($\mathrm{Sym}^{n_1 n_2}(Z)$).

We also need the following associated map:

Definition 2.6. The **graph map** of a multi-valued map $f : X \to_n Y$ is given by

$$\mathrm{gr}_f : X \to_n X \times Y, \quad (\text{i.e., } X \to \mathrm{Sym}^n(X \times Y)), \quad x \mapsto \{(x,y) \mid y \in f(x)\}.$$

The motivation is that in a chain of superpositions, we want later composing functions to have access not just to the output of the preceding function, but to the outputs of all functions before that.

2.2. Representability via superposition

These considerations let us define a notion of representing sequences of algebraic functions (i.e., via the superposition of their graph maps) that captures the intuitive idea behind both the quadratic formula and Cardano's solution.

Definition 2.7. Let $\mathcal{S}$ be a set of algebraic functions (over k). An algebraic function f of m variables and n values is **representable by** $\mathcal{S}$ or $\mathcal{S}$**-representable** if there exist Φ_i of the form

$$\Phi_i = \varphi_i \circ q_i : k^{m+i-1} \dashrightarrow_{n_i} k \quad \text{such that} \quad \mathrm{gr}_f \subseteq \pi \circ \mathrm{gr}_{\Phi_N} \circ \cdots \circ \mathrm{gr}_{\Phi_1}$$

holds as a multiset inclusion over each point in the (dense open) intersection of the two sides' domains. Each $q_i : k^{m+i-1} \dashrightarrow k^{\ell_i}$ is a rational map, each $\varphi_i : k^{\ell_i} \dashrightarrow_{n_i} k$ is in $\mathcal{S}$, and

$$\pi : \mathrm{Sym}^{n_1 \cdots n_N}(k^m \times k^N) \to \mathrm{Sym}^{n_1 \cdots n_N}(k^m \times k)$$

is the projection which preserves the first m variables and maps the last component to the last component. If $\subseteq$ can be made a multiset equality everywhere by some choice of $\{\Phi_i\}$, then f is **faithfully representable by** $\mathcal{S}$.

Definition 2.8. An algebraic function f is ℓ**-representable** if it is $\mathcal{S}$-representable, where $\mathcal{S}$ is taken to be the set of ℓ-variable (and fewer) algebraic functions.

The best way to understand these formal definitions is by example:

Example 2.9. Consider the algebraic function φ_4 over $\mathbb{C}$ implicitly defined by

$$\Phi_4 = \Phi_2 + \Phi_3, \quad \Phi_3 = \sqrt[3]{b_2 - \Phi_1}, \quad \Phi_2 = \sqrt[3]{b_2 + \Phi_1}, \quad \Phi_1 = \sqrt[2]{b_1^3 + b_2^2}.$$

Then Φ_4 is an algebraic 18-valued function of 2 variables b_1, b_2. In fact, our description shows that f is representable by the (1-variable) radical functions $\{\sqrt[n]{-}\}$ as follows:

$$\begin{aligned} q_1(b_1, b_2) &= b_1^3 + b_2^2, \quad \varphi_1(z) = \sqrt{z} \\ q_2(b_1, b_2, \Phi_1) &= b_2 + \Phi_1, \quad \varphi_2(z) = \sqrt[3]{z} \\ q_3(b_1, b_2, \Phi_1, \Phi_2) &= b_2 - \Phi_1, \quad \varphi_3(z) = \sqrt[3]{z} \\ q_4(b_1, b_2, \Phi_1, \Phi_2, \Phi_3) &= \Phi_2 + \Phi_3, \quad \varphi_4(z) = z. \end{aligned}$$

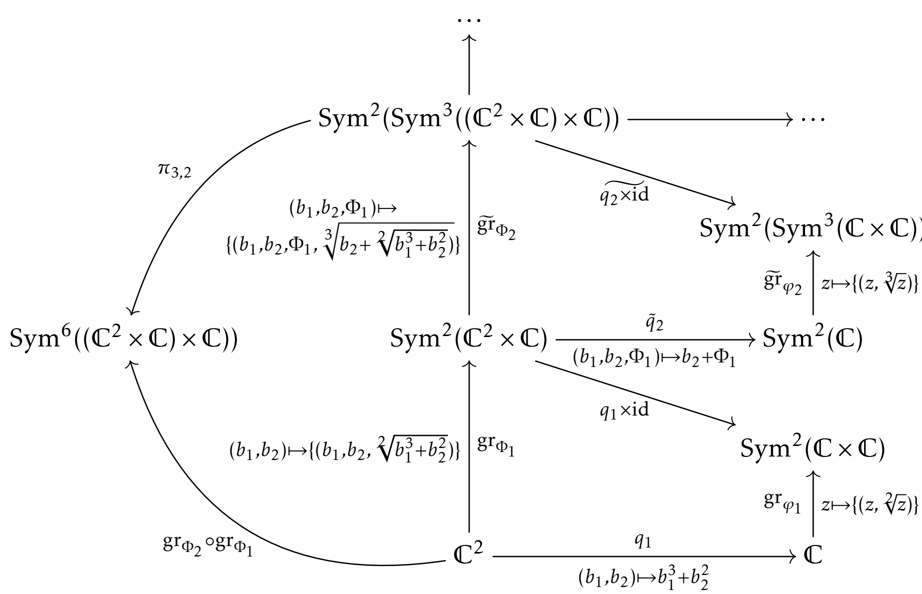
Taking q_i, φ_i, and Φ_i to be those in the definition of representability, one sees that:

