SOBOLEV SPACES IN SECOND-ORDER ELLIPTIC PDE

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Abstract. We explore Sobolev Spaces in 1 and n dimensions. The functional analytic properties of these spaces make them appropriate spaces for applying the Lax-Milgram theorem. We use these techniques to describe some properties of weak solutions to second-order elliptic partial differential equations.

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1. Introduction

As a motivating example, we can consider the following ordinary differential equation,

$$\begin{cases} -(pu')' + qu = f & \text{on } (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

for $p, q, f : (0,1) \to \mathbb{R}$ continuous, p continuously differentiable with $p \geq \alpha > 0$, and $q > 0$. This is the Sturm-Liouville problem. Many of its properties, such as existence and uniqueness of solutions, are within the realm of the classical theory of ordinary differential equations. However, there is a more abstract approach to studying this and related problems that is primarily based on linear functional analysis. We begin with a fair amount of this functional analysis and the theory of Sobolev spaces in one dimension, which are the natural spaces on which to apply the abstract results. Once we have explored the classical one-dimensional Sturm-Liouville problem, we will move to a generalization in higher dimensions. Denoting partial derivatives by subscripts, the partial differential equation will look like

$$-\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

in some open set $U \subseteq \mathbb{R}$ subject to the boundary condition that $u : \bar{U} \to \mathbb{R}^n$ vanishes on ∂U. The generalization of the condition that $p \geq \alpha > 0$ is the requirement that

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the matrix $(a^{ij}(x))$ is positive-definite for almost every $x \in U$. For this equation, the abstract principles we developed in one dimension will have to be (somewhat laboriously) generalized, but they will provide powerful results on an extraordinarily general class of equations known as the elliptic partial differential equations.

2. Some Functional Analysis and the Lax-Milgram Theorem

We first recall some definitions from linear algebra.

Definitions 2.1. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let V be a vector space over $\mathbb{F}$. A **norm** is a function $V \to \mathbb{F}$ denoted $||u||$ for $u \in \mathbb{F}$ such that

(1) $||\lambda u|| = |\lambda|||u||$ for all $\lambda \in \mathbb{F}$ and $u \in V$,

(2) $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$,

(3) $||u|| = 0$ if and only if $u = 0$.

An **inner product** is a function $V \times V \to \mathbb{F}$ denoted (u, v) for $u, v \in V$ such that

(1) $(\lambda u + v, w) = \lambda((u, w) + (v, w))$ for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$,

(2) $(u, v) = \overline{(v, u)}$ for all $u, v \in V$, where $\overline{z}$ is the complex conjugate of z.

(3) $(u, u) \geq 0$ with equality if and only if $u = 0$.

Remark 2.2. A norm on V induces a metric d on V defined by $d(u, v) = ||u - v||$. This metric induces a topology. When we speak of open and closed sets in a normed space, they are with respect to this topology. An inner-product induces a norm given by $||v|| = (v, v)$.

We now define the most important objects and maps in linear functional analysis. We define them over $\mathbb{R}$ or $\mathbb{C}$, but every example we consider will be real.

Definition 2.3. A **Hilbert space** H is a real or complex vector space equipped with an inner product $(,)$ such that every sequence that is Cauchy with respect to the norm, $|| \cdot ||$, induced by this inner product converges in H. This property is called **completeness.**

Definition 2.4. A **bounded operator** T is a linear map $H \to H$ such that there is some constant $M < \infty$ such that $||Tu|| \leq M||u||$ for all $u \in H$.

We denote the kernel of a bounded operator by $N(T)$, the image by $R(T)$, and the identity operator by I.

Proposition 2.5. *A linear map $H \to H$ is bounded if and only if it is continuous with respect to the metric topology (often called the strong topology) induced by the norm.*

Proof. Suppose $T : H \to H$ is bounded. Then $||T(u-v)|| < M||u-v||$, so choosing $\delta < \frac{\epsilon}{M}$ gives continuity. Conversely, if T is continuous, it is in particular continuous at 0. Therefore if $||v|| < \delta$, we have $||Tv|| < 1$. Then, for a general $v \in H$,

$$||Tv|| = \left\| T\left(\frac{2\delta||v||v}{2\delta||v||}\right) \right\| = \frac{2||v||}{\delta} \left\| T\left(\frac{\delta}{2} \cdot \frac{v}{||v||}\right) \right\| < \frac{2||v||}{\delta}.$$

□

In view of the preceding proposition we switch between the terminology of bounded and continuous operators to suit convenience and convention. We now begin working up to the Lax-Milgram theorem, which will be our most important result from functional analysis when studying differential equations. We will need several definitions and results to state and prove it. The first two results are fundamental and as such can be found in most elementary textbooks on functional analysis (such as [3, ch. 4]), so we omit their proofs for space.

Theorem 2.6. *(Riesz Representation Theorem) Let H be a real Hilbert space and H^* its dual space. Then for each $u^* \in H^*$ there exists a unique $u \in H$ such that $\langle u^*, v\rangle = (u, v)$ for all $v \in H$.*

This theorem contains the magic words "there exists a unique," and indeed the Riesz Representation Theorem will be an essential ingredient in the Lax-Milgram theorem, which will be our most powerful tool for finding weak solutions to differential equations. The Riesz Representation Theorem can even be used in place of the Lax-Milgram theorem in very simple differential equations.

Definition 2.7. Let $S \subseteq H$ be a subspace of H. The **orthogonal complement** of S, denoted $S^\perp$, is the set of all $u \in H$ such that $(u, v) = 0$ for all $v \in S$.

Lemma 2.8. *Let H be a Hilbert space and S a closed subspace of H. Then, $H = S \oplus S^\perp$.*

Definition 2.9. We say that an operator $T : H \to H$ is **bounded below** if there exists $c > 0$ such that $||Tu|| \geq c||u||$ for all $u \in H$.

