Well-foundedness of Countable Ordinals and the Hydra Game

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1 Abstract

An argument involving the Hydra game shows why $\mathbf{ACA}_0$ is insufficient for a theory of ordinals in which collections of ordinals are well-founded. This is then briefly explained in light of Gentzen-style proof theory.

2 Introduction

Reverse mathematics is the study of the strength of mathematical results, and poses questions of the form "What is the weakest system of axioms required to prove a given result?" Reverse mathematics exhibits many different logical systems; however, there are a few canonical ones, simply because many classical results in mathematics are equivalent to a member of this collection. Most of this introduction is based on Simpson's *Subsystems of Second Order Logic* [6].

This paper will not delve too deeply into the universe of systems of Reverse mathematics. Briefly, though a system of Reverse mathematics is a set-theoretic construct $M = (|M|, S_M, +_M, \cdot_M, 1_M, 0_M, 1_M, <_M)$, where $|M|$ is the natural numbers of the model M, S_M is the universe of sets of M, $+_M$ and $\cdot_M$ are the addition and multiplication operations of $|M|$, respectively, 0_M and 1_M are the additive and multiplicative identities of $|M|$, respectively, and $<_M$ is a binary operation on $|M|$. M always follows the basic axioms of arithmetic and the set-theoretic induction axiom.

The axioms of the various systems differ, though, (for the purposes of this paper) in which elements S_M is guaranteed to contain.[1] There are the three canonical systems we will be working with in this paper:

- $\mathbf{RCA}_0$, for *Recursive Comprehension*, specifies that S_M contains all recursive sets, that is the comprehension $\{x \in |M| \mid \phi(x)\}$ exists whenever ϕ is a recursive formula. $\mathbf{RCA}_0$ is sufficient to develop a theory of $\mathbb{R}$, a theory of complete separable metric spaces, and algebraic closures of countable fields.

[1] Some systems differ in the system of natural numbers used as well. In the case where $|M| = \mathbb{N}$, M is called an ω-model; when $|M| \neq \mathbb{N}$, M is a non-ω-model.

- $\mathbf{ACA}_0$, for *Arithmetical Comprehension*, specifies that S_M contains all arithmetical sets, that is the comprehension $\{x \in |M| \mid \phi(x)\}$ exists whenever ϕ is an arithmetical formula (that is, quantifying only over first-order free variables). Assuming $\mathbf{RCA}_0$, one can prove that $\mathbf{ACA}_0$ is equivalent to König's Lemma on trees, Bolzano-Weierstrass, and Arzela-Ascoli.
- $\mathbf{ATR}_0$, for *Arithmetical Transfinite Recursion*, which is $\mathbf{ACA}_0$ plus the existence of sets defined by transfinite recursion of an arithmetical formula over a well-order. Assuming $\mathbf{RCA}_0$, $\mathbf{ATR}_0$ is equivalent to the Cantor-Bendixson Theorem.

The system $\mathbf{ATR}_0$ is also equivalent to a statement about countable well-orders:

Theorem 2.1. *Under* $\mathbf{RCA}_0$*, the following are equivalent:*

1. $\mathbf{ATR}_0$
2. *Given any two α and β countable well-orderings, there exists an $f : \alpha \to \beta$ order preserving, injective, and mapping α to an initial segment of β, or a $g : \beta \to \alpha$ order preserving, injective, and mapping β to an initial segment of α.*

Because of this equivalence, Simpson has remarked that $\mathbf{ATR}_0$ is the weakest system containing a sensible theory of ordinals [6]. Hirst has proved a few other such equivalences supporting this claim, such as the equivalence of $\mathbf{ATR}_0$ to the Cantor Normal Form theorem and Sherman's inequality in [2] and to an ordinal division algorithm in [3].

However, there *are* a few weaker results about comparison of countable ordinals and countable ordinal arithmetic that hold in $\mathbf{ACA}_0$, and some that even hold in $\mathbf{RCA}_0$. These will be discussed in subsequent sections.