$$\begin{aligned} &(\pi \circ \mathrm{gr}_{\Phi_4} \circ \mathrm{gr}_{\Phi_3} \circ \mathrm{gr}_{\Phi_2} \circ \mathrm{gr}_{\Phi_1})(b_1, b_2) \\ &= \left\{ \left(b_1, b_2, \sqrt[3]{b_2 + \sqrt[2]{b_1^3 + b_2^2}} + \sqrt[3]{b_2 - \sqrt[2]{b_1^3 + b_2^2}} \right) \right\}, \end{aligned}$$

that is, the multiset of 18 elements determined by making one of 2 choices for the value of the inner square root, and then running through the $3 \cdot 3$ choices for the outer cube roots. When viewed as a set instead of a multiset, this gives Cardano's "cubic formula" ν_3 (Example 1.7). Therefore, the 2-variable, 3-valued algebraic function ν_3 is 1-representable via radicals. However, we have not exhibited faithful representability, as our output gives six copies of each of the three roots.

Here is an instructive diagram that makes explicit the role of graph maps in the representability construction:

Example 2.10. Take q_i and φ_i as in Example 2.9, such that $\Phi_i = \varphi_i \circ q_i$. We have



where the notations are consistent with those in Definition 2.5. That is, $\pi_{3,2}$ is the projection map that collapses the nested symmetric powers, and e.g., $\tilde{q}_2 : \mathrm{Sym}^2(\mathbb{C}^2 \times \mathbb{C}) \to \mathrm{Sym}^2(\mathbb{C})$ means

$$\tilde{q}_2 : \{(w_1, w_2, w_3), (z_1, z_2, z_3)\} \mapsto \{q_2(w_1, w_2, w_3), q_2(z_1, z_2, z_3)\}.$$

Ultimately, multi-valued maps and superpositions are classical formalisms that let us pose problems of function representability in a manner closest to their original statements.

2.3. The viewpoint from algebraic geometry

We now look at equivalent perspectives that place us in more familiar fields of mathematics. A modern mathematician might note that under very specific constraints, a multi-valued map $f : X \to_n Z$ can be viewed as a continuous surjection $Z \to X$ with generically n-element fibers. For our purposes, it suffices to say that if $Z = X \times Y$, there is a natural set of correspondences:

$$f : X \dashrightarrow_n Y \qquad \longleftrightarrow \qquad \mathrm{gr}_f : X \dashrightarrow_n X \times Y \qquad \longleftrightarrow \qquad p_f : \Gamma_f \subseteq X \times Y \to X.$$

where the objects on the right-hand side are:

Definition 2.11. The **hypersurface** Γ_f of a multi-valued function f is

$$\Gamma_f := \overline{\{(x,y) \mid x \in \mathrm{dom}(f), y \in f(x)\}} \subseteq X \times Y.$$

Intuitively, this is the closure of the graph of the function (however, we cannot simply say "$\Gamma_f = \mathrm{im}(\mathrm{gr}_f)$" because gr_f's outputs are multisets).

Definition 2.12. The **branched covering map** p_f of an n-valued function f is the partial map

$$p_f : \Gamma_f \dashrightarrow X, \quad (x,y) \mapsto x,$$

where p_f is only defined when $x \in \mathrm{dom}(f)$. The fibers of p_f have at most n elements.

Something special happens when the originating multi-valued map f is an algebraic function. Here, f takes $(a_1, \dots, a_m) \in k^m$ to the multiset of solutions z of

$$\alpha_f(z) = z^n + c_1(a_1, \dots, a_m)z^{n-1} + \cdots + c_n(a_1, \dots, a_m) = 0$$

for some c_i.

Definition 2.13. We call $\alpha_f \in k(x_1, \dots, x_m)[z]$ the **associated polynomial** to the algebraic function f. Observe that we can write $\alpha_f \in k(x_1, \dots, x_m)[z]$ uniquely as a rational expression

$$\alpha_f = \frac{\beta_f(x_1, \dots, x_m, z)}{\gamma_f(x_1, \dots, x_m)}$$

where $\beta_f \in k[x_1, \dots, x_m, z]$, $\gamma_f \in k[x_1, \dots, x_m] \subseteq k[x_1, \dots, x_m, z]$, and the fraction is in lowest terms. Here, γ_f is the lowest common multiple of the denominators of the c_i.