Lemma 2.10. *The range of a continuous operator $T : H \to H$ that is bounded below on a Hilbert space is closed.*

Proof. Since T is injective, it is an isomorphism onto its image $T(H)$. Therefore, there is a well-defined inverse $T^{-1} : T(H) \to H$. We pause to point out that $T(H)$ is in fact a linear subspace of H. The zero vector is certainly in the image ($T0 = 0$). If $x, y \in T(H)$, then there exist α, β such that $T(\alpha) = x$ and $T(\beta) = y$, so $x + y = T(\alpha + \beta) \in T(H)$. Finally, if $c \in \mathbb{R}$, then $cx = T(c\alpha) \in T(H)$. Then T^{-1} is a linear mapping between Hilbert spaces. To show that T^{-1} is continuous, we first note that $T^{-1}u = v$ implies $Tv = u$. Now we use the fact that $||Tv|| \geq c||v||$ for all $v \in H$ to obtain

$$||T^{-1}u|| = ||v|| \leq \frac{1}{c}||Tv|| = \frac{1}{c}||u||.$$

Then $T(H) = (T^{-1})^{-1}(H)$, and so $T(H)$ is the preimage of the closed set H under the continuous mapping T^{-1} and is hence closed. □

We now move to proving the most important tool in answering questions about existence and uniqueness of solutions to differential equations. Before stating the theorem we require another definition.

Definition 2.11. A **continuous, coercive bilinear form** on H is a function $B : H \times H \to \mathbb{R}$ such that

(1) B is linear in both variables,
(2) There exists $M < \infty$ such that $\big|B[u, v]\big| \leq M||u||_H||v||_H$ for all $u, v \in H$,
(3) There exists $N > 0$ such that $B[u, u] \geq N||u||_H$.

The second condition is continuity, the third coerciveness.

Theorem 2.12. *(Lax-Milgram) Let $B : H \times H \to \mathbb{R}$ be a continuous, coercive bilinear form. Let f be a bounded linear functional on H. Then there exists a unique $u \in H$ such that $B[u, v] = \langle f, v\rangle$ for all $v \in H$.*

Proof. By continuity of B, the mapping $v \mapsto B[u, v]$ is a bounded linear functional, so the Riesz Representation Theorem guarantees a unique element $w \in H$ with $B[u, v] = (w, v)$. This w is completely determined by u. We define the (purely set-theoretic for now) mapping $A : H \to H$ by $Au = w$.

We now show that A is a bounded linear operator. We note that our definition gives us $B[u, v] = (Au, v)$. Now, let $\lambda \in \mathbb{R}$ and $u_1, u_2 \in H$. Then,

$$\begin{aligned}(A(\lambda u_1 + u_2), v) &= B[\lambda u_1 + u_2, v] \\ &= \lambda B[u_1, v] + B[u_2, v] \\ &= \lambda(Au_1, v) + (Au_2, v) \\ &= (\lambda Au_1 + Au_2, v),\end{aligned}$$

for each $v \in H$. Technically we have only shown that $(A(\lambda u_1 + u_2), v) = (\lambda Au_1 + Au_2, v)$, but subtracting one side from the other and invoking bilinearity tells us $A(\lambda u_1 + u_2) - \lambda Au_1 + Au_2 = 0$, so A is linear. To show boundedness, we compute using the boundedness of B. That is,

$$||Au||^2 = (Au, Au) = B[u, Au] \leq M||u||||Au||.$$

Therefore, $||Au|| \leq M||u||$.

Because B is coercive, we have the estimate

$$N||u||^2 \leq B[u, u] = (Au, u) \leq ||Au||||u||,$$

by Cauchy-Schwarz. Then, we have that $||Au|| \geq N||u||$, and A is thus bounded below. Applying Lemma 2.10, we see that the range, $R(A)$, of A is closed. Moreover, the estimate tells us that A is injective, for if $Au = Au'$, then $0 = ||A(u - u')|| \geq N||u - u'||$, so $||u - u'|| = 0$, and the definition of norm implies $u = u'$.

We now show that $R(A) = H$. If not, then $R(A)^\perp$ has a nonzero element z, by Lemma 2.8. But then

$$N||z||^2 \leq B[z, z] = (Az, z) = 0,$$

contradicting $z \neq 0$.

The Riesz Representation theorem guarantees that there is a $w \in H$ such that $\langle f, v\rangle = (w, v)$. Since $R(A) = H$, there is some $u \in H$ such that $Au = w$, so

$$\langle f, v\rangle = (w, v) = (Au, v) = B[u, v].$$

Since A is injective, u is unique. □

The Lax-Milgram theorem is our most important result from functional analysis. Much of our later work will be devoted to finding appropriate Hilbert spaces and bilinear forms on which to verify the hypotheses of the Lax-Milgram theorem. The appropriate spaces part will take some work, but we can already see the form that B will take. Consider the Sturm-Liouville problem:

$$\begin{cases} -(pu')' + qu = f & \text{on } (0, 1) \\ u(0) = u(1) = 0, \end{cases}$$

Now multiply each side by of the differential equation by v, a smooth and compactly supported function on $(0,1)$, and integrate by parts. Considering the resulting equation as a function of u and v, we have the bilinear form

$$B[u,v] = \int_I pu'v' + \int_I quv = \int_I fv.$$

Then the work will be to find a Hilbert space that captures the boundary conditions and to verify the hypotheses of the Lax-Milgram theorem.

There is a special class of bounded operators that will be useful, the compact operators. In a Hilbert space, compact operators are the limit of finite-rank operators (i.e. those with finite dimensional range space).

Definition 2.13. A bounded operator $K : H \to H$ is **compact** if for every bounded set $U \subset H$, $\overline{K(U)}$ is compact.

Our next theorem makes precise the notion that, roughly speaking, compact operators behave like linear operators on a finite-dimensional vector space. To save space and focus more on differential equations than abstract functional analysis, we skip the proof. We state the familiar notion of adjoint for completeness.

Definition 2.14. Given a bounded operator $T : H \to H$, its **adjoint** is the unique bounded operator T^* such that $(Tu, v) = (u, T^*v)$.

Theorem 2.15. *(Fredholm Alternative) Let $K : H \to H$ be a compact operator. Then*

(1) $N(I-K)$ *is finite dimensional,*

(2) $R(I-K)$ *is closed,*

(3) $R(I-K) = N(I-K^*)^{\perp}$,

(4) $N(I-K) = \{0\}$ *if and only if* $R(I-K) = H$,

(5) $\dim N(I-K) = \dim N(I-K^*)$.

The Fredholm alternative deals with solvability of the equations $u - Ku = f$ and $u - Ku = 0$ for $f \in H$. In particular from the fourth assertion of the theorem, we see that either the former equation has a unique solution for each $f \in H$ or there are nontrivial (i.e. not the zero vector) solutions to the latter. A restatement of this dichotomy in terms of elliptic PDE will be our last theorem.