With this in mind, it is quite reasonable to claim that $\mathbf{ACA}_0$ is insufficient to develop a proper theory of ordinals, and natural to ask how insufficient $\mathbf{ACA}_0$ is. Perhaps, for instance, $\mathbf{ACA}_0$ is sufficient to prove that any (countable) collection of ordinals is well-founded. This paper will demonstrate that this is not the case; in fact, we will show that under $\mathbf{ACA}_0$, if the countable ordinals are well-founded, then $\mathbf{ACA}_0$ is consistent (that is, there exists a model of $\mathbf{ACA}_0$), and hence $\mathbf{ACA}_0$ cannot prove that collections of (countable) ordinals are well-founded. This will be done by showing that the well-foundedness of ordinals is sufficient over $\mathbf{ACA}_0$ to prove that the Hydra game is winnable, and then invoking a result about the strength of systems able to show that the Hydra game is winnable. (See Section 3 for a definition of the Hydra game.)

These results will then be explained in the context of Gentzen-style proof theory, and why these results would be expected given that the proof theoretic ordinal of $\mathbf{ACA}_0$ is ϵ_0.

Remark 2.2. Reverse mathematical ordinals are *not* the same as the ordinals of **ZF**; rather, under the Reverse mathematical systems, ordinals are codes for particular countable well-orders, usually encoded as Kleene's $\mathcal{O}$. When we say countable ordinals in this paper, we really mean *codes* for countable well-orderings.

3 The Hydra Game

We think of a hydra as a tree:

Definition 3.1 ($\mathbf{RCA}_0$)**.** A *hydra* is a finitely branching finite tree with an assigned root node. We call terminal nodes "heads".

In an imaginary fight with Hercules and a hydra, on Hercules' nth turn, he cuts off any head that he wishes. When he does so, the hydra spawns new heads in the following manner: If Hercules cut off a head immediately above root of the hydra, the hydra spawns no new heads for turn $n+1$. Otherwise, if head h was removed, navigate from h two nodes closer to the head, and call this new node p. Identify $T(p)$ with the subtree of the hydra with root node p. Then the hydra at turn $n+1$ will have n additional copies of $T(p)$ starting at node p.

Hercules wins the Hydra game if, given a Hydra and some strategy, Hercules reduces the Hydra to the trivial tree (having only its root node) after finitely many steps. Classically, Hercules will always succeed:

Theorem 3.2. *Under* $\mathbf{ZF}$*, for every Hydra, every strategy is a winning strategy.*

Proof strategy. Each Hydra is assigned an ordinal in such a way that cutting off a head strictly reduces the associated ordinal. Then playing out a Hydra game results in a smallest such ordinal, which must be 0. □

This paper will develop the necessary definitions and theorems to prove this in $\mathbf{ZF}$; actually, this result will be proven in the weaker system $\mathbf{ACA}_0 + WF(COrd)$, where $WF(COrd)$ is the statement that every collection of countable well-orders is well-founded. Assigning hydras to ordinals will require a development of the theory of countable ordinals in $\mathbf{ACA}_0$.

4 Basic Ordinal Arithmetic

Classically, given two ordinals α and β, we say that $\alpha \leq \beta$ if α is isomorphic to an initial segment of β. However, there is another notion of ordering that is useful in weaker Reverse mathematical systems [4].

The classical definition of ordering on ordinals pulls neatly into $\mathbf{RCA}_0$:

Definition 4.1 ($\mathbf{RCA}_0$)**.** Let α and β be countable ordinals. Then α is strongly less than or equal to β, written $\alpha \leq_s \beta$, if there exists an order preserving injection $f : \alpha \to \beta$ that maps α onto an initial segment of β.

There is, however, another notion that is more useful when working in systems weaker that $\mathbf{ATR}_0$:

Definition 4.2 ($\mathbf{RCA}_0$)**.** Let α and β be countable ordinals. Then α is weakly less than or equal to β, written $\alpha \leq_w \beta$, if there exists an order preserving injection $f : \alpha \to \beta$.