Proposition 2.14. *The associated polynomial α_f is well-defined (i.e., the c_i are uniquely determined). In fact, we have the bijection*

$$\left\{\begin{array}{c}\textit{algebraic functions of } m\\ \textit{variables and } n \textit{ values}\\ \textit{over } k\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{monic } n\textit{-th degree}\\ \textit{polynomials with}\\ \textit{coefficients in } k(x_1, \dots, x_m)\end{array}\right\}$$

Proof. Let f be our algebraic function. Note that ρ_n is a homeomorphism (continuous correspondence between roots and coefficients of a monic polynomial) and thus invertible. We then take $c = \rho_n^{-1} \circ f$. The open set on which f is defined determines the denominator γ_f, and f's values determine β_f. Then the polynomial $\alpha_f = \beta_f/\gamma_f$ is uniquely determined, and exists by the definition of an algebraic function. Thus, $c = \rho_n^{-1} \circ f$ is uniquely expressible componentwise as rational functions on k^m. □

These observation let us view the hypersurface and branched covering maps of algebraic functions in the category of algebraic varieties and dominant rational morphisms:

Corollary 2.15. $\Gamma_f = V(\beta_f) \subseteq \mathbb{A}^{m+1}$*, where* $\mathbb{A}^{m+1}$ *has coordinates* $(a_1, \dots, a_m, z)$*.*

Corollary 2.16. $p_f : \Gamma_f \dashrightarrow \mathbb{A}^m$ *is the dominant rational morphism given by restricting the domain of the projection map* $(a_1, \dots, a_m, z) \mapsto (a_1, \dots, a_m)$ *away from* $V(\gamma_f)$*.*

With this dictionary, we can now use the additional data endowed by the algebraic perspective:

Theorem 2.17. *The following categories are dually equivalent [Sta16, Tag 0BXM]:*

$$\left\{\begin{array}{c}\textit{varieties over } k \textit{ and}\\ \textit{dominant rational}\\ \textit{morphisms}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{finitely-generated field}\\ \textit{extensions of } k \textit{ and field}\\ \textit{inclusions}\end{array}\right\}.$$

In our case, the equivalence is manifested by the dominant rational morphism

$$p_f : \Gamma_f \dashrightarrow \mathbb{A}^m$$

giving an induced inclusion of function fields

$$p_f^* : k(\mathbb{A}^m) \to k(\Gamma_f).$$

This observation lets us define the key algebraic invariant we will need to resolve the position of ρ_5 in the hierarchy:

Definition 2.18. The **monodromy group Mon(f) of an algebraic function** f is

$$\mathrm{Mon}(f) := \mathrm{Gal}(L/k(x_1, \ldots, x_m))$$

where L is the Galois closure of $k(\Gamma_f)$.

Proposition 2.19. *Let f be an n-valued algebraic function. Then $Mon(f) \subseteq S_n$.*

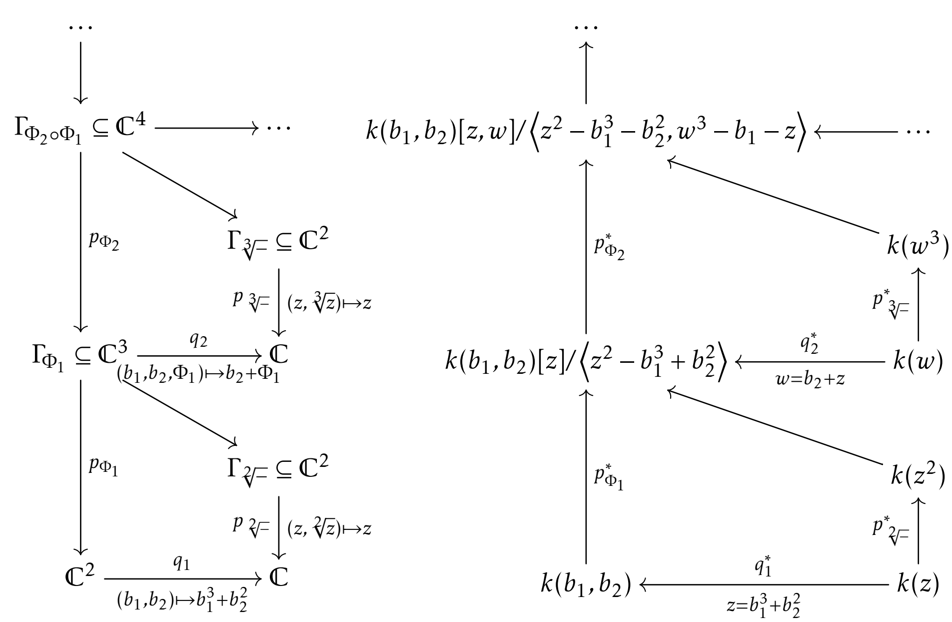
Proof. It suffices to consider the case where α_f is irreducible. Then the function field of Γ_f is given by

$$k(x_1, \ldots, x_m)[z]/\left\langle \alpha_f \right\rangle.$$

The Galois closure involves adjoining all n roots of f, and so the field automorphisms of L that fix $k(x_1, \ldots, x_m)$ are a subgroup of the permutation group on the roots of f, i.e., S_n. □

While this viewpoint abandons the perspective that equations like the quadratic and cubic formulas are functions in their own right, we gain by retrieving a more topologically natural perspective.

