MILNOR'S CONSTRUCTION OF EXOTIC 7-SPHERES

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Abstract. In this paper, I will provide a detailed explanation of Milnor's construction of exotic 7-spheres. The candidate manifolds will be constructed as total spaces of S^3 bundles over S^4, denoted $M_{h,l}$. The subset of these candidates satisfying the condition $h + l = \pm 1$ will be shown to be topological spheres by Morse Theory. A subset of these that do not satisfy $(h-l)^2 \equiv 1 \pmod 7$ will be shown to not be differential spheres, by the Hirzebruch Signature Theorem and some other results from the theory of characteristic classes. Finally, I will discuss some interesting applications of exotic smooth structure throughout mathematics.

Contents

1. Motivation 1
2. Hopf Fibrations 2
2.1. The Complex Hopf fibration 2
2.2. The Quaternionic Hopf fibration 2
3. S^3 bundles over S^4 3
4. The exotic sphere is a topological 7-sphere 5
4.1. Morse Theory 5
4.2. Reeb's Theorem 6
4.3. Proof that $M_{h,l}$ is a topological sphere if $h + l = -1$ 6
5. Characteristic Classes 8
5.1. Euler Class 8
5.2. Chern Class 8
5.3. Pontryagin Class 9
6. $M_{h,l}$ is not Diffeomorphic to the 7-sphere 9
6.1. Calculating $p_1(\xi_{h,l})$ 9
6.2. Calculating the first Pontryagin class of $K_{h,l}$ 12
6.3. Applying the Hirzebruch Signature Theorem 13
7. Conclusion 14
8. Further discussion of Exotic Structure 14
Acknowledgments 15
References 15

1. Motivation

The existence of exotic smooth structure is a sign of the limits of human intuition. Geometric intuition is based on generalization of everyday objects. Surfaces, or 2-manifolds, are a class of such objects. One of the most 'natural' spaces is a sphere. We can give several different 'natural' definitions of spheres. A selection follow.

Definition 1.1. A **standard sphere** is a hypersurface cut out by an equation of the form

$$\sum_{i=1}^{n} x_i^2 = 1 \tag{1.2}$$

where the x_i are all real numbers.

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Definition 1.3. A **homotopy sphere** is a topological manifold M of dimension n with $H^0(M,\mathbb{Z}) = H^n(M,\mathbb{Z}) = \mathbb{Z}$, $H^*(M,\mathbb{Z}) = 0$ otherwise and $\pi_1(M) = 0$.

Definition 1.4. A **topological sphere** is a topological manifold that is homeomorphic to a standard sphere.

The intuition behind the definition of a topological sphere is that a topological sphere can be continuously deformed into a standard sphere.

Definition 1.5. A **differential sphere** is a smooth manifold that is diffeomorphic to a standard sphere.

In other words, a differential sphere can be 'smoothly deformed' into a standard sphere or, equivalently, the calculus done on a differential sphere is equivalent to the calculus done on a standard sphere.

For the case of surfaces, homotopy spheres, topological spheres, and differential spheres are all equivalent. Therefore, it seems obvious that all of these topological equivalences will hold in higher dimensions. In fact, by the Poincare Conjecture, homotopy spheres and topological spheres are equivalent in all dimensions. On the other hand, topological spheres and differential spheres are not. The simplest known example of a smooth manifold that is a topological sphere, but is not a differential sphere, occurs in 7 dimensions. In this paper, we will construct this example.

2. Hopf Fibrations

In this section, we will discuss Hopf fibrations, which are fiber bundles in which the base space, the fibers, and the total space are all spheres.

2.1. **The Complex Hopf fibration.**

We will start with the classical Hopf fibration, which can be realized in 3 dimensions. This is the fiber bundle with fibers homeomorphic to S^1 and base space S^2. The total space of this fiber bundle is a 3-sphere. So, we represent this fiber bundle with the following diagram:

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

The 1-sphere S^1 is just the set of complex numbers with norm 1, and the 3-sphere is the set of ordered pairs of complex numbers with norm 1. The 2-sphere is the one point compactification of the complex plane, which is the complex projective line $\mathbb{CP}^1$.

We can explicitly write down the projection map in the complex Hopf fibration as $\pi : (z_1, z_2) \mapsto [z_1, z_2]$. For example, note that the preimage of $[1;0] \in \mathbb{CP}^1$ under the projection map π is $\{z_1 \in \mathbb{C} \big| \|z_1\|^2 = 1\}$, which is a copy of S^1, and therefore is a fiber embedded in the total space.

Note that every two fibers are linked with linking number one in the total space. This is a qualitative difference from the relationship between the fibers in the fiber bundle whose fibers and base space are copies of S^1 and whose total space is a torus. In that case, the linking number of any two fibers is zero.

2.2. **The Quaternionic Hopf fibration.**

There are in fact only four Hopf fibrations, which correspond to the four division algebras, which is a result of Adams. The four division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The quaternions are denoted $\mathbb{H}$, and they have non-commutative multiplication. Formally,

Definition 2.1. $\mathbb{H}$ is the set of numbers of the form $a + bi + cj + dk$, where $ij = k = -ji$ and addition is component-wise.

The octonions are a division algebra with non-associative multiplication. So, as the 'size' of the division algebra increases, the division algebra has fewer basic properties like commutativity.

The quaternionic Hopf fibration is formed exactly analogously to the complex, or classical, Hopf fibration. So it has the following diagram:

$$\begin{array}{ccc} S^3 & \hookrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

where S^3 is the set of unit quaternions and S^7 is the set of pairs of quaternions with unit norm. The maps are also analogous to the complex case: $\pi : (z_1, z_2) \mapsto [z_1, z_2]$. The preimage of the point $[1; 0]$ under the projection map is $\{z_1 \in \mathbb{H} \big| \|z_1\|^2 = 1\}$, which is a 3-sphere embedded in the total space. The only difference is in the definition of projective space. Projective space is defined by putting an equivalence relation on any two points that are on the same line through the origin. That is:

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2) \tag{2.2}$$

Since multiplication in the quaternions is not commutative, the definition of quaternionic projective spaces includes a choice of default multiplication. We choose left-multiplication. This is a degree of freedom that did not occur in the complex Hopf fibration and is the reason why the simplest known example of exotic smooth structure occurs in seven dimensions, instead of three dimensions.