The following is then clear:

Theorem 4.3 ($\mathbf{RCA}_0$). *$\alpha \leq_s \beta \implies \alpha \leq_w \beta$.*

The following is slightly less clear:

Theorem 4.4 ($\mathbf{ATR}_0$). *$\alpha \leq_w \beta \implies \alpha \leq_s \beta$.*

Proof sketch. Given an order preserving injection $f : \alpha \to \beta$, use arithmetical transfinite recursion to build a bijection g from α to an initial segment of β by "pulling down" elements of α that map "too far over". □

Thankfully, $\mathbf{ACA}_0$, and even $\mathbf{RCA}_0$, are sufficient to encapsulate some important behavior of ordinals. For instance:

Theorem 4.5 ($\mathbf{RCA}_0$). *Let α and β be ordinals. Then the usual definitions of $\alpha + \beta$ and $\alpha \cdot \beta$ are well-defined and ordinals [3].*

5 Ordinal Addition

We will need the following theorem:

Theorem 5.1. *Let α, β, and γ be countable ordinals where $\beta <_w \gamma$. Then $\alpha + \beta <_w \alpha + \gamma$.*

This proof is done in two parts. First, it is shown that if $\beta \leq_w \gamma$, then $\alpha + \beta \leq \alpha + \gamma$. Then, it is shown that if $\alpha + \beta \equiv_w \alpha + \gamma$, then $\beta \equiv_w \gamma$. This proves Theorem 5.1.

Proof. Suppose that $\beta \leq_w \gamma$ with $f : \beta \to \gamma$ being an order preserving injection. Then the map $f' : \alpha + \beta \to \alpha + \gamma$ defined by $f'(a) = a$ whenever $a \in \alpha$ and $f'(x) = f(x)$ otherwise is an order preserving injection from $\alpha + \beta$ to $\alpha + \gamma$.

Then, suppose that $\alpha+\beta \equiv_w \alpha+\gamma$ with $f : \alpha+\beta \to \alpha+\gamma$ and $g : \alpha+\gamma \to \alpha+\beta$ being order preserving injections, and suppose that the set $S = \{x \in \beta \mid f(x) \in \alpha\}$ (which exists by arithmetical comprehension) is nonempty. Then S has a β-least element b.

We now have two cases. If b is not the smallest element of β, then for some $b' < b$, $f(b') > f(b)$ which contradicts that f is order preserving. If b is the smallest element of β, then since f is order preserving and injective, $f(b)$ must be the greatest element of α. But then $f(f(b)) = f(b)$, so f is not injective, which contradicts the injectivity of f.

Therefore, S is empty, and hence the image of β under f is contained in γ. Thus, f restricted to β witnesses $\beta \leq_w \gamma$.

By the same argument, g restricted to γ witnesses $\gamma \leq_w \beta$.

Therefore, $\beta \equiv_w \gamma$ as desired. □

6 Ordinal Multiplication

The following draws heavily from [3].

For the classical proof of the Hydra game's winnability, ordinal multiplication must satisfy some regular properties. Thankfully, ordinal multiplication satisfies the most basic monotonicity in $\mathbf{RCA}_0$.

Theorem 6.1 ($\mathbf{RCA}_0$). *If α, β, γ, and δ are countable well-orderings such that $\alpha \leq_w \gamma$ and $\beta \leq_w \delta$, then $\alpha\gamma \leq_w \beta\delta$.*

7 Ordinal exponentiation

The classical definition of ordinal exponentiation pulls cleanly down to $\mathbf{RCA}_0$:

Definition 7.1 ($\mathbf{RCA}_0$). Let α and β be countable well-orders. Then α^β is the structure given by the set S of all finite sequences $(b_0, a_0), \ldots, (b_n, a_n)$ with $a_i \neq 0$ for all applicable i and $k < l \implies b_k >_\beta b_l$. Order is defined as the lexicographic ordering.