Example 2.20. This is the diagram from Example 2.10 adapted to the algebro-geometric perspective. By dealing with varieties and morphisms, we no longer need the extraneous passage from $\mathrm{Sym}^{n_2} \circ \mathrm{Sym}^{n_1}$ to $\mathrm{Sym}^{n_1 n_2}$. On the left, we replace the graph maps of algebraic functions with dominant rational morphisms. On the right, we replace them with function field inclusions:



These diagrams show how the perspective from algebraic geometry vindicates the graph construction all the way back from superposition (which here is now played by Γ_f and $k(\Gamma_f)$), via the following observation:

Proposition 2.21. *The squares in the diagrams of Example 2.20 are categorical pullbacks and pushouts respectively. That is, Γ_{Φ_i} is the fiber product along the dominant rational morphisms q_i and p_{φ_i}, and $k(\Gamma_{\Phi_i})$ is the compositum of fields along the field inclusions q_i^* and $p_{\varphi_i}^*$.*

Proof (sketch). The first half of the proposition is an exercise in translating gr_{Φ_i}, gr_{ϕ_i} into p_{Φ_i}, p_{ϕ_i}, constructing the pullback of p_{ϕ_i} along q_i, then showing it coincides with the map p_{Φ_i}. The second half follows from this and Theorem 2.17. □

Remark. Various technical considerations have been glossed over for simplicity of exposition. One involves the irreducibility of α_f. Another involves the implicit passage from the product topology of k^m to the Zariski topology of $\mathbb{A}^m$ when we make the assertions in Corollary 2.15 and Corollary 2.16 (for example, for $k = \mathbb{C}$ the first topology would be the strictly finer **complex analytic topology**).

2.4. Equivalences and further notions

From the diagrams of Example 2.20, we assert that:

Theorem 2.22. *The following notions are equivalent to the $\mathcal{S}$-representability of an algebraic function f:*

- *The branched covering map p_f is birationally equivalent to a branched cover produced by successive pullbacks along k-rational maps q_i and branched covers p_{φ_i} (where $\varphi_i \in \mathcal{S}$).*
- *The field $k(\Gamma_f)$ is isomorphic to the field produced by successive pushouts along function field inclusions q_i^* and $p_{\varphi_i}^*$ (where $\varphi_i \in \mathcal{S}$).*

As far as the author is aware, this unified algebro-geometric viewpoint is not explicitly found in the literature. However, each of the implicit viewpoints (superposition, covering maps, and function field extensions) does appear:

- The superposition viewpoint is the classic perspective used implicitly by Cardano, Abel, Ruffini, and Hilbert [Hil27]. Arnold makes the relevant constructions rigorous in [Arn70b].
- The covering map viewpoint was used by [Żoł00], paraphrasing lectures by Arnold [Ale04], to prove Abel-Ruffini without recourse to Galois theory. Arnold and Lin [Lin76] start with superpositions and then pass to the covering map perspective to prove results on faithful representability.
- The field-theoretic viewpoint is used by Galois in his proof of the Abel-Ruffini theorem. Brauer [Bra75] phrased Hilbert's 13th problem in terms of iterated function field extensions, along with Arnold and Shimura in their retrospective on Hilbert's problem [Bro76].

However, this does not mean other viewpoints are redundant to the algebro-geometric viewpoint. For example, the covering map viewpoint would define monodromy in a solely topological manner:

Definition 2.23 (informal). The **(topological) monodromy group $\mathbf{Mon}_T(f)$** of an m-variable, n-valued map f is given by the image of

$$\pi_1(\Gamma_f \backslash M_f, x) \to S_n$$

where the image of a path γ is the induced permutation on the n-element fibers. (M_f is the branch locus of the branched covering map $p_f : \Gamma_f \dashrightarrow k^m$, which we excise here, and some technical requirements related to the genericity of the fiber and the irreducibility of f need to hold).

It is a *theorem* that this topological monodromy group coincides with our definition of monodromy for algebraic functions (see [Har79] for the case $k = \mathbb{C}$). However, by working topologically one can now ask questions such as representability with respect to non-algebraic functions like the complex logarithm. These are highly relevant, e.g., in the theory of representability for solutions to differential equations. See Picard-Vessiot theory and the contemporary work of Khovanskii [Kho14].