3. S^3 bundles over S^4

To construct the exotic 7-sphere, we want to look at different ways of constructing sphere bundles of S^3 over S^4. We will need some tools from algebraic topology to do this, which will be developed in this section. We will construct this family explicitly and show that each element of the family corresponds to an element of the group $\mathbb{Z} \oplus \mathbb{Z}$. We will show that this is $\pi_3(SO(4))$.

First, we note that S^4 can be covered by an atlas with only two charts, just like any other topological sphere. An $\mathbb{R}^4$ bundle over each chart is trivial since the charts are contractible, so the only way to get a non-trivial bundle is to have some 'twisting' in the gluing maps between the fibers. We want to construct a family of such gluing maps that give a variety of total spaces.

We will choose the two charts to be all of S^4 except the south pole, which will be denoted U_1 and all of S^4 except the north pole, which will be denoted U_2. Now I define a map that takes each chart to $\mathbb{R}^4$, which I can identify with the quaternions. I will write this correspondence in the following diagrams to emphasize to the reader the importance of this correspondence for this construction. The maps are called ϕ_1 and ϕ_2 respectively.

$$\phi_1 : U_1 \to \mathbb{R}^4 = \mathbb{H} \tag{3.1}$$
$$[z; 1] \mapsto z \tag{3.2}$$

$$\phi_2 : U_2 \to \mathbb{R}^4 = \mathbb{H} \tag{3.3}$$
$$[1; w] \mapsto w \tag{3.4}$$

Note that the transition map is:

$$\phi_2 \circ \phi_1^{-1} : \mathbb{H} - \{0\} \to \mathbb{H} - \{0\} \tag{3.5}$$
$$z \mapsto \frac{1}{z} \tag{3.6}$$

which is exactly what we expect. We will start with the simplest example, which is that of the standard quaternionic Hopf fibration of an S^3 bundle over S^4. The total space of this fiber bundle is given by

$$\{\big((x, y), [z; w]\big) \big| xw = zy; z, w \neq 0\} \subset \mathbb{H}^2 \times \mathbb{HP}^1 \tag{3.7}$$

The projection from the total space to S^4 is denoted π.

We want to construct the local trivializations of the fiber bundle. These need to be done on each chart of the base space and then matched.

First, we will consider the preimage of the the first chart of the base space in the total space.

$$\pi^{-1}(U_1) = \{((x, y), [z; 1]) \mid x = \lambda z, y = \lambda \text{ for some } \lambda \in \mathbb{H}\} \tag{3.8}$$

For the first chart, the local trivialization is:

$$\rho_1 : \pi^{-1}(U_1) \to \phi_1(U_1) \times \mathbb{H} \tag{3.9}$$
$$((x, y), [z, 1]) \mapsto (z, y) \tag{3.10}$$

Similarly the preimage of U_2 in the total space of the Hopf fibration is:

$$\pi^{-1}(U_2) = \{((x, y), [1; w]) \mid x = \lambda, y = \lambda w \text{ for some } \lambda \in \mathbb{H}\} \tag{3.11}$$

The local trivialization is:

$$\rho_2 : \pi^{-1}(U_2) \to \phi_2(U_2) \times \mathbb{H} \tag{3.12}$$

$$((x, y), [1; w]) \mapsto (w, x) \tag{3.13}$$

Now we want to compute the transition map of the trivializations

$$\rho_2 \circ \rho_1^{-1} : \phi_1(U_1 \cap U_2) \times \mathbb{H} \to \phi_2(U_1 \cap U_2) \times \mathbb{H} \tag{3.14}$$

$$(z, y) \mapsto \left(\frac{1}{z}, yz\right) \tag{3.15}$$

So what we are doing is gluing (z, y) to $(\frac{1}{z}, yz)$. The coordinate on the base space is actually the same, so we are essentially gluing the fibers, which are quaternion lines, in a twisted way. We actually want the fiber to be the unit quaternions, or 3-spheres, so we need to normalize the coordinate on the fiber.

$$\rho_2 \circ \rho_1^{-1} : \phi_1(U_1 \cap U_2) \times S^3 \to \phi_2(U_1 \cap U_2) \times S^3 \tag{3.16}$$

$$(z, y) \mapsto \left(\frac{1}{z}, \frac{yz}{\|z\|}\right) \tag{3.17}$$

This gives a total space that is a 7-sphere. Note that the non-commutativity on the quaternions and the fact that S^3 is the unit quaternions gives a second gluing with the multiplication in the second coordinate reversed. This is another way of saying that the 'inside-out' Hopf fibration is isomorphic to the Hopf fibration. Now we construct an entire family of such gluing maps

$$f_{h,l} : \phi(U_1 \cap U_2) \times S^3 \to \phi(U_1 \cap U_2) \times S^3 \tag{3.18}$$

$$(z, y) \mapsto \left(\frac{1}{z}, \frac{z^h y z^l}{\|z\|^{h+l}}\right) \tag{3.19}$$

The indices h and l are elements of $\mathbb{Z}$, so this family of gluing maps is labeled by the group $\mathbb{Z} \oplus \mathbb{Z}$. We have not yet shown that this is the family of all possible gluing maps, nor have we shown that every gluing map $f_{h,l}$ gives a distinct fiber bundle. To show these two facts, we consider the following special case of the classification theorem of fiber bundles.

Theorem 3.20. *There is a bijection between the isomorphism classes of vector bundles of real dimension m over S^n and the homotopy classes of maps from $S^{n-1} \to GL_m(\mathbb{R})$.*

Proof. See [2] □

The case $m = n = 4$ corresponds to $\mathbb{R}^4$ bundles over S^4. We only want the unit vectors in these bundles, so we only want orthogonal matrices with determinant 1, which gives the following special case of the theorem:

Theorem 3.21. *There is a bijection between isomorphism classes of S^3 bundles over S^4 with structure group $SO(4)$ and homotopy classes of maps from S^3 to $SO(4)$.*

So, to classify our candidate manifolds, we need to find $\pi_3(SO(4))$.