3. Sobolev Space in One Dimension

Our goal for this section is to work out existence, uniqueness, and regularity for the Sturm-Liouville problem using the abstract techniques developed in the previous section, in particular the Lax-Milgram theorem. Sobolev Spaces will be the correct spaces to attempt to apply the Lax-Milgram theorem. We state a few important definitions needed to define these spaces. We start by working in one dimension, but a number of results generalize with the same proofs to n dimensions.

Definition 3.1. Let $U \subseteq \mathbb{R}$ be an open set. A function $\varphi : \mathbb{R} \to \mathbb{R}$ is said to **compactly-supported** in U if the closure of the set of points $x \in \mathbb{R}$ such that $\varphi(x) \neq 0$ is a compact subset of U. Such a function is a **compactly-supported**

smooth or **test** function if it is compactly-supported and has continuous derivatives of all orders. We denote the space of compactly-supported smooth functions on U by $C_0^\infty(U)$.

Definition 3.2. A measurable function $u : U \to \mathbb{R}$ is said to be **locally summable** if for every compact subset V of U, $\int_U |v| < \infty$. We denote the space of locally-summable functions by $L^1_{loc}(U)$.

Definition 3.3. Let $U \subseteq \mathbb{R}$ be an open set. A **weak derivative** of a function $u \in L^1_{loc}(U)$ is a function $v : \mathbb{R} \to \mathbb{R}$ such that for all $\varphi \in C_0^\infty(U)$,

$$\int_U u\varphi' = -\int_U v\varphi.$$

We now record a very useful result, and omit its proof for space.

Proposition 3.4. *(Fundamental Lemma of the Calculus of Variations) Let $u \in L^1_{loc}(U)$. If*

$$\int_U uv = 0$$

for all $v \in C_0^\infty(U)$, then $u = 0$ a.e.

This result lets us prove the following

Proposition 3.5. *Suppose $u \in U \to \mathbb{R}$ has a weak derivative u'. If u has another weak derivative u_0', then $u' = u_0'$ almost everywhere in U.*

Proof. Since both u' and u_0' are weak derivatives of u, we have $\int_U uv' = -\int_U u'v = -\int_U u_0'v$ for all test functions v. Then $\int_U (u' - u_0')v = 0$, so applying the fundamental lemma of the calculus of variations, we have $u' - u = 0$ a.e. $\square$

Note that the same proof works if $u : U \to \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$.

Definition 3.6. The **Sobolev Space** $H^k(U)$ is the space of all locally-summable functions $u : U \to \mathbb{R}$ such that u has k weak derivatives, all of which belong to $L^2(U)$.

There is an extremely important subspace of $H^1(U)$.

Definition 3.7. Let $H_0^1(U)$ be the closure of $C_0^\infty(U)$ in $H^1(U)$.

This space captures the boundary condition that $u(0) = u(1) = 0$, so if we find a function satisfying the differential equation (perhaps in the weak sense) in $H_0^1(U)$, then we have found a solution to the boundary value problem.

Definition 3.8. The **Sobolev norm** is given by

$$||u||_{H^k(U)} = \left(\sum_{i=0}^{k} ||u^{(i)}||^2_{L^2(U)} \right)^{1/2},$$

and the **Sobolev inner-product** is given by

$$(u, v)_{H^k(U)} = \sum_{i=0}^{k} (u^{(i)}, v^{(i)}).$$

Remark 3.9. An equivalent norm on $H^k(U)$ is given by

$$||u||_{H^k(U)} = \sum_{i=0}^{k} ||u^{(i)}||_{L^2}.$$

Proposition 3.10. *$H^1(U)$ is a Hilbert space.*

Proof. Let (u_n) be a Cauchy sequence. Then for every $\epsilon > 0$, there exists N such that $n, m > N$ implies $||u_n - u_m||_{H^1(U)} < \epsilon$. Therefore

$$||u_n - u_m||_{L^2(U)} + ||u_n' - u_m'||_{L^2(U)} < \epsilon,$$

so both (u_n) and (u_n') are Cauchy in L^2. Since $L^2(U)$ is complete, we have limits u and u' in $L^2(U)$, respectively of our sequences. Then in the limit

$$\int u_n v' = -\int_U u_n' v$$

becomes

$$\int_U uv' = -\int_U u'v.$$

□

This proof also generalizes to $\mathbb{R}^n$ very easily. The next result is a very powerful generalization of the fundamental theorem of calculus to Sobolev spaces. We omit the proof for space, but it can be found in [1, pp. 204-206].

Theorem 3.11. *Let $u \in H^1(I)$ for I some open subset of $\mathbb{R}$. There exists $\tilde{u} \in C(\bar{I})$ such that $u = \tilde{u}$ almost everywhere in I, and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)\,dt$$

for all x and y in I.

This result does generalize to higher dimensions but not in a necessarily obvious way. We make no use of the generalization, however, so we point the interested reader to [1, p. 282]. We need one more result before moving to the Sturm-Liouville problem. We will prove it in its full, n-dimensional generality in the next section.

Theorem 3.12. *(Poincaré Inequality) Let $u \in H_0^1(U)$ and U be a bounded open subset of $\mathbb{R}$. Then there exists a constant C (depending only on U) such that*

$$||u||_{H_0^1(U)} \le C||u'||_{L^2(U)}.$$

Example 3.13. Let I be the open unit interval, $p \in C^1(\bar{I})$ with $p \ge \alpha > 0$, $q \in C(\bar{I})$ with $q > 0$, and $f \in L^2(I)$. Then there exists a unique $u \in H^2(I) \cap H_0^1(I)$ satisfying

$$\begin{cases} -(pu')' + qu = f & \text{on } I \\ u(0) = u(1) = 0. \end{cases}$$

Moreover, if $f \in C(\bar{I})$, then u is a classical solution.