One problem with a purely topological viewpoint in the study of algebraic functions is that the related constructs (cohomology, homotopy groups) are invariant under homeomorphism and even homotopy, but certainly not under birational equivalence. Remarkably, the difference is encoded by yet another classical notion from superposition:

Proposition 2.24. *An algebraic function f is faithfully representable by $\mathcal{S}$ if and only if the branched covering map p_f is homeomorphic to a branched cover produced by successive pullbacks along k-rational maps q_i and branched covers p_{φ_i} (where $\varphi_i \in \mathcal{S}$).*

That is, the divide between faithful and classical representability is the divide between isomorphic and birational geometry in algebraic geometry. Arnold computed the cohomological properties of the braid groups [Arn70a], which can be connected to the cohomology of the unbranched cover $\Gamma_f \backslash M_f \to \mathbb{C}^m \backslash p_f(M_f)$ ([Arn70b], [Lin76]) to show that the faithful representability hierarchy of algebraic functions over $\mathbb{C}$ is straightforward. That is:

Theorem 2.25. *For $n \geq 3$, ρ_n is not faithfully representable by algebraic functions of $n-2$ variables or fewer.*

While the proof is elegant (arguing via the use of a trivial Stiefel-Whitney class in the cohomology of the classifying space BO_n which is then contradictorily propagated to a non-trivial class in the cohomology of the algebraic function), it does not generalize to classical representability because the classes are not preserved under birational equivalence. Working around this to find an invariant more powerful than monodromy but with well-understood properties under rational morphisms is a topic of further research.

3. The representability of ρ_2 to ρ_5

In this section, we work to establish the algebraic hierarchy over $\mathbb{C}$. At first glance, $\rho_n : \mathbb{C}^n \to_n \mathbb{C}$ appears to be the most natural n-variable, n-valued function one can take. While this is true, we show that there are transformations that prove ρ_n is representable in fewer variables. Finally, we will observe that the two main proofs of Abel-Ruffini (the radical non-representability of ρ_5) are the same via the notional equivalence described in Theorem 2.22. One uses restricted sets of field extensions (i.e., radical extensions), and the other uses restricted sets of covering maps ("radical covers").

3.1. Tschirnhaus transformations

We work over $\mathbb{C}$ to allow for arbitrary division by constants. Inspired by the first step of Cardano's solution (Example 1.7), we see that given

$$\rho_n : k^n \to \mathrm{Sym}^n(k), \quad (a_1, \dots, a_n) \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ z^n + a_1 z^{n-1} + \cdots + a_n \end{matrix} \right\},$$

we can take $w = z + \frac{a_1}{n}$ to get a new polynomial (and algebraic function)

$$\nu_n : k^{n-1} \to \mathrm{Sym}^n(k), \quad (b_1, \dots, b_{n-1}) \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ w^n + b_1 w^{n-2} + \cdots + b_{n-1} \end{matrix} \right\}$$

such that $\rho_n(a_1, \cdots, a_n) = \nu_n(b_1(\vec{a}), \dots, b_{n-1}(\vec{a})) - \frac{a_1}{n}$. This already shows ρ_n (for $n \geq 2$) is representable by algebraic functions ν_n of $n-1$ variables, as we have merely pre-composed by polynomial maps b_i and post-composed by a linear shift of $\frac{a_1}{n}$.

In general, Tschirnhaus envisioned a transformation $w = z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1}$ for rational maps α_i in the $a_1, \dots, a_n$, chosen such that all the non-constant coefficients vanish, so that one is left with

$$\nu_n : k \to \mathrm{Sym}^n(k), \quad b \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ w^n - b = 0 \end{matrix} \right\}.$$

This is the wishful hope that all algebraic functions of n values are, up to solving an $(n-1)$-th degree polynomial, the n-th radical function. This would allow an inductive argument that all ρ_n are radically representable, since solving it would only be contingent on ρ_{n-1} and $\sqrt[n]{-}$.

To derive the coefficients α_i, we (anachronistically) use the theory of resultants from algebraic geometry [Tig01], which arose from the more classic elimination theory of the 1700s. Motivation and proofs of the relevant assertions can be found in [GKZ94].

Definition 3.1. The **resultant** $R(p, q)$ of two univariate polynomials

$$p(z) = s_0 z^m + s_1 z^{m-1} + \cdots + s_m$$
$$q(z) = t_0 z^n + t_1 z^{n-1} + \cdots + t_n$$

is given by the determinant of a $(m+n) \times (m+n)$ matrix:

$$R(p,q) := \det \begin{pmatrix} s_0 & s_1 & \cdots & s_m & 0 & \cdots & 0 \\ 0 & s_0 & \cdots & \cdots & s_m & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & s_0 & s_1 & \cdots & s_m \\ t_0 & t_1 & \cdots & t_n & 0 & \cdots & 0 \\ 0 & t_0 & \cdots & \cdots & t_n & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & t_0 & t_1 & \cdots & t_n \end{pmatrix}.$$

In the same way that the **discriminant** $\Delta(p)$ of a polynomial is zero exactly when p has repeated roots (i.e., not squarefree), the resultant has the following defining property up to a factor:

Proposition 3.2. $R(p,q) = 0$ *if and only if the polynomials* p, q *have a common root.*

Remark. In fact, the discriminant $\Delta(p)$ of a polynomial p is $R(p, p')$ up to a factor.