From now on, we treat S^3 as the unit quaternions. So we can treat S^3 as a group with the group operation given by multiplication in the quaternions. We build the following map Ψ:

$$\Psi : S^3 \times S^3 \to SO(4) \tag{3.22}$$

$$(u, v) \mapsto \{\psi_{uv} : x \mapsto uxv^{-1}\} \tag{3.23}$$

The map ψ_{uv} can be viewed as a linear isometry from $\mathbb{H}$ to $\mathbb{H}$, or from $\mathbb{R}^4$ to $\mathbb{R}^4$, since $\mathbb{R}^4$ and $\mathbb{H}$ are isomorphic as $\mathbb{R}$-vector spaces. Ψ is actually a group homomorphism with kernel $\{(1, 1), (-1, -1)\}$. Therefore, $S^3 \times S^3$ is a double cover of $SO(4)$. Now we apply the following theorem from algebraic topology:

Theorem 3.24. *For two topological spaces X and Y, and a covering map between them $F : X \to Y$, then for $n \geq 2$, $\pi_n(X) \cong \pi_n(Y)$.*

Proof. This follows from the mapping lifting theorem and the homotopy lifting lemma. □

Therefore, $\pi_3(S^3 \times S^3) \cong \pi_3(SO(4))$. Clearly, $\pi_3(S^3 \times S^3) = \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$, so we have found that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. The isomorphism is given explicitly as follows. For any $(h, l) \in \mathbb{Z} \oplus \mathbb{Z}$, we define

$$\tilde{f}_{h,l} : S^3 \to S^3 \times S^3 \tag{3.25}$$

$$u \mapsto (u^h, u^{-l}) \tag{3.26}$$

This is not the most natural definition of $\tilde{f}_{h,l}$, but this will result in the nicest form of $f_{h,l}$, which we define as $f_{h,l} = \Psi \circ \tilde{f}_{h,l}$.

$$f_{h,l} : S^3 \to SO(4) \tag{3.27}$$

$$u \mapsto \{x \mapsto u^h x u^l\} \tag{3.28}$$

where x is an element of $\mathbb{H}$.

For each gluing map $f_{h,l}$ we call the corresponding total space $M_{h,l}$ and the entire fiber bundle $\xi_{h,l}$. So we have constructed a family of fiber bundles $\xi_{h,l}$ as follows

$$\begin{array}{ccc} S^3 & \hookrightarrow & M_{h,l} \\ & & \downarrow \\ & & S^4 \end{array}$$

which are classified by $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

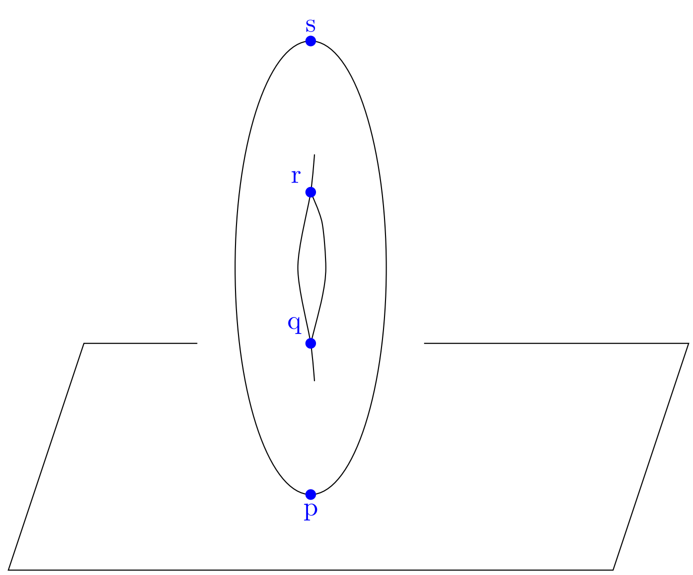
4. The exotic sphere is a topological 7-sphere

Our fundamental tool in this part of the proof will be Morse Theory, specifically Reeb's Theorem, which is a method of detecting topological spheres in any dimension.

4.1. **Morse Theory.** Morse Theory is a way of translating the homotopy type of a manifold into statements about critical points of particular functions. Therefore, the primary difficulty of applying Morse Theory lies in finding a function with a set of critical points that is easy to study. In particular, we want a minimal set of critical points with well-behaved Hessians.

We will discuss an illustrative example of Morse Theory, which is the motivating example in Milnor's book on the subject. [5]

Consider a torus T over a plane P such that the hole of the torus is perpendicular to the plane, as in the following figure:



Consider a height function on T that cuts the torus at the height specified. There are clearly 4 points where the homotopy type of the portion of the torus below the level curve changes. The first is the lowest point of the torus over the plane p. When the height function reaches this point, the cell structure goes from empty to a 0-cell. The second is the point q which is where a 1-cell is added. At heights between the height of p and the height of q, the space can be shrunk to a 0-cell by deformation retract. At the point r a 1-cell is added, and at the point s a 2-cell is added.

The negative gradient of the height function given above is the flow of 'rain' on the torus. At the point s, all of the water will be flowing away from the point, since there is no neighborhood of the torus 'uphill' from s. Similarly, at the point p, all of the water will be flowing towards the point, since there is no neighborhood of p containing a point downhill from p. The points r and s both have one direction along which the water is flowing inward and one along which it is flowing outward.

We can express these statements about 'flow of water' in terms of axes of expansion and contraction of a vector field, which is the negative gradient of the height function. Flow outward is expansion and flow inward is contraction. Now, the correspondence is obvious: the number of linearly independent axes of expansion of the vector field around a critical point is equal to the dimensionality of the cell added at that point under a height function. Although we have only shown this for the example of the torus, it is true in general.

4.2. **Reeb's Theorem.** Now we will prove Reeb's Theorem, which is the main tool in the next subsection. We will give a proof using a handlebody construction, which only holds if the critical points are non-degenerate, but the theorem is more generally true. Milnor's book gives a more detailed proof of this case [5].