Several nice properties of exponentiation can be proven in $\mathbf{RCA}_0$:

Theorem 7.2. *Let α, β, and γ be countable well-orderings. Then*

1. $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
2. $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$
3. *If α, β, and γ are nonzero and $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$ and $\gamma^\alpha \leq \gamma^\beta$. This holds whether $\leq$ is strong or weak.*

However, $\mathbf{ACA}_0$ is required to show that ordinal exponentiation produces ordinals. These are Definitions 2.1 and 2.2 and Theorems 2.3, 2.4, and 2.6 in [2]:

Theorem 7.3 ($\mathbf{RCA}_0$). *The following are equivalent:*

1. $\mathbf{ACA}_0$
2. *If α and β are countable well-orders, then so is α^β.*

We will need that ordinals are strictly monotonic under exponentiation. Thankfully, this holds in $\mathbf{ACA}_0$:

Theorem 7.4 ($\mathbf{ACA}_0$). *If α and β are countable well-orders, then $\alpha \leq_w \beta$ if and only if $\omega^\alpha \leq_w \omega^\beta$.*

Proof. This is Lemma 4.3 in [2]. □

Corollary 7.5 ($\mathbf{ACA}_0$). *If α and β are countable well orders with $\alpha <_w \beta$, then $\omega^\alpha <_w \omega^\beta$.*

Proof. If $\alpha <_w \beta$, then $\omega^\alpha \leq_w \omega^\beta$. To see that this inequality is strict, if it happens that $\omega^\alpha \equiv_w \omega^\beta$, then by the preceding theorem $\alpha \equiv_w \beta$, which contradicts $\alpha <_w \beta$. □

Some ordinals have a special property that gives a weakened version of strict monotonicity of right addition.

Definition 7.6 ($\mathbf{RCA}_0$)**.** An ordinal α is said to be *indecomposable* if for all final segments β of α, $\alpha \leq_w \beta$.

Theorem 7.7 ($\mathbf{ACA}_0$)**.** *Suppose that α, β, and γ are countable ordinals with α, $\beta \neq 0$, $\alpha, \beta \geq_w \gamma$, and $\alpha >_w \beta$. Then $\omega^\alpha + \gamma >_w \omega^\beta + \gamma$.*

Proof. By Theorem , $\omega^\alpha + \gamma \geq_w \omega^\beta + \gamma$.

Now, suppose $\omega^\alpha + \gamma \equiv_w \omega^\beta + \gamma$. Then let $f : \omega^\alpha + \gamma \to \omega^\beta + \gamma$ be an order preserving injection. If $f|_{\omega^\alpha}$ maps into ω^β, then $\omega^\alpha \leq_w \omega^\beta$. Else, f maps a final segment of ω^α into γ. By Corollary 3.6 of [2], ω^α is indecomposable, and hence $\omega^\alpha \leq_w \gamma$. So $\omega^\alpha \leq_w \omega^\beta$. By a symmetric argument, $\omega^\beta \leq_w \omega^\alpha$. Hence, $\omega^\alpha \equiv_w \omega^\beta$. □

We will also need one final result:

Theorem 7.8 ($\mathbf{ACA}_0$)**.** *Let δ be a countable ordinal, and let n be a finite ordinal. Then $\omega^{\delta+1} >_w \omega^\delta n$.*

Proof. $\omega^{\delta+1} \equiv_w \omega^\delta \omega$, and by Theorem 13.2 of [3], $\omega^\delta \omega >_w \omega^\delta n$. □

8 Ordinal Arithmetic and the Hydra Game

This section takes place in the context of $\mathbf{ACA}_0$. The particular encoding scheme is inspired by [1] and [5].

Given a hydra, we wish to encode it into an ordinal. We do this as follows: Given a hydra T, we begin by assigning the ordinal 0 to each head. Then for each node N on the tree with child nodes $C_1, \ldots, C_k$ with assigned ordinals $\alpha_1 \geq_w \cdots \geq_w \alpha_k$, we assign the ordinal $\sum_{i=1}^k \omega^{r_i}$ to node N. Since the tree is finite, this process will eventually assign a unique ordinal α to the root node. We then associate the hydra T with the ordinal α and, for convenience, will denote $\alpha := o(T)$.