Returning to Tschirnhaus' desired transformation, consider the resultant $R(p,q)$ of the following polynomials in z:

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$
$$q(z) = z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1} - w.$$

Our goal is to choose $\alpha_1, \dots, \alpha_{n-1}$ such that z is a common root to both p and q. To do this, note that $R(p,q)$ is a multinomial in $a_1, \dots, a_n, \alpha_1, \dots, \alpha_{n-1}, w$. We then view $R(p,q)$ as an n-th degree polynomial in w:

$$R(p,q) = w^n + \beta_1 w^{n-1} + \cdots + \beta_n$$

where β_i are multinomials in $a_1, \dots, a_n, \alpha_1, \dots, \alpha_{n-1}$. Recall that our goal was to take $\beta_1 = \cdots = \beta_{n-1} = 0$. In fact, one can show that this is a system of $n-1$ equations of degrees $1, \dots, n-1$ in the α_i (recall that the a_i are already given to us), and it was proved by Bézout that solving the system can be reduced (think successive eliminations via a similar resultant technique) into a polynomial of one variable in degree $(n-1)!$ [Tig01]. If we choose to only eliminate the first $\beta_1 = \cdots = \beta_m$, then this polynomial is of degree $m!$ in general.

In the interest of space, we summarize what happens for $\rho_2, \dots, \rho_5$:

- For ρ_2, the linear transformation $\alpha_1 = -\frac{a_1}{2}$ suffices. This is exactly the method of **completing the square**, which gives the quadratic formula.
- For ρ_3, we have $m! = (n-1)! = 2$. If one works out the method described (i.e., taking $\beta_1 = \beta_2 = 0$ in the above), one gets **Cardano's solution**. More precisely, Cardano's solution is the above method in steps. Performing the linear transformation $w = -\frac{a_1}{3}$ is equivalent to setting $\beta_1 = 0$, the inner square root corresponds to the quadratic Tschirnhaus transformation giving $\beta_2 = 0$, and the outer cube roots come from solving the form $w^3 + b$.
- For ρ_4, we have $m! = (n-1)! = 6$. At face value, this suggests that Tschirnhaus' method requires solving a harder polynomial than we originally had (i.e., a sextic). However, it turns out that for $m = n-1 = 3$, this sextic is a product of degree-2 factors, each of whose coefficients solve a cubic. Thus, there is a transformation that will give $w^4 + b$, albeit one that is difficult to write down.
- For ρ_5, via a quadratic Tschirnhaus transformation (that eliminates $\beta_1 = \beta_2 = 0$), we can assume that we have the depressed quintic:

  $$z^5 + a_1 z^2 + a_2 z + a_3 = 0.$$

  Since we allow ourselves rational maps, in turns out that a further *fractional* cubic transformation

  $$w = \frac{y^3 + \alpha_1 y^2 + \alpha_2 y + \alpha_3}{y + \alpha_4}$$

  suffices to give a form than can be rationally transformed to

  $$w^5 + bw + 1 = 0,$$

  where we assume the constant term is 1 by a further substitution in terms of radicals. This is known as a **Bring-Jerrard form**.

Finally, it is known that the method for ρ_5 generalizes for all n. That is, given

$$z^n + a_1 z^{n-1} + \cdots + a_n$$

one can always take a Tschirnhaus transformation to retrieve an expression

$$w^n + b_1 w^{n-4} + \cdots + b_{n-4} w + 1$$

i.e., taking $\beta_1 = \beta_2 = \beta_3 = 0$ and $\beta_n = 1$ [Vit04]. Hence, ρ_n is always at least $(n-4)$-representable for $n \geq 5$. (The key is that such a transformation requires solving a $3! = 6$ degree polynomial that happens to be a composition of a quadratic and cubic, and thus can be solved using radical functions; since radicals are 1-variable, this does not affect our ℓ-representability bound.)

Remark. This above observation is exactly why the original statement of Hilbert's problem on the solvability of the depressed septic $z^7 + a_1 z^3 + a_2 z^2 + a_3 z + 1 = 0$ using algebraic functions of 2 variables (Question 1.10) was equivalent to asking if ρ_7 is 2-representable.

3.2. Radical extensions and solvability

To summarize, for $\rho_2, \ldots, \rho_5$ there is a body of work that uses methods like Tschirnhaus' transformations to show that up to rational maps over $\mathbb{C}$, they are all 1-representable. Furthermore, we claimed that $\rho_2, \ldots, \rho_4$ are representable by the radical algebraic functions, writing so explicitly for ρ_2 and ρ_3 (Example 1.6, Example 1.7).

We now show that ρ_5 is not **radically representable** ($\mathcal{S}$-representable where $\mathcal{S} = \{\sqrt[n]{-}\}$). We claimed that there is a Bring-Jerrard form $w^5 + bw + 1$ that one can use Tschirnhaus transforms to retrieve; the assertion now is that one cannot turn this form into $w^5 + b$. We will show this by producing a monodromy invariant that is preserved by taking radicals, but one that ρ_5 does not have.