Theorem 4.1. *If M is a compact smooth manifold of dimension n, and f is a differentiable function on M with only two nondegenerate critical points, then M is homeomorphic to a standard sphere.*

Proof. Let M and f be as in the theorem statement. Then one of the critical points, p, must be a maximum and the other, q, must be a minimum. By definition, in any small neighborhood around p, the vector field must be directed outward in every direction. So the point p corresponds to an n-cell. By definition, a minimum has the vector field directed inward everywhere. So q corresponds to a 0-cell. M is the union of an n-cell and a 0-cell and therefore is homeomorphic to an n-sphere. □

4.3. **Proof that $M_{h,l}$ is a topological sphere if $h+l=-1$.** The main difficulty of this proof is constructing a suitable function so that Reeb's Theorem can be applied. We have already constructed our candidate manifold, which is the total space in the following fiber bundle:

$$\begin{array}{ccc} S^3 & \hookrightarrow & M_{h,l} \\ & & \downarrow \\ & & S^4 = \mathbb{HP}^1 \end{array}$$

We, with a regrettable lack of motivation, define the following function g. This function is defined on each coordinate chart of S^4 to match with the local trivializations.

$$g \circ \rho_1^{-1} : \phi_1(U_1) \times S^3 \to \mathbb{R} \tag{4.2}$$

$$(z, v) \mapsto \frac{Re(v)}{\sqrt{1 + \|z\|^2}} \tag{4.3}$$

$$g \circ \rho_2^{-1} : \phi_2(U_2) \times S^3 \to \mathbb{R} \tag{4.4}$$

$$(w, u) \mapsto \frac{Re(wu^{-1})}{\sqrt{1 + \|wu^{-1}\|^2}} \tag{4.5}$$

We check that this agrees on the intersection and find that it does only if the condition $h + l = -1$ holds. That is, we verify this diagram:

$$\begin{array}{ccc} (z, v) & \xrightarrow{g\circ\rho_1^{-1}} & \frac{Re(v)}{\sqrt{1+\|z\|^2}} \\ \downarrow{\scriptstyle \rho_2\circ\rho_1^{-1}} & & \downarrow{\scriptstyle =} \\ \left(\frac{1}{z}, \frac{z^h v z^l}{\|z\|^{h+l}}\right) & \xrightarrow{g\circ\rho_2^{-1}} & \frac{Re(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}} \end{array}$$

We calculate

$$\begin{aligned} u^{-1} &= \frac{\bar{u}}{\|u\|^2} \\ &= \left(\frac{z^h v z^l}{\|z\|^{h+l}}\right)^{-1} \\ &= \frac{\overline{z^h v z^l}}{\|z\|^{h+l}} \frac{1}{\|v\|^2} \\ &= \frac{\bar{z}^l \bar{v} \bar{z}^h}{\|z\|^{h+l}} \end{aligned}$$

Now:

$$\begin{aligned} wu^{-1} &= \frac{1}{z} \frac{\bar{z}^l \bar{v} \bar{z}^h}{\|z\|^{h+l}} \\ &= \frac{\bar{z}^{l+1} \bar{v} \bar{z}^h}{\|z\|^{h+l+2}} \end{aligned}$$

We want to take the real part of this to find the numerator. If $h+l=-1$, we have:

$$\begin{aligned} Re\left(\frac{\bar{z}^{l+1} \bar{v} \bar{z}^h}{\|z\|^{h+l+2}}\right) &= \frac{Re(\bar{z}^{l+1} \bar{v} \bar{z}^h)}{\|z\|^{h+l+2}} \\ &= \frac{Re(\bar{z}^{l+1} \bar{v} \bar{z}^{-1-l})}{\|z\|^{-1-l+l+2}} \\ &= \frac{Re(\bar{v})}{\|z\|} \\ &= \frac{Re(v)}{\|z\|} \end{aligned}$$

This used the following computation. For any $x, y \in \mathbb{H}$, $Re(xyx^{-1}) = Re(y)$, since

$$\begin{aligned} 2Re(xyx^{-1}) = xyx^{-1} + \overline{xyx^{-1}} &= xyx^{-1} + \frac{\overline{xy\bar{x}}}{\|x\|^2} \\ &= xyx^{-1} + \frac{x\bar{y}\bar{x}}{\|x\|^2} \\ &= xyx^{-1} + x\bar{y}x^{-1} \\ &= x(y+\bar{y})x^{-1} \\ &= x(2Re(y))x^{-1} \\ &= 2Re(y) \end{aligned}$$

Now we compute the denominator:

$$\begin{aligned} \left\|\frac{\bar{z}^{l+1} \bar{v} \bar{z}^h}{\|z\|^{h+l+2}}\right\|^2 &= \frac{\|v\|^2}{\|z\|^2} \\ &= \frac{1}{\|z\|^2} \end{aligned}$$

So the denominator is

$$\frac{1}{\sqrt{1+\|wu^{-1}\|^2}} = \frac{1}{\sqrt{1+\frac{1}{\|z\|^2}}}$$

Putting these together we get

$$\frac{Re(v)}{\|z\|} \frac{1}{\sqrt{1+\frac{1}{\|z\|^2}}} = \frac{Re(v)}{\sqrt{1+\|z\|^2}}$$

and so we have the desired equality.

Now we want to find the critical points of this function. For a fixed z, the critical points are where the condition $Re(v) = \pm 1$ holds. So let's consider the points of the form $(z, \pm 1)$. If $\|z\| \neq 0$, then

$$\nabla\left(\frac{\pm 1}{\sqrt{1+\|z\|^2}}\right) = \frac{\pm 2\|z\|}{\sqrt{1+\|z\|^2}} \neq 0 \tag{4.6}$$

Therefore $(0, 1)$ and $(0, -1)$ are the two critical points in the first chart. We can easily check the second chart and find that it has no critical points.
So, the function g has only two critical points on $M_{h,l}$ for $h + l = -1$ and therefore we can apply Reeb's Theorem to show that $M_{h,l}$ is homeomorphic to S^7 under this condition. By Lemma 6.14, we have that $M_{h,l}$ is homeomorphic to S^7 if $h + l = 1$ as well.