Theorem 8.1. *Any move in the hydra game reduces the associated ordinal. That is, if T and T' are hydras, associated to ordinals $o(T)$ and $o(T')$, respectively, with T' being the result of cutting a head off of T at stage n of the game. then $o(T') <_w o(T)$.*

Proof. If $T \to_n T'$ is cutting off a node at the root, then $o(T) = \alpha + 1$ for some ordinal α and $o'(T) = \alpha$, and clearly $o(T') <_w o(T)$.

Otherwise, we will examine the subtree S that consists of all branches from the node below which the head h in T is cut off to produce T'. We will call S

without h by the name V, and the tree above the root node of V by V'. Then $o(S) = \omega^{o(V')+1}$ and $o(V) = \omega^{o(V)}n$. By Theorem 7.8, $o(S) <_w o(V)$.

Then, by Theorems 5.1, 7.7, and 7.5, $o(T') <_w o(T)$. □

Theorem 8.2. *Under* $\mathbf{ACA}_0 + WF(COrd)$*, the hydra game is always winnable in finitely many steps.*

Proof. Let T be a hydra, σ be a strategy within $\mathbf{ACA}_0$, and consider the collection A of all countable ordinals produced applying σ to T over finitely many steps. By Well-foundedness of Countable Ordinals, A has a minimal member ρ. If $\rho \neq 0$, then the hydra R associated to ρ has a head, and thus for the hydra R' produced by applying strategy σ at step n to R, $\rho >_w o(R')$ by Theorem 8.1. This contradicts minimality of ρ, and thus $\rho = 0$. □

9 Consistency Results

Now that we know that the Hydra game, within $\mathbf{ACA}_0 + WF(COrd)$, is always winnable, we are poised to show that $WF(COrd)$ is independent of $\mathbf{ACA}_0$.

Kirby and Paris [5] showed the following result:

Lemma 9.1 ($\mathbf{ACA}_0$)**.** *If the hydra game is winnable, then the first-order system* **PA** *is consistent, meaning that there exists a model* $M \models \mathbf{PA}$.

From there, a conservation result on models from [6] translates this into a statement about $\mathbf{ACA}_0$:

Lemma 9.2 ($\mathbf{ACA}_0$)**.** **PA** *is the first order part of* $\mathbf{ACA}_0$*; that is, given any model* M *of* $\mathbf{ACA}_0$, $M \models \mathbf{PA}$*, and given a model* $M \models \mathbf{PA}$*, there exists a model* M' *with* $M \subseteq_\omega M'$ *and* $M' \models \mathbf{ACA}_0$.

Proof. Theorem IX.1.5 and Corollary IX.1.6 in [6]. □

Corollary 9.3 ($\mathbf{ACA}_0$)**.** *If the Hydra game is winnable, then* $\mathbf{ACA}_0$ *is consistent.*

Proof. Apply Lemma 9.1, and then invoke Lemma 9.2. □

A simple application of Gödel's Incompleteness Theorem then gives the desired result:

Theorem 9.4 (Independence of $WF(COrd)$ from $\mathbf{ACA}_0$)**.** $WF(COrd)$ *is independent of* $\mathbf{ACA}_0$.

Proof. $WF(COrd)$, by Theorem 9.3, sufficient to prove $Con(\mathbf{ACA}_0)$. Gödel's Incompleteness Theorem then gives that $WF(COrd)$ is independent of $\mathbf{ACA}_0$. □

Remark 9.5. In a sense, it is reasonable to suspect that ordinals are not necessarily well-founded in $\mathbf{ACA}_0$. The construction of hydras and their associations with ordinals gives a collection of ordinals S with a (classical) supremum of ϵ_0. Then the well-foundedness of S would imply the well-foundedness of ϵ_0, which by a classical result of Gentzen, is the smallest ordinal that $\mathbf{ACA}_0$ cannot prove is well-founded.

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