Definition 3.3. A group is **solvable** if it has a **subnormal series** where the factor groups are abelian. That is, there exists a finite sequence $G_0, \ldots, G_r$ such that

$$G = G_0 \geq \cdots \geq G_r = \{e\}, \quad G_k \text{ normal in } G_{k+1}$$

where the G_k/G_{k+1} are abelian.

Remark. The reason such groups are called solvable was exactly because they were Galois' group invariant for showing the unsolvability of the general quintic.

Definition 3.4. The **commutator subgroup** $G^{(1)} = [G, G]$ of a group G is generated by $[a, b] = aba^{-1}b^{-1}$ for all $a, b \in G$. The **derived subgroups** are given inductively by $G^{(k)} = (G^{(k-1)})^{(1)}$.

For any G, one has that $G^{(1)}$ is a normal subgroup with abelian quotient $G/G^{(1)}$. Intuitively, having $G^{(1)}$'s elements satisfy $aba^{-1}b^{-1} = e$ after quotienting is the minimal set of relations required for commutativity (this is equivalent to $ab = ba$). Hence we have a subnormal series:

Definition 3.5. The **derived series** of a group is the subnormal series

$$G \geq G^{(1)} \geq G^{(2)} \geq G^{(3)} \geq \cdots$$

In fact, one sees from this "universal property" of $G^{(1)}$ that the quotient G/H is abelian if and only if $H \geq G^{(1)}$. It follows that checking if this derived series is finite suffices:

Proposition 3.6. *A group G is solvable if and only if its derived series is finite, i.e., $G^{(k)}$ is trivial for some k.*

In the proof of Abel-Ruffini, both viewpoints use the solvability of a group as an invariant which happens to be preserved upon "taking radicals" in the respective category (varieties/topological spaces with branched covers, function fields with inclusions). This is possible because of solvability's closure under a number of group operations. The next two results are in [Żoł00].

Proposition 3.7. *Solvability is closed under taking direct products, taking subgroups, and taking surjective homomorphisms.*

Proof. For direct products, take the direct product of each term in their subnormal series. For subgroups, restrict the derivative series to the subgroup. For surjections $G \to H$, we induce respective surjections $G^{(r)} \to H^{(r)}$. □

Proposition 3.8. *Abelian groups are solvable. S_n for $n \geq 5$ is unsolvable.*

Proof. Abelian groups have the trivial derivative series $G \geq \{e\}$. Meanwhile, since $A_n \leq S_n$ with quotient $\mathbb{Z}/2\mathbb{Z}$, it suffices to show A_n is unsolvable for $n \geq 5$. Let $\sigma = (123)$, $\tau = (345)$ in $H \subseteq A_n$, and observe that these have exactly one common element. One computes that $[\sigma, \tau] = (143)$ and $[\sigma^{-1}, \tau^{-1}] = (253)$, which are in $H^{(1)} \subseteq A_n^{(1)}$ by construction. Repeating this argument shows $A_n^{(j)}$ has two cycles with one common element. Hence $A_n^{(j)}$ is never trivial, and thus the derived series $S_n^{(j)}$ including these never becomes trivial either (i.e., it is not finite). □

3.3. Two proofs of Abel-Ruffini

Here is a dictionary between the ingredients of the proof:

[template]	Geometry	Field theory
Viewpoint (on irreducible algebraic functions $\mathbb{C}^m \dashrightarrow_n \mathbb{C}$)	A degree n branched cover of $\mathbb{C}^m$ (by an algebraic hypersurface in $\mathbb{C}^{m+1}$)	Field extension given by adjoining a single root of a degree n polynomial in $\mathbb{C}(x_1, \ldots, x_m)$
Map type	Dominant rational morphism	Field inclusion
Associated construction	Successive covering maps via pullback (fiber product)	Successive field extensions via pushout (compositums)
Universal construction step	Pullback	Pushout
Radical solvability	Only radical (and trivial) coverings allowed	Only radical (and trivial) extensions allowed
Associated group	Monodromy group of a branched covering map	Galois group of the Galois closure of an extension

We know from Section 2 that this dictionary is straightforward due to the categorical equivalence (Theorem 2.17) from algebraic geometry. The purpose of this section is to highlight that one might learn two proofs of Abel-Ruffini without realizing they are equivalent or undergirded by a unified notion of algebraic function.

- Galois' classical proof works by equating radical solvability directly with taking field extensions that are **radical** (adjoining the roots of equations $w^n - a$), and thus taking the Galois group directly. See [Edw84] for an exposition.

- The topological proof was outlined by Arnold in a series of lectures to gifted high school students. He constructs Riemann surfaces (analogous to the total spaces of our covers Γ_f) and defines monodromy topologically (Definition 2.23). He then takes successive Riemann surfaces corresponding to field operations and radicals. See [Ale04] for a full treatment, which is paraphrased by [Żoł00].