5. Characteristic Classes

Characteristic classes are a way to generalize the linking number mentioned in the section on the Hopf fibration above. We want to describe the way in which the fibers over $M_{h,l}$ are linked together analogously to the linking number of the complex Hopf fibration. This generalization will be called the Euler class. Then we want to find other characteristic classes that allow us to distinguish our exotic spheres, that is to give more detail about how the fibers are twisted together than just a generalized linking.

5.1. **Euler Class.** The Euler class of a fiber bundle is a way of generalizing the linking number of fibers in a bundle where the fibers are one dimensional circles. Suppose we have the following orientable real fiber bundle of rank n, which we call ξ:

$$\begin{array}{ccc} \mathbb{R}^n & \hookrightarrow & E \\ & & \downarrow p \\ & & M^m \end{array}$$

where M^m is a compact orientable manifold of dimension m, E is a smooth manifold of dimension $m + n$, and p is the projection map. We define the zero-section of M :

$$s_0 : M \hookrightarrow E$$

by sending every x in M to the zero vector of $p^{-1}(x)$. So the zero-section gives an embedding of the base space in the total space. We can perturb s_0 by some small amount to get a new embedding s_0', such that the images of s_0 and s_0' intersect transversely.

$$N := s_0(M) \cap s_0'(M) \tag{5.1}$$

So N is a closed submanifold of dimension $m - n$. It is called the $\mathbf{self-intersection}$ of M inside E since both $s_0(M)$ and $s_0'(M)$ are homeomorphic to M.

We denote the homology class in $H_{m-n}(M, \mathbb{Z})$ represented by the closed chain N as $[N]$. Now we can define the **Euler class** of ξ, $e(\xi)$ as the Poincare dual of $[N] \in H_{m-n}(M, \mathbb{Z})$.

In other words, the Euler class of a vector bundle can be represented by a differential n-form supported on a tubular neighborhood of $N \subset M$ such that the integral of $e(\xi)$ along any vertical slice of the tube orthogonal to N is 1. More precisely, the tubular neighborhood is a disk bundle over the submanifold N. The Euler class can be represented as some bump function times a $(\dim M - \dim N)$-form in the $(\dim M - \dim N)$ dimensions of the disc and the integral of this bump function over a disc in the bundle is 1.

5.2. **Chern Class.** If we consider vector bundles whose fibers are complex vector spaces, then we can define Chern classes, which are analogous to the Euler class. The top Chern class, that is the Chern class with the dimensionality of the fiber, is equivalent to the Euler class.

Definition 5.2.

$$c_n(\xi) := e(\xi_{\mathbb{R}}) \in H^{2n}(M, \mathbb{Z}) \tag{5.3}$$

We also define the first Chern class in this section. Our proof of the existence of exotic structure will only rely on the first and top Chern classes, so we do not discuss the other Chern classes in this paper. See [6] for a complete treatment.

To begin, we construct a line bundle $\wedge^n \xi$ associated to the vector bundle ξ that is known as the determinant line bundle of ξ.

$$\wedge^n E := \{v_1 \wedge v_2 \wedge ... \wedge v_n \big| v_1, ..., v_n \text{ all lie in the same fiber}\} \tag{5.4}$$

We define the projection

$$\wp : \wedge^n E \to M \tag{5.5}$$

by sending every $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ to the point $p(v_1) = \cdots = p(v_n) \in M$.

For every $x \in M$, we have $\wp^{-1}(x) = \wedge^n \left(p^{-1}(x)\right)$. Note that $p^{-1}(x)$ is a complex vector space of dimension n, so $\wp^{-1}(x)$ is a complex vector space of dimension 1. Hence, $\wp : \wedge^n E \to M$ actually forms a complex line bundle over M. We denote this complex line bundle by $\wedge^n \xi$.

Now, we define **the first chern class** $c_1(\xi)$ to be the same as the Euler class of the complex line bundle $\wedge^n \xi$, i.e.,

$$c_1(\xi) = e\left((\wedge^n \xi)_{\mathbb{R}}\right) \tag{5.6}$$

So we have $c_1(\xi) \in H^2(M, \mathbb{Z})$.

5.3. **Pontryagin Class.** Pontryagin classes are defined in terms of Chern classes.

Definition 5.7. $p_k(\xi) := (-1)^k c_{2k}$, where c_{2k} is the $(2k)^{th}$ Chern class of the complex vector bundle $\xi \underset{\mathbb{R}}{\otimes} \mathbb{C}$.

So the k^{th} Pontryagin class is an element of the $4k^{th}$ cohomology group, $H^{4k}(X, \mathbb{Z})$. Note that the Pontryagin class of $\xi_{h,l}$ does not depend on the orientation of $\xi_{h,l}$.

6. $M_{h,l}$ is not Diffeomorphic to the 7-sphere

We will prove this by contradiction. The general outline of the proof is as follows. First, we will assume that $M_{h,l}$ is diffeomorphic to the 7-sphere and use this assumption to construct a particular manifold $K_{h,l}$. Then we will combine the Hirzebruch Signature Theorem and a particular lemma about Pontryagin classes to get a contradiction for some combinations of h and l.

We can construct a manifold $N_{h,l}$ that consists of all the vectors in $\xi_{h,l}$ with length less than or equal to 1. The boundary of this manifold is clearly $M_{h,l}$. Assume that $M_{h,l}$ is diffeomorphic to S^7. Then, since the boundary of an 8-disk is a 7-sphere, we can glue an 8-disk to $N_{h,l}$ to create a new 8-manifold, $K_{h,l}$, that has no boundary. This manifold has a smooth structure, since $N_{h,l}$ had a smooth structure by construction, an 8-disk has a natural smooth structure, and the gluing is diffeomorphic.

6.1. **Calculating $p_1(\xi_{h,l})$.** We want to calculate the first Pontryagin class of $K_{h,l}$, in terms of h and l. We will do this by finding the Pontryagin class of $\xi_{h,l}$.

Now, we will consider the relationship between the following three objects $\pi_3(SO(4))$, $\pi_4(BSO(4))$, and $H^4(S^4, \mathbb{Z})$. By section 3, we know that $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$, and that the elements (h, l) correspond to fiber identifications $f_{h,l}$. $BSO(4)$ is the classifying space of S^3 bundles with structure group $SO(4)$, or equivalently, the classifying space of rank 4 real oriented vector bundles.