Proof. We begin by showing that there is a unique weak solution in $H_0^1(I)$. Multiplying both sides of the differential equation by $v \in C_0^\infty$ and integrating over I yields

$$\int_I -(pu')'v + \int_I quv = \int_I fv.$$

(A weak solution is a function $u \in H_0^1(I)$ satisfying the above equality). An integration by parts gives the bilinear form

$$B[u,v] = \int_I pu'v' + \int_I quv = \int_I fv.$$

Of course we have not shown that any such u exists satisfying the above equality or the one before it. For this, we use Lax-Milgram. Note that the mapping $v \mapsto \int_I fv$ is a bounded (by Cauchy-Schwarz) linear functional $H_0^1(I) \to \mathbb{R}$. Then if B is continuous and coercive, there will exist a unique u satisfying $B[u,v] = \langle f, v\rangle = \int_I fv$ for all $v \in H_0^1(I)$, by Lax-Milgram. For continuity, we have that

$$\begin{aligned}\left|\int_I pu'v'\right| &\leq \max_{x\in I} p(x) \int_I |u'v'| \\ &\leq M||u'||_{L^2(I)}||v'||_{L^2(I)} \\ &\leq M||u||_{H_0^1}||v||_{H_0^1}.\end{aligned}$$

Similar bounds hold for the other integral, so B is continuous. Showing that B is coercive relies on the Poincaré inequality. Using the assumption that $q \geq 0$ and $p \geq \alpha > 0$, we have

$$\begin{aligned}B[u,u] = \int_I pu'^2 + \int_I qu^2 \geq \int_I pu'^2& \\ \geq \min_{x\in I} p(x) \int_I u'^2& \\ = c||u'||^2_{L^2(I)}& \\ \geq c||u||^2_{H_0^1}&.\end{aligned}$$

Then by Lax-Milgram, we have a $u \in H_0^1$ satisfying the weak formulation of the differential equation. That is a u such that

$$\int_I (pu')v' + \int_I quv = \int_I fv,$$

for all test functions v. Note then that $pu' \in H_0^1$ as

$$\int_I (pu')v' = -\int_I (f - qu)v,$$

for all test functions v, so pu' has a weak derivative $f - qu$. Then $u' = (1/p)(pu')$ is also in H^1. Thus $u \in H^2$. Thus by our theorem above $u \in C^2(\bar{I})$. Now we arrange our weak formulation to obtain

$$\int_I (-(pu')' + qu - f)v = 0$$

for all $v \in C_0^\infty(\bar{I})$, so $-(pu')' + qu = f$ a.e. in I. Since $u \in C^2(\bar{I})$, $-(pu')' + qu = f$ everywhere on I. □

4. Sobolev Spaces in N Dimensions

We begin by generalizing the definition of Sobolev spaces to n dimensions in the obvious way. We can clearly generalize $C_0^\infty(U)$ to be the set of compactly-supported smooth functions $U \to \mathbb{R}^n$ for $U \subseteq \mathbb{R}^n$.

Definition 4.1. Let $U \subseteq \mathbb{R}^n$ be open. The **Sobolev space of order *k*** is the set of all functions $u \in L^2(U)$ such that for every multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^2(U)$. In set-builder notation, this definition is

$$H^k(U) = \left\{ u \in L^2(U) \,|\, \forall |\alpha| \leq k \; \exists g \in L^2(U), \int_U u D^\alpha v = (-1)^{|\alpha|} \int_U gv \quad \forall v \in C_0^\infty(U) \right\}.$$

Here $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex, so $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1} ... \frac{\partial^{\alpha_n}}{\partial x_n}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. Then the Sobolev Space $H^1(U)$ for example is the space of L^2 functions with weak first partial derivatives in $L^2(U)$. Similar to the one dimensional case we define $H_0^1(U)$ to be the closure of $C_0^\infty(U)$ in $H^1(U)$. Our next theorem is the most important inequality in Sobolev space theory. The proof will use a generalized form of Hölder's inequality.

Theorem 4.2. *(Generalized Hölder Inequality) Let* $1 \leq p_1, ..., p_m \leq \infty$, *with* $\sum_{i=1}^m \frac{1}{p_i} = 1$ *and* $u_k \in L^{p_k}(U)$ *for* $k = 1, ..., m$. *Then,*

$$\int_U |u_1 \cdots u_m| \leq \prod_{k=1}^m ||u_k||_{L^{p_k}(U)}.$$

Theorem 4.3. *(Gagliardo-Nirenberg-Sobolev) Let* $1 \leq p < n$. *There exists a constant* $C(p,n)$ *such that*

$$||u||_{L^{p^*}}(\mathbb{R}^n) \leq C||Du||_{L^p(\mathbb{R}^n)},$$

for all $u \in C_0^1(\mathbb{R}^n)$, *where* $p^* = \frac{np}{n-p}$.

Proof. We begin with $p = 1$. Using the fundamental theorem of calculus (the traditional one since everything is smooth and compactly supported), we have

$$\begin{aligned} |u(x)| &= \left| \int_{-\infty}^{x_i} u_{x_i}(x_1, ..., x_{i-1} y_i, x_{x_i+1}, ..., x_n) \, dy_i \right| \\ &\leq \int_{-\infty}^{\infty} \left| u_{x_i}(x_1, ..., x_{i-1} y_i, x_{x_i+1}, ..., x_n) \right| \, dy_i \\ &\leq \int_{-\infty}^{\infty} \left| Du(x_1, ..., x_{i-1} y_i, x_{x_i+1}, ..., x_n) \right| \, dy_i. \end{aligned}$$

Since this holds for all $1 \leq i \leq n$ and $x \in \mathbb{R}^n$, we have

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1, ..., y_1, ..., x_n)| \, dy_i \right)^{\frac{1}{n-1}}.$$

Integrating each side with respect to x_1 and denoting $Du(x_1, ..., y_i, ..., x_n)$ simply by Du for convenience gives

$$\begin{aligned} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} \, dx_1 &\leq \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du| \, dy_i \right)^{\frac{1}{n-1}} \, dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\prod_{i=2}^n \int_{-\infty}^{\infty} |Du| \, dy_i \right)^{\frac{1}{n-1}} \, dx_i \end{aligned}$$

Now we apply the generalized Hölder inequality with the product functions as $\left(\int_{-\infty}^{\infty} |Du|\, dy_i\right)^{\frac{1}{n-1}}$ and $p_i = n-1$, so that $\sum_{i=2}^{n} \frac{1}{p_i} = 1$. Then

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 \leq \left(\int_{-\infty}^{\infty} |Du|\, dy_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du| dx_1 dy_i\right)^{\frac{1}{n-1}}, \qquad (*)$$

where we also used Fubini's theorem to change the order of integration. Now we integrate with respect to x_2. Applying the estimate $(*)$ and the fact that $Du(x_1, y_2, ..., x_n)$ is constant with respect to x_2, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 dx_2 \leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}}\, dx_2,$$

where $I_1 = \int_{-\infty}^{\infty} |Du|\, dy_1$ and for $3 \leq i \leq n$, $I_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\, dx_2 dy_i$. Now we apply the generalized Hölder inequality yet again to obtain