Theorem 3.9 (Abel-Ruffini). *There is no general solution for polynomials of degree $n \geq 5$ in radicals.*

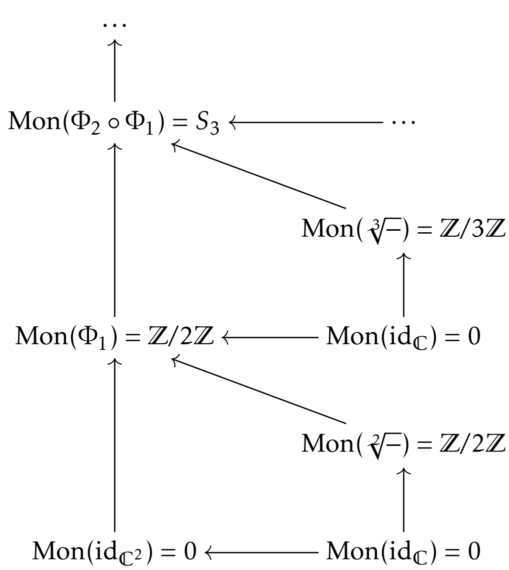
Proof. We want to show that ρ_n is not radically representable when $n \geq 5$. Suppose such a representation

$$\mathrm{gr}_{\rho_n} \subseteq \pi \circ \mathrm{gr}_{\Phi_N} \circ \cdots \circ \mathrm{gr}_{\Phi_1}$$

exists. Recall our *[viewpoint]* and the notion of representability in the respective category (Theorem 2.22). Here, every intermediate *[map type]* in our *[associated construction]* is a *[universal construction step]* along the *[map type]* of a radical function (the *[radical solvability]* condition). It follows that the *[associated group]* of the intermediate map is $\mathbb{Z}/k\mathbb{Z}$ for some k. This is a solvable group.

Solvability is closed under direct products, subgroups, and surjective homomorphisms (Proposition 3.7), so the *[associated groups]* of $\mathrm{gr}_{\Phi_N} \circ \cdots \circ \mathrm{gr}_{\Phi_1}$ and $\pi \circ \mathrm{gr}_{\Phi_N} \circ \cdots \circ \mathrm{gr}_{\Phi_1}$ are solvable. But the latter contains gr_{ρ_n}. However, ρ_n's *[associated group]* is S_n, which is not solvable for $n \geq 5$ (Proposition 3.8). This is a contradiction. ☐

Example 3.10. Let us use Cardano's solution ν_3 as an explicit example. Taking either of the diagrams of Example 2.20 and passing to monodromy groups on the main column gives the following commuting diagram of groups:



The arrows are always inclusions, but squares are no longer necessarily pushouts (although they are in this case; in general, monodromy passes to Galois closure before taking the Galois group). Regardless, the upper-left corners are always subgroups of the direct product of two solvable groups, and hence solvable. In fact, $\mathrm{Mon}(\rho_3) = S_3$, which one can verify is identical to $\mathrm{Mon}(\pi \circ \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1)$ *and* to $\mathrm{Mon}(\Phi_2 \circ \Phi_1)$, despite

$$\mathrm{gr}_{\rho_3} \subsetneq \pi \circ \mathrm{gr}_{\Phi_4} \circ \mathrm{gr}_{\Phi_3} \circ \mathrm{gr}_{\Phi_2} \circ \mathrm{gr}_{\Phi_1}$$

(due to superfluous roots). This gives two interesting final observations:

- Nested radicals are *inevitable*, since one can only get $\mathbb{Z}/k\mathbb{Z}$ from taking the monodromy of the pushout along a radical (because radical extensions are Galois over $\mathbb{C}$, as $\mathbb{C}$ contains all roots of unity).
- Monodromy cannot be used to prove statements about *faithful* representability. This is related to birational vs. topological invariants (Section 2.4), where function fields are invariants of the birational type.

Finally, we saw that Tschirnhaus transformations can still transform the general quintic to the form w^5+bw+1 where the transformation is reversible by taking radicals.

Definition 3.11. The **Bring radical** BR is the algebraic function of 1 variable and 5 values given by

$$BR : k \to \mathrm{Sym}^5(k), \quad b \mapsto \left\{ \begin{matrix} \text{roots of the polynomial} \\ w^5 + bw + 1 \end{matrix} \right\}.$$

Thus, we can now justify the placement of ρ_5 in Figure 1:

Corollary 3.12. *Over $\mathbb{C}$, we have that ρ_5 is not radically representable, but ρ_5 is representable by the radicals and BR.*

Remark. With BR, it turns out that $\mathrm{Mon}(BR) = A_5$, which leads to the relevant invariant of **nearly-solvable** monodromy mentioned in Figure 1. This is a slight generalization of solvability, where the factor groups are also allowed to be A_5. See [DM89] for an exposition.

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