Our second fact is the following theorem, which is used to prove Thm 3.20, and therefore was already implied:

Theorem 6.1. *$\pi_4(BSO(4))$ and $\pi_3(SO(4))$ are isomorphic as groups.*

Proof. See p.126 of [4] □

We also use the following theorem, which is also a precursor to Thm 3.20:

Theorem 6.2. *Every rank 4 real vector bundle over S^4 is the pullback of the natural rank 4 vector bundle over $BSO(4)$ with respect to some continuous map from S^4 to $BSO(4)$.*

Proof. See [2] □

This implies that there is a one-to-one correspondence between homotopy classes of maps from S^4 to $BSO(4)$ and rank 4 real vector bundles over S^4.

From the discussion of Euler class, we have the following map:

$$\{\text{rank 4 vector bundles over } S^4\} \to H^4(S^4, \mathbb{Z})$$
$$\xi \mapsto e(\xi)$$

So, we now want to find a group homomorphism between $\pi_4(BSO(4))$ and $H^4(S^4, \mathbb{Z})$ so that we can have a diagram that commutes.

$$\begin{array}{ccc} \pi_4(BSO(4)) & \longrightarrow & H^4(S^4, \mathbb{Z}) \\ \downarrow \cong & \nearrow & \\ \pi_3(SO(4)) & & \end{array}$$

Now we introduce characteristic classes into the discussion. Since characteristic classes are natural transformations, we have the following fact:

Lemma 6.3. *For a vector bundle ξ, a characteristic class κ and a map on cohomology classes induced by the pullback of a bundle f^*, $f^*(\kappa(\xi)) = \kappa(f^*(\xi))$*

To find the group homomorphism we want, we need to look at the group structure of $\pi_4(BSO(4))$. So, we know that

$$\pi_3(SO(4)) \cong \pi_4(BSO(4)) \cong \mathbb{Z} \oplus \mathbb{Z} \tag{6.4}$$

We also know that

$$H^4(S^4, \mathbb{Z}) \cong \mathbb{Z} \tag{6.5}$$

Now consider the wedge of two 4-spheres $S^4 \vee S^4$. We define two maps c_i by collapsing the i^{th} summand where $i = 1, 2$. We have the following isomorphism:

$$\phi : H^4(S^4, \mathbb{Z}) \times H^4(S^4, \mathbb{Z}) \to H^4(S^4 \vee S^4, \mathbb{Z}) \tag{6.6}$$
$$(\alpha, \beta) \mapsto c_1^*(\alpha) + c_2^*(\beta) \tag{6.7}$$

We define the map τ to be the map from S^4 to $S^4 \vee S^4$ by collapsing the equator. So we have

$$H^4(S^4, \mathbb{Z}) \times H^4(S^4, \mathbb{Z}) \xrightarrow{\phi} H^4(S^4 \vee S^4, \mathbb{Z}) \xrightarrow{\tau^*} H^4(S^4, \mathbb{Z}) \tag{6.8}$$
$$(\alpha, \beta) \mapsto c_1^*(\alpha) + c_2^*(\beta) \mapsto \alpha + \beta \tag{6.9}$$

We now consider $\pi_4(BSO(4))$. Suppose $[f]$, $[g]$ are two elements in $\pi_4(BSO(4))$, where f, g are continuous maps from S^4 to $BSO(4)$, and $[f]$ (resp. $[g]$) represents the class of maps which are based homotopic to f (resp. g). Then by definition, the sum $[f]+[g]$ in $\pi_4(BSO(4))$ is given by the homotopy class of the following composition:

$$S^4 \xrightarrow{\tau} S^4 \vee S^4 \xrightarrow{f \vee g} BSO(4) \tag{6.10}$$

Here, the map $f \vee g$ is defined on $S^4 \vee S^4$ such that the restriction of it on the first (resp. second) wedge summand is equal to f (resp. g).

We denote the above composition by $f + g$. Then $[f] + [g] = [f + g]$ in $\pi_4(BSO(4))$. Suppose now we fix an element ν in $H^4(BSO(4), \mathbb{Z})$. We can define the following map:

$$\Theta : \pi_4(BSO(4)) \longrightarrow H^4(S^4, \mathbb{Z}) \tag{6.11}$$
$$[h] \mapsto h^*(\nu) \tag{6.12}$$

For any $[f]$, $[g]$ in $\pi_4(BSO(4))$,

$$\begin{aligned}\Theta([f]+[g]) = \Theta([f+g]) &= (f+g)^*(\nu) = \tau^* \circ (f \vee g)^*(\nu) \\ &= (\tau^* \circ \phi) \circ (\phi^{-1} \circ (f \vee g)^*)(\nu) \\ &= (\tau^* \circ \phi)(f^*(\nu), g^*(\nu)) \\ &= f^*(\nu) + g^*(\nu) \\ &= \Theta([f]) + \Theta([g])\end{aligned}$$

This shows that Θ is actually a group homomorphism.

If we choose ν to be the Euler class of the universal bundle over $BSO(4)$, then this shows that

$$e(\xi_{hl}) = (x \cdot h + y \cdot l)\alpha \tag{6.13}$$

where α is a generator of $H^4(S^4, \mathbb{Z}) \cong \mathbb{Z}$.