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 dx_2 \\ &\quad \leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\, dx_1 dy_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\, dy_1 dx_2\right)^{\frac{1}{n-1}} \\ &\qquad\qquad \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\, dx_1 dx_2 dy_i\right)^{\frac{1}{n-1}}. \end{aligned}$$

Proceeding in this manner and integrating over each of x_i for $3 \leq i \leq n$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} &\leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} |Du|\, dx_1 ... dy_i ... dx_n\right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n} |Du|\, dx\right)^{\frac{n}{n-1}}, \end{aligned} \qquad (**)$$

which is the theorem for $p = 1$. If $1 < p < n$, we apply $(**)$ on the function $v = |u|^{\gamma}$, for some $\gamma > 1$. We obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}\, dx}\right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |D|u|^{\gamma}|\, dx \\ &= \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du|\, dx \\ &\leq \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\, dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\, dx\right)^{\frac{1}{p}}, \end{aligned}$$

where we applied Hölder's inequality with $L^{\frac{p}{p-1}}$ and L^p in the last inequality. Now we wish to choose γ to allow us to make a cancellation in the above estimate. In this case, we need $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$. Solving we have $\gamma = \frac{p(n-1)}{n-p}$. Note that this choice gives us $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = \frac{np}{n-p} =: p^*$. So we have

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{n-1}{n}} \leq \left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^{p}\, dx\right)^{\frac{1}{p}},$$

dividing through we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}} \leq C||Du||_{L^p(\mathbb{R}^n)}$$
$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}.$$

□

This estimate lets us derive two more useful inequalities.

Theorem 4.4. *Let U be a bounded, open subset of $\mathbb{R}^n$. If $u \in H_0^1(U)$, then $u \in L^q(U)$ for all $q \in [1, 2^*]$, and*

$$||u||_{L^q(U)} \leq C||Du||_{L^2(U)}.$$

Proof. By construction of $H_0^1(U)$, there exists a sequence of functions $u_m \in C_0^\infty(U)$ converging to u in $H_0^1(U)$. Then we can define each u_m to be 0 on $\mathbb{R}^n - \bar{U}$. Since each function is smooth and compactly supported inside U, this extension produces a smooth and compactly supported function on $\mathbb{R}^n$. Then we can apply Gagliardo-Sobolev-Nirenberg to obtain by approximation (and dominated convergence) $||u||_{L^{2^*}(U)} \leq C||Du||_{L^2(U)}$. Since U is bounded (and hence finite measure), we have $||u||_{L^q(U)} \leq C||Du||_{L^2(U)}$ for $1 \leq q \leq 2^*$. □

Corollary 4.5. *(Poincaré Inequality) Let U be a bounded, open subset of $\mathbb{R}^n$. If $u \in H_0^1(U)$, then there exists a constant C (depending only on U) such that*

$$||u||_{L^2(U)} \leq C||Du||_{L^2(U)}.$$

Proof. Since $2^* = \frac{2n}{n-2}$, we have $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$, hence $2^* > 2$ and the above theorem applies. □

Our next theorem will be of great importance when we wish to show that certain operators are in fact compact in order to apply the Fredholm theory from section 2. We first recall some definitions and results from advanced calculus.

Definition 4.6. Let G and H be Hilbert spaces with $G \subset H$. We say that G is **compactly embedded** in H (denoted $G \subset\subset H$) if $||u||_H \leq C||u||_G$ for some constant C and each bounded sequence in G has a subsequence converging to a limit in H.

Definitions 4.7. A sequence (f_n) of functions $\mathbb{R}^n \to \mathbb{R}$ is said to **uniformly bounded** if there exists an M such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$. The sequence is **uniformly equicontinuous** if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$.

Theorem 4.8. *Let (f_n) be a sequence of smooth functions $\mathbb{R}^n \to \mathbb{R}$ such that the sequence (Df_n) is uniformly bounded. Then (f_n) is uniformly equicontinuous.*

Theorem 4.9. *(Arzelà-Ascoli) Let (f_n) be a sequence of functions $\mathbb{R}^n \to \mathbb{R}$. If (f_n) is uniformly bounded and uniformly equicontinuous, then there exists a subsequence (f_{n_k}) of (f_n) and a continuous function f such that $f_{n_k} \to f$ uniformly on compact subsets of $\mathbb{R}^n$.*

We must introduce the very useful technique of mollification to prove our theorem. See [2, p. 713-716] for more information on mollification and proofs of the results stated here.

Definitions 4.10. Define $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where C is such that $\int_{\mathbb{R}^n} \eta = 1$. For each $\epsilon > 0$ define $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ by $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$. Let $U \subseteq \mathbb{R}^n$ be open and define $U_\epsilon = \{x \in U | \operatorname{dist}(x, \partial U) > \epsilon\}$ and $B(x, r)$ to be the ball of radius r centered at x. Given $f : U \to \mathbb{R}$ locally integrable, define its **mollification** by

$$f^\epsilon = \int_U \eta_\epsilon(x-y) f(y)\, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y)\, dy$$

for $x \in U_\epsilon$.

Theorem 4.11. *(1) $\eta \in C^\infty(\mathbb{R}^n)$.*

(2) $\eta_\epsilon \in C^\infty(U_\epsilon)$.

(3) $f^\epsilon \in C^\infty(U_\epsilon)$.

(4) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.