We now introduce another fact

Lemma 6.14. *There exists an orientation-reversing isomorphism of real vector bundles. That is,* $\xi_{h,l} \cong \xi_{-l,-h}$

This implies that $e(\xi_{h,l}) = -e(\xi_{-l,-h})$. We can simplify equation 6.13 by considering the standard Hopf fibration where $(h, l) = (0, 1)$. This isomorphism gives that there is an 'inside-out' standard Hopf fibration, that is isomorphic to the standard one. So we have:

$$x \cdot h + y \cdot l = -(x \cdot (-l) + y \cdot (-h)) \tag{6.15}$$

The Euler class of the standard Hopf fibration is $-\alpha$, so

$$x \cdot 0 + y \cdot 1 = -1 \tag{6.16}$$

Combining these two gives $x = y = -1$. So, for some generator α in the relevant cohomology class,

$$e(\xi_{hl}) = -(h + l)\alpha \tag{6.17}$$

If we choose ν to be the first Pontryagin class of the universal bundle of $BSO(4)$, then, as before, we have an equation of the form

$$p_1(\xi_{hl}) = a \cdot h + b \cdot l \tag{6.18}$$

Since Pontryagin classes are independent of the orientation of the bundle,

$$ah + bl = a(-l) + b(-h) \tag{6.19}$$

Combining, this gives

$$p_1(\xi_{hl}) = a(h - l)\alpha \tag{6.20}$$

where α is a generator in $H^4(S^4, \mathbb{Z})$.

We can also calculate the Chern classes for this vector bundle, and we find that

$$c_2(\xi_{0,-1}) = -\alpha \tag{6.21}$$

$$c_1 = 0 \tag{6.22}$$

Now we cite two lemmas about Chern classes from [6]

Lemma 6.23. $\xi_{0,1} \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong \xi_{0,1} \oplus \overline{\xi_{0,1}}$

Proof. See Lemma 15.4 in [6] □

Lemma 6.24. $c_k(\bar{\omega}) = (-1)^k c_k(\omega)$

Proof. See Lemma 14.9 in [6] □

Now we calculate

$$1 + c_2(\xi_{0,1} \underset{\mathbb{R}}{\otimes} \mathbb{C}) = (1 + c_2(\xi_{0,1})) \smile (1 + c_2(\overline{\xi_{0,1}})) \tag{6.25}$$

$$= (1 - \alpha) \smile (1 - \alpha) \tag{6.26}$$

$$= 1 - 2\alpha \tag{6.27}$$

So, using the correspondence between Chern and Pontryagin classes,

$$p_1(\xi_{0,1}) = -c_2(\xi_{0,1} \underset{\mathbb{R}}{\otimes} \mathbb{C}) = 2\alpha \tag{6.28}$$

This generalizes to

$$p_1(\xi_{h,l}) = 2(h - l)\alpha \tag{6.29}$$

6.2. **Calculating the first Pontryagin class of $K_{h,l}$.** However, the goal of this entire computation is to find the first Pontryagin class of $K_{h,l}$, not $\xi_{h,l}$.

We will use multiple fiber bundles to get the Pontryagin class we want.

$\xi_{h,l}$ is the real vector bundle of rank 4 over S^4 that corresponds to $f_{h,l} \in \pi_3(SO(4))$. We denote the total space of this bundle by E.

$$\begin{array}{ccc} \mathbb{R}^4 & \hookrightarrow & E \\ & & \downarrow \\ & & S^4 \end{array}$$

Next we have $\delta_{h,l}$, the fiber bundle over S^4 with fiber $\mathbb{D}^4$, the closed unit disk in $\mathbb{R}^4$ that consists of all the vectors with length ≤ 1 in $\xi_{h,l}$. The total space of this bundle is denoted $N_{h,l}$, and the projection map is denoted π.

$$\begin{array}{ccc} \mathbb{D}^4 & \hookrightarrow & N_{h,l} \\ & & \downarrow \\ & & S^4 \end{array}$$

Now we have a third fiber bundle, $\sigma_{h,l}$, which is a sphere bundle. The total space of this bundle is what was previously denoted by $M_{h,l}$. Note that previously both $\xi_{h,l}$ and $\sigma_{h,l}$ were denoted $\xi_{h,l}$

$$\begin{array}{ccc} S^3 & \hookrightarrow & M_{h,l} \\ & & \downarrow \\ & & S^4 \end{array}$$

For any smooth manifold M, we denote the tangent bundle $\tau(M)$ and the total space of the tangent bundle TM. For any smooth manifold M, we denote the trivial real vector bundle of rank n over M by ε^n

$$\begin{array}{ccc} \mathbb{R}^n & \hookrightarrow & \mathbb{R}^n \times M \\ & & \downarrow \\ & & M \end{array}$$

Also, we can pull back $\xi_{h,l}$ under π to get a new vector bundle $\pi^*(\xi_{h,l})$. We can pull the tangent bundle back in the same way to get a new bundle $\pi^*(\tau(S^4))$. This is a real vector bundle of rank 4 over $N_{h,l}$.
We have the following isomorphism between the two rank 8 real vector bundles over $N_{h,l}$:

$$\tau\left(N_{h,l}\right) \cong \pi^*\left(\xi_{h,l}\right) \oplus \pi^*\left(\tau(S^4)\right).$$

Noting that $\tau(S^4) \oplus \varepsilon^1 \cong \varepsilon^5$ as vector bundles over S^4, we have:

$$\begin{aligned}\tau\left(N_{h,l}\right) \oplus \varepsilon^1 &\cong \pi^*\left(\xi_{h,l}\right) \oplus \pi^*\left(\tau(S^4)\right) \oplus \varepsilon^1 \\ &\cong \pi^*\left(\xi_{h,l}\right) \oplus \pi^*\left(\tau(S^4) \oplus \varepsilon^1\right) \\ &\cong \pi^*\left(\xi_{h,l}\right) \oplus \pi^*\left(\varepsilon^5\right) \\ &\cong \pi^*\left(\xi_{h,l}\right) \oplus \varepsilon^5\end{aligned}$$

Now we apply the following lemma,

Lemma 6.30. *For any real vector bundle ξ over a manifold X and for any $n \in \mathbb{N}$, we have*

$$\kappa(\xi) = \kappa(\xi \oplus \varepsilon^n) \tag{6.31}$$

where κ is the Euler class, Stiefel-Whitney class, or Pontryagin class. If we consider complex vector bundles, this also holds for Chern classes.