Theorem 4.12. *(Rellich-Kondrachov Compactness Theorem) If U is a bounded open subset of $\mathbb{R}^n$ for $n \geq 2$, then*

$$H_0^1(U) \subset\subset L^2(U).$$

Proof. The definition of the Sobolev norm gives us $||u||_{L^2(U)} \leq ||u||_{H_0^1(U)}$. Then we must show that a bounded sequence (u_m) in $H_0^1(U)$ has a convergent subsequence. Before beginning we note that we can take $U = \mathbb{R}^n$ (just by defining each u_m to be 0 outside U) and assume that each u_m has support contained in some bounded set $V \subset \mathbb{R}^n$ (since they do by construction of $H_0^1(U)$. Finally the boundedness condition means that $\sup_m ||u_m||_{H_0^1(U)} \leq \infty$. We begin by considering the mollifications of u_m and wish to show that $u_m^\epsilon \to u_m$ in $L^2(V)$ as $\epsilon \to 0$ uniformly in m. First assume u_m is smooth (since the definition of $H_0^1(U)$ guarantees that smooth functions are dense in $H_0^1(U)$, our result for smooth functions will hold in general by approximation). At any rate

$$\begin{aligned} u^\epsilon(x) - u_m(x) &= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x-z}{\epsilon}\right) u_m(z)\, dz - u_m(x) \int_{B(x,\epsilon)} \eta\left(\frac{x-z}{\epsilon}\right) \\ &= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))\, dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}(u_m(x-\epsilon t y))\, dtdy \\ &= -\epsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon t y) \cdot y\, dtdy. \end{aligned}$$

Integrating we therefore find,

$$\begin{aligned}\int_V |u_m^\epsilon(x) - u_m(x)|\, dx &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x - \epsilon ty)|\, dxdtdy \\ &\leq \epsilon \int_V |Du_m(w)|\, dw.\end{aligned}$$

Then, $||u_m^\epsilon - u_m||_{L^1(V)} \leq \epsilon ||Du_m||_{L^1(V)} \leq \epsilon C ||Du_m||_{L^2(V)} \leq \epsilon CM$, where we used the facts that V is bounded to pass to the L^2 norm and that our sequence is bounded in H_0^1 and hence L^2 norm for the last inequality. We therefore find that $u_m^\epsilon \to u_m$ in $L^2(V)$ as $\epsilon \to 0$ uniformly in m.

We next wish to show that (u_m^ϵ) is uniformly bounded and equicontinuous for each fixed $\epsilon > 0$. This is a straightforward application of the theorem that uniform boundedness of the derivative implies equicontinuity. Anyway,

$$\begin{aligned}|u^\epsilon(x)| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|\, dy \\ &\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(V)} \leq \frac{CM}{\epsilon^n},\end{aligned}$$

and

$$\begin{aligned}|Du_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|\, dy \\ &\leq ||D\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(U)} \leq \frac{CM}{\epsilon^{n+1}}.\end{aligned}$$

Then we have that (u_m^ϵ) is uniformly bounded and equicontinuous for each fixed $\epsilon > 0$.

Now we select $\epsilon > 0$ so that $||u_m^\epsilon - u_m||_{L^2(U)} \leq \frac{\delta}{2}$ for all m. Then since each u_m^ϵ is compactly supported (on the fixed set V), we can invoke Arzelà-Ascoli to find subsequence a $(u_{m_k}^\epsilon)$ converging uniformly on V. Then

$$\limsup_{k,j\to\infty} ||u_{m_k}^\epsilon - u_{m_j}^\epsilon||_{L^2(V)} = 0,$$

so combining the previous estimate gives (by the triangle inequality)

$$\limsup_{k,j\to\infty} ||u_{m_k} - u_{m_j}||_{L^2(V)} \leq \delta.$$

Finally, we consider the subsequences $(u_{m_k,i})$ associated to $\delta = \frac{1}{i}$. Then construct (u_{m_l}) by $u_{m_l} = u_{m_l,l}$. So, for example, the u_{m_2} is the second entry in the subsequence $(u_{m_k,2})$ which satisfies $\limsup\limits_{j,k\to\infty} ||u_{m_k} - u_{m_j}||_{L^2(V)} \leq \frac{1}{2}$. This process (known as a diagonal argument) yields a subseqence (u_{m_l}) satisfying

$$\limsup_{l,j\to\infty} ||u_{m_l} - u_{m_j}||_{L^2(U)} = 0.$$

$\square$

5. Existence of Weak Solutions to Second-Order Elliptic Equations

Definition 5.1. We say a differential operator L of the form

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u$$

is **elliptic** if there is a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in U$.

Note that the ellipticity condition means that the matrix $(a^{ij}(x))$ is positive-definite with smallest eigenvalue greater than or equal to θ. The problem we wish to study is the boundary value problem

$$\begin{cases} Lu = f & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{$*$}$$

We assume that U is a bounded subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$. We assume that the coefficient functions are in $L^\infty(U)$ and satisfy the ellipticity condition. We make further assumptions on their regularity when appropriate.

Similar to the one-dimensional case, we now define a weak formulation of our PDE.

Definition 5.2. The **weak formulation** of $(*)$ is to find $u \in H_0^1(U)$ such that

$$\int_U \sum_{i,j}^{n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i}v + c(x)uv\,dx = \int_U fv\,dx,$$

for all $v \in C_0^\infty(U)$. Moreover the bilinear form associated to $(*)$ is

$$B[u,v] = \int_U \sum_{i,j}^{n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i}v + c(x)uv\,dx.$$

.

With an eye towards using the Lax-Milgram theorem, we can consider $v \mapsto \int_U fv$ as a bounded linear functional on H_0^1, and thus succinctly write the weak formulation as

$$B[u,v] = \langle f, v\rangle.$$

Formally, $\langle , \rangle$ represents a pairing of $H_0^1(U)$ and its dual space, but we can work with just the map without concerning ourselves with the algebraic properties of the dual space.

We now try to apply Lax-Milgram to $(*)$. Barring some serious pathology, we will always have continuity of B, but coerciveness is a much more delicate question. The next result begins to make this notion precise.

Theorem 5.3. *There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B[u,v]| \leq \alpha||u||_{H_0^1}||v||_{H_0^1}$$

and

$$\beta||u||_{H_0^1}^2 \leq B[u,u] + \gamma||u||_{L^2}^2$$

Before beginning the proof, note that if $\gamma > 0$, we cannot apply Lax-Milgram.

Proof. Continuity is straightforward. We apply Hölder's inequality to obtain

$$|B[u,v]| \leq \sum_{i,j=1}^{n} ||a^{ij}||_{L^\infty} \int_U |u_{x_i} v_{x_j}|\, dx + \sum_{i=1}^{n} ||b^i||_{L^\infty} \int_U |u_{x_i} v|\, dx$$

$$+||c||_{L^\infty} \int_U |u||v|\, dx$$

$$\leq \alpha \left(\sum_{i,j=1}^{n} ||u_{x_i}||_{L^2} ||v_{x_j}||_{L^2} + \sum_{i=1}^{n} ||u_{x_i}||_{L^2} ||v||_{L^2} + ||u||_{L^2} ||v||_{L^2} \right).$$

Now we point out that the product of the Sobolev norms of u and v is equal to

$$||u||_{H_0^1} ||v||_{H_0^1} = (||u||_{L^2} + ||Du||_{L^2})(||v||_{L^2} ||Dv||_{L^2}).$$

Then the observation that $|u_{x_i}| \leq |Du|$ allows us to say that each summand in our inequality is smaller than the Sobolev norm times some constant. Thus,

$$|B[u,v]| \leq \alpha ||u||_{H_0^1} ||v||_{H_0^1}.$$

Proving that B is coercive requires us to use the ellipticity condition. Letting $\xi = Du$ and integrating we have

$$\begin{aligned}
\theta \int_U |Du|^2 &\leq \int_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \\
&= B[u,u] - \int_U \sum_{i=1}^{n} b^i u_{x_i} u + \int_U cu^2 \\
&\leq B[u,u] + \sum_{i=1}^{n} ||b^i||_{L^\infty} \int_U |Du||v|\, dx + ||c||_{L^\infty} \int_U u^2\, dx.
\end{aligned}$$

We now invoke the general inequality that for $a, b, \epsilon > 0$ we have

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

This is known as Cauchy's Inequality with ϵ. We apply it to $|Du||u|$ to obtain

$$\int_U |Du||u|\, dx \leq \epsilon \int_U |Du|^2\, dx + \frac{1}{4\epsilon} \int_U u^2\, dx.$$

Choosing $\epsilon < \theta/(2\sum_{i=1}^{n} ||b^i||_{L^\infty})$, we return to our estimate and have

$$\begin{aligned}
\theta \int_U |Du|^2\, dx &\leq B[u,u] + \frac{\theta}{2} \int_U |Du|^2\, dx + C \int_U u^2\, dx \\
\frac{\theta}{2} \int_U |Du|^2\, dx &\leq B[u,u] + C \int_U u^2\, dx.
\end{aligned}$$

Now we add $\frac{\theta}{2} \int_U u^2\, dx$ to both sides to obtain

$$\frac{\theta}{2} \left(\int_U |Du|^2 + u^2\, dx \right) \leq B[u,u] + \left(C + \frac{\theta}{2} \right) \int_U u^2\, dx.$$

Or

$$\beta ||u||^2_{H_0^1(U)} \leq B[u,u] + \gamma ||u||^2_{L^2(U)},$$

as desired. □

We can now prove an existence theorem for weak solutions to elliptic PDE.

Theorem 5.4. *There exists a constant $\gamma \geq 0$ such that for $\mu \geq \gamma$ and $f \in L^2(U)$, there is a unique $u \in H_0^1(U)$ such that*

$$\int_U \sum_{i,j}^n a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} v + c(x)uv + \mu uv \, dx = \int_U fv \, dx.$$

That is, a weak solution to

$$\begin{cases} Lu + \mu u = f & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{5}$$

Proof. Define $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ by $B_\mu[u,v] = B[u,v] + \mu \int_U uv \, dx$. Note that $B_\mu[u,v]$ is the bilinear form corresponding to (5). From our previous theorem, we have

$$\begin{aligned} B_\mu[u,u] &= B[u,u] + \mu ||u||^2_{L^2(U)} \\ &\geq \beta ||u||^2_{H_0^1(U)} + (\mu - \gamma)||u||^2_{L^2(U)} \\ &\geq \beta ||u||^2_{H_0^1(U)}. \end{aligned}$$

Thus B_μ is coercive. It is also clearly continuous given the continuity of B. Finally, we note that $\langle f, v \rangle = \int_U fv$ is a bounded linear functional on $H_0^1(U)$ (apply Cauchy-Schwarz in L^2), so the hypotheses of the Lax-Milgram theorem are satisfied with B_μ and $\langle f, v \rangle$. $\square$

Our next existence result depends upon the theory of compact operators, namely the Fredholm alternative discussed in section 2. Before beginning, we define the notion of a formal adjoint to the differential operator L.

Definition 5.6. The **formal adjoint** L^* of L is given by

$$L^* v = -\sum_{i,j=1}^n (a^{ij} v_{x_j})_{x_i} - \sum_{i=1}^n b^i v_{x_i} - \sum_{i=1}^n b^i_{x_i} v + cv,$$

where we require $b^i \in C^1(\bar{U})$ for each i.

The adjoint is formal because we defined L merely as a notational convenience rather than an operator on a Hilbert space. However, since we always take U to be bounded, $a^{ij}, c \in L^\infty(U)$, and here $b^i \in C^1(\bar{U})$, Lu will lie in L^2 by a calculation analogous to showing that $B[u,v]$ is bounded. Then, we can compute the adjoint using the L^2 inner product.

$$\begin{aligned} (u, L^*v)_{L^2(U)} &= -\int_U \sum_{i,j=1}^n (a^{ij} v_{x_j})_{x_i} u - \int_U \sum_{i=1}^n b^i v_{x_i} u - \int_U \sum_{i=1}^n b^i_{x_i} vu + \int_U cvu \\ &= \int_U \sum_{i,j=1}^n a^{ij} v_{x_j} u_{x_i} - \int_U \sum_{i=1}^n b^i v_{x_i} u + \int_U \sum_{i=1}^n b^i (vu)_{x_i} + \int_U cvu \\ &= \int_U \sum_{i,j=1}^n a^{ij} v_{x_j} u_{x_i} - \int_U \sum_{i=1}^n b^i v_{x_i} u + \int_U \sum_{i=1}^n b^i (v_{x_i} u + v u_{x_i}) + \int_U cvu \\ &= \int_U \sum_{i,j=1}^n a^{ij} v_{x_j} u_{x_i} + \int_U \sum_{i=1}^n b^i u_{x_i} v + \int_U cuv \\ &= (Lu, v)_{L^2(U)}. \end{aligned}$$

We emphasize here that despite the above calculation, L is not an operator on $L^2(U)$ since we needed additional differentiability (at least in the weak sense) assumptions on v. (We also used the product rule without justification; the calculation is valid for $u, v \in H^1(U) \cap L^\infty(U)$; See [1, p. 269]) We can, however, define the weak formulation of the formal adjoint in the same manner as we did for L. The calculation above shows that the bilinear form associated to the weak formulation is just $B[u, v]$. We denote this bilinear form by $B^*[v, u]$. Then we say that $u \in H_0^1(U)$ is a weak solution to the adjoint problem

$$\begin{cases} L^*v = f & \text{on } U \\ v = 0 & \text{on } \partial U \end{cases}$$

if

$$B^*[v, u] = \int_U fu = \langle f, u \rangle,$$

for all $u \in H_0^1(U)$. We can now state our next theorem.