We apply this theorem to $N_{h,l}$ and use the above decomposition:

$$\begin{aligned}p_1(N_{h,l}) &= p_1(\tau(N_{h,l})) \\ &= p_1\left(\tau\left(N_{h,l}\right) \oplus \varepsilon^1\right) \\ &= p_1\left(\pi^*\left(\xi_{h,l}\right) \oplus \varepsilon^5\right) \\ &= p_1\left(\pi^*\left(\xi_{h,l}\right)\right) \\ &= \pi^*\left(p_1\left(\xi_{h,l}\right)\right)\end{aligned}$$

In other words, the tangent bundle $\tau(N_{h,l})$ can be split into two 'directions', one of which is trivial. The Pontryagin class only depends on the nontrivial direction.

Now we use the fact that $\pi : N_{h,l} \to S^4$ is a homotopy equivalence to get the following induced isomorphism on the cohomology classes:

$$\pi^* : H^4(S^4, \mathbb{Z}) \to H^4(N_{h,l}, \mathbb{Z}) \tag{6.32}$$

Both are clearly isomorphic to $\mathbb{Z}$. Now we apply equation 6.29 to get

$$p_1(N_{h,l}) = \pi^*(p_1(\xi_{h,l})) = \pi^*(2(h-l)\alpha) = 2(h-l)\pi^*(\alpha) \tag{6.33}$$

where $\pi^*\alpha$ is the the generator of $H^4(N_{h,l}, \mathbb{Z})$. Now we look at the following injection:

$$i : N_{h,l} \longhookrightarrow K_{h,l}$$

and induces the following isomorphism on cohomology groups

$$i^* : H^4(K_{h,l}, \mathbb{Z}) \to H^4(N_{h,l}, \mathbb{Z}) \tag{6.34}$$

and

$$i^*(\tau(K_{h,l})) \cong \tau(N_{h,l}) \tag{6.35}$$

So now we can again apply the previous lemma about the first Pontryagin class to get

$$p_1(K_{h,l}) = 2(h-l)\beta \tag{6.36}$$

where β is a generator of $H^4(K_{h,l}, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$.

6.3. **Applying the Hirzebruch Signature Theorem.** The next big tool is the Hirzebruch Signature Theorem, a major result of the theory of characteristic classes, which is proved on p. 224 of [6]. It relates the signature of a manifold to the Pontryagin numbers of the manifold

Theorem 6.37. ***Hirzebruch Signature Theorem*** *Let M be a closed orientable smooth manifold of dimension 8, with signature $\tau(M)$. Then*

$$\tau(M) = \frac{1}{45}(7p_2(M) - p_1^2(M)) \tag{6.38}$$

Remark 6.39. Note that the numbers in this formula come from a computation of multiplicative sequences. The 7 has nothing to do with the fact that we are considering manifolds homeomorphic to 7-spheres!

It is a fact from linear algebra that any symmetric matrix is congruent to a matrix such that all of the diagonal entries are ± 1 or 0, and the off-diagonal entries are all 0. The sum of the diagonal terms is defined to be the signature. This quantity can be defined for any symmetric bilinear form, not just symmetric matrices. For example, consider the map that was discussed above

$$H^4(K_{h,l},\mathbb{Z}) \times H^4(K_{h,l},\mathbb{Z}) \to H^8(K_{h,l},\mathbb{Z}) \tag{6.40}$$

$$(\alpha,\beta) \mapsto \alpha \smile \beta \tag{6.41}$$

This is a symmetric bilinear form, so a signature can be defined on it. However, this map actually corresponds to a 1-by-1 matrix and therefore the signature must be ± 1.
Now we are ready to apply the Hirzebruch Signature Theorem to our manifold $K_{h,l}$. We have

$$\pm 1 = \frac{1}{45}(7p_2(K_{h,l}) - (\pm 2(h-l))^2) \tag{6.42}$$

We mod out by 7 to eliminate $p_2(K_{h,l})$.

$$3 = \pm 4(h-l)^2 (\bmod 7) \tag{6.43}$$

which simplifies to

$$(h-l)^2 = 1 (\bmod 7) \tag{6.44}$$

This is a necessary condition for $K_{h,l}$ to have a differentiable structure. Since a disk can always be given a differentiable structure and $N_{h,l}$ clearly has differentiable structure, the faulty assumption must have been that $M_{h,l}$ could be glued to S^7 smoothly. Therefore, $M_{h,l}$ is not diffeomorphic to S^7 if $(h-l)^2 \neq 1 (\bmod 7)$. So, $M_{0,1}$ and $M_{1,0}$, which are the base spaces of the standard quaternionic Hopf fibrations are both diffeomorphic to S^7, as they should be, but $M_{3,-2}$, for instance, is not.

7. Conclusion

To summarize, we have constructed a family of manifolds that are homeomorphic but not diffeomorphic to the 7-sphere. These are the simplest examples of exotic smooth structure known. The tool used to construct this manifold is the quaternionic Hopf fibration. The non-commutativity of the quaternions causes the set of possible S^3 bundles over S^4 to be classified by $\mathbb{Z} \oplus \mathbb{Z}$. This gives enough 'room' in the set of candidate manifolds for exotic structure to exist. The homeomorphism between the base spaces $M_{h,l}$ and S^7 was shown for $h + l = \pm 1$ by Morse Theory, and, in particular, Reeb's Theorem. The non-diffeomorphism for $(h-l)^2 \neq 1 (\mathrm{mod} 7)$ is shown by constructing a smooth manifold from $M_{h,l}$ and then using the Hirzebruch Signature Theorem to find a contradiction, in the above cases.

8. Further discussion of Exotic Structure

Here we list some more facts about exotic structure in general, to encourage the reader to further explore this subject. See [3] for more details.
(1) There are 28 distinct smooth structures on the 7-sphere and they form a group under connected sum.
(2) There are no exotic smooth structures on n-spheres for spheres of dimension $n = 1, 2, 3, 5, 6$.
(3) It is unknown whether or not there exist exotic smooth structures on the 4-sphere. However, it is known that there are either infinitely many exotic smooth structures on the 4-sphere or none. Proving which of these two alternative holds would resolve the smooth Poincare conjecture.
(4) $\mathbb{R}^4$ has uncountably many exotic smooth structures. No other Euclidean spaces have any exotic smooth structures.
(5) We can explicitly write down an equation for the exotic 7-spheres as the points in $\mathbb{C}^5$ satisfying

$$a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0 \tag{8.1}$$

See [1] for more details.

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