Theorem 5.7. *(Fredholm Alternative for Elliptic PDE) For every $f \in L^2(U)$, there either exists a unique weak solution $u \in H_0^1(U)$ to*

$$\begin{cases} Lu = f & \text{on } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

or there exists a nontrivial (i.e. not the 0 vector) solution u of the homogeneous problem

$$\begin{cases} Lu = 0 & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Proof. For this proof, We use $(,)$ to denote the usual L^2 inner product; it is equivalent to $\langle , \rangle$ in this context, but we wish to perform arithmetic with other L^2 inner products. We recall from our first existence result that for some γ such that for $g \in L^2(U)$, there is a unique $u \in H_0^1(U)$ satisfying $B[u, v] + \gamma(u, v) =: B_\gamma[u, v] = (g, v)$ for all $v \in H_0^1(U)$. This u is a weak solution of differential operator $L_\gamma u = Lu + \gamma u$. Then we have a well-defined mapping $L^2(U) \to H_0^1(U)$ given by $g \mapsto u$. We denote this mapping by L_γ^{-1}, so that $L_\gamma^{-1} g = u$. Linearity of this mapping follows easily from the linearity of our differential equation.

Note that if u is a weak solution to $Lu = f$, then we have $B[u, v] = (f, v)$. Adding $\gamma(u, v)$ to each side we obtain $B_\gamma[u, v] = (f + \gamma u, v)$. Since we could just as well have started there and subtracted, we see that u is a weak solution of $Lu = f$ if and only if $B_\gamma[u, v] = (\gamma u + f, v)$ for all $v \in H_0^1(U)$. This means, however, that u is a weak solution to $L_\gamma u = \gamma u + f$. Then we have $u = L_\gamma^{-1}(\gamma u + f)$. We now define $K := \gamma L_\gamma^{-1} u$. Note that since L_γ^{-1} maps from $L^2(U)$, K is an operator $L^2(U) \to L^2(U)$. Also define $h := L_\gamma^{-1} f$, so that we have the equation

$$u - Ku = h.$$

If we can show that $K : L^2(U) \to L^2(U)$ is a compact operator then we can apply the Fredholm alternative to the above equation. We first have to show that K is bounded. Note that $Kg = u$, so we need to bound the L^2 norm of u by some constant times the L^2 norm of g. We have trivially that $||Kg||_{L^2(U)} \leq ||Kg||_{H_0^1(U)}$. Then, using Theorem 5.3, we have the existence of some $\beta > 0$ such that

$$\beta ||u||_{H_0^1(U)}^2 = B[u, u] = (g, u) \leq ||g||_{L^2(U)} ||u||_{L^2(U)} \leq ||g||_{L^2(U)} ||u||_{H_0^1(U)},$$

so then
$$||Kg||_{L^2(U)} \leq ||Kg||_{H_0^1(U)} \leq \frac{1}{\beta}||g||_{L^2(U)},$$
hence K is bounded. Moreover, from the Rellich-Kondrachov compactness theorem, we know that $H_0^1(U) \subset\subset L^2(U)$, so K is a compact operator.

Then if for each $h \in L^2(U)$, we have a unique $u \in L^2(U)$ satisfying $u - Ku = h$, by our previous calculations we have $L_\gamma u - \gamma u = L_\gamma h$, which implies (just by unraveling notation) that $Lu = f$, and hence a weak solution. Alternatively, if there is a nontrivial solution to $u - Ku = 0$, then by an analogous calculation, we have a nonzero weak solution to the homogeneous problem. □

Corollary 5.8. *Suppose that there is a nontrivial solution to the homogeneous problem*
$$\begin{cases} Lu = 0 & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases}$$
Then, the subspace $N \subset H_0^1(U)$ of weak solutions to the homogeneous problem is finite-dimensional with dimension that of the subspace $N^* \subset H_0^1(U)$ of weak solutions to the homogeneous adjoint problem
$$\begin{cases} L^*v = 0 & \text{on } U \\ v = 0 & \text{on } \partial U. \end{cases}$$

Proof. Using the Fredholm alternative, we just have to verify (using the notation of the previous proof) that $v - K^*v = 0$ implies that v is a weak solution to the homogeneous adjoint problem. For $g \in L^2(U)$, we recall that $L_\gamma^{-1}g = u$ if and only if $B_\gamma[u, v] = (g, v)$ for all $v \in H_0^1(U)$, so $Kg = \gamma u$ if and only if $B_\gamma[u, v] = (g, v)$ for all $v \in H_0^1(U)$. Similar to our construction of K, we define K^* by $K^*f = \gamma v$ if and only if $B_\gamma^*[v, u] = (f, u)$ for all $u \in H_0^1(U)$. Now we just have to verify that the operator K^* we have constructed is in fact the adjoint of K. Suppose $Kg = \gamma u$ and $K^*f = \gamma v$. Then
$$(Kg, f) = \gamma(u, f) = \gamma B_\gamma^*[v, u] = \gamma B_\gamma[u, v] = (g, \gamma v) = (g, K^*g).$$
Thus the analogous construction for the adjoint problem will produce the adjoint of K, and the Fredholm theory gives the result. □

Corollary 5.9. *Let $N^* \subset H_0^1(U)$ be the subspace of weak solutions to the adjoint problem. The problem*
$$\begin{cases} Lu = f & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases}$$
has a weak solution if and only if $(f, v) = 0$ for all $v \in N^*$.

Proof. We recall that the third assertion of the Fredholm Alternative is that $R(I - K) = N(I - K^*)^\perp$. Then $u - Ku = h$ has a solution if and only if $(h, v) = 0$ for all v satisfying $v - K^*v = 0$. However,
$$(h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^*v) = \frac{1}{\gamma}(f, v).$$
Thus we have a weak solution to
$$\begin{cases} Lu = f & \text{on } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

if and only if $(f, v) = 0$. $\square$

We conclude by pointing out that although we have some powerful results concerning weak solutions, we know little about the regularity of the solutions. In n dimensions, this is a somewhat delicate (or at least computationally involved) problem. Details can be found in [2, Sec. 6.3].

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