

OPERATIONS AND METHODS IN FUZZY LOGIC SYSTEMS

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ABSTRACT. This paper looks to introduce the reader to the idea of a fuzzy logic system and how this type of system differs from traditional propositional logic systems by way of summarizing the foundational ideas presented in Hájek's *Metamathematics of Fuzzy Logic*. Because a fuzzy system has different laws from a standard propositional one, propositional logic operations such as conjunction and implication need to have different meanings in a fuzzy system. By examining these operations, we can discover some differences between the two. One such difference we will discover is the failure of the deduction theorem as well as issues of completeness.

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1. WHAT IS A FUZZY LOGIC SYSTEM?

Traditional propositional logic systems are ways to translate English sentences into formulas with precise rules about formation and syntax. These formulas are comprised of series of symbols in order to denote ideas and the relationships between them; formulas can also be comprised of other formulas. The most basic unit of a logic system is a propositional symbol. Therefore:

Definition 1.1. Formulas are the smallest sets of strings satisfying the following conditions:

- (1) If s is a propositional symbol then s is a formula.
- (2) If φ and ψ are formulas then so are $(\varphi \& \psi)$ and $(\varphi \rightarrow \psi)$.

We evaluate these symbols and the formulas they comprise by assigning them a number in $\{0, 1\}$. Propositional logic provides us with rules that allow the truth value of formulas to be calculated quickly and easily. In fuzzy logic we instead have:

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Definition 1.2. An evaluation e is a map which associates to each formula φ a value in $[0,1]$, and which satisfies:

- (1) $e((\varphi \& \psi)) = e(\varphi) * e(\psi)$
- (2) $e((\varphi \rightarrow \psi)) = e(\varphi) \Rightarrow e(\psi)$
- (3) $e(0) = 0$

where $*$ and \Rightarrow will be defined later. For now, though note that $\&$ corresponds with $*$ and \rightarrow corresponds with \Rightarrow .

Therefore, \rightarrow and $\&$ are operations on formulas and variables while \Rightarrow and $*$ are operations on values of those formulas and variables.

These systems are foundational ideas in both mathematics and computer science. Throughout this paper we will follow Hájek's exposition on fuzzy logic, using many of his definitions and ideas as foundations. Hájek notes that fuzzy systems differ by discarding the traditional binary system in favor of a spectrum. The most important distinction between traditional and fuzzy systems is that truth values may take on any value between and including 0 and 1 in a fuzzy system. From this basic fact follow many rules and rule changes which will comprise the bulk of this paper. For now, though, we shall adapt Hájek's definition to say:

A fuzzy system is a logical system whose elements may take truth values in the interval $[0,1]$.

2. WHY ARE FUZZY SYSTEMS IMPORTANT?

One important distinction to make is that fuzzy logic is NOT probability. Although both employ values between 0 and 1 that represent something about the symbol or event, it is the meaning of this number that differs. In probability, the number represents the likelihood of an event's occurrence. In fuzzy logic, the number represents a truth value, or the extent to which a formula is true. In accordance with Hájek, we can evaluate how true symbols are by assigning to each symbol some value in $[0,1]$. We let the function $e(\varphi)$ assign a truth value to some φ .

Remark 2.1. If $e(\varphi) = 0.25$ this does NOT mean that φ is true one-quarter of the time. Instead, it means that ψ is one-quarter true.

Fuzzy logic is applied heavily in machine learning and automated systems. One important example of the use in fuzzy logic is the rapid visual screening procedure (RSVP) used to identify buildings at risk of damage from earthquakes. According to [5], multiple parameters of a building are evaluated based on how they affect the structural system. Traditionally, yes/no decisions must be made quickly and compiled in order to determine if a building needs further investigation or is at risk. Using fuzzy logic, however, allows interactions between parameters to affect the screening of the building. Now, instead of only a yes or no, 1 or 0 answer, there is a spectrum of possibilities. Additionally, this makes sense to employ, because some parameters might be more dangerous than others, and some parameters might only be dangerous if combined with others.

A further application of fuzzy logic is used in edge detection in images. To simplify our discussion of the idea, let us observe a grey-scale image. Analyzing this image, the fuzzy system evaluates how dark the grey is in relation to other grey pixels around it. Then, using a set of fuzzy rules comparing four different values based on the darkness of the greys, a formula gives the "degree of membership" of that pixel. We can think of this as "how much is this pixel a member of one

group as opposed to an adjacent group?” thus establishing a boundary between these groups.

3. FUZZY OPERATIONS

We begin our reconnaissance of fuzzy systems with an introduction to one of the most fundamental aspects of any system: the t-norm. Hájek writes:

Definition 3.1. A triangular norm (which we will henceforth refer to as a t-norm) is defined to be a function $*$: $[0, 1]^2 \rightarrow [0, 1]$ satisfying:

i) $*$ is commutative and associative or $\forall x, y, z \in [0, 1]$

$$x * y = y * x$$

$$(x * y) * z = x * (y * z)$$

ii) $*$ is non-decreasing in both arguments or

$$x_1 \leq x_2 \text{ implies } x_1 * y \leq x_2 * y$$

$$y_1 \leq y_2 \text{ implies } x * y_1 \leq x * y_2$$

iii) $\forall x \in [0, 1]$ $1 * x = x$ and $0 * x = 0$

A t-norm is an operation $*$ which corresponds to the logical operation ‘&’, which one may read in the traditional sense of “and”. Three examples of t-norms are:

i) Lukasiewicz t-norm: $x * y = \max(0, x + y - 1)$

ii) Gödel t-norm: $x * y = \min(x, y)$

iii) Product (Goguen) t-norm: $x * y = x \cdot y$

Looking at these different t-norms we can see how differently fuzzy systems can act. For instance, if $e(\varphi) = 0.9$ then $e(\varphi) * e(\varphi)$ is 0.8 in the Lukasiewicz t-norm, 0.9 in the Gödel t-norm, and 0.81 by the Product t-norm.

It is even more important to know what system and definitions one is using because each t-norm has a unique operation called a residuum. Hájek provides a lemma defining the residuum as follows:

Lemma 3.2. Each continuous t-norm has a unique function \Rightarrow : $[0, 1]^2 \rightarrow [0, 1]$ satisfying: $\forall x, y, z \in [0, 1]$, $(x * z) \leq y$ iff $z \leq (x \Rightarrow y)$, namely $x \Rightarrow y = \max\{z \mid x * z \leq y\}$. This operation \Rightarrow , called the residuum, also satisfies:

i) $x \leq y$ iff $(x \Rightarrow y) = 1$

ii) $(1 \Rightarrow x) = x$

This means that φ is slightly less true than ψ iff $\varphi \rightarrow \psi$ is completely true, and also that a completely true statement implying another statement is only as true as the latter statement.

Corresponding examples of residua are as follows:

i) Lukasiewicz implication: $x \Rightarrow y = 1 - x + y$

ii) Gödel implication: $x \Rightarrow y = y$

iii) Goguen implication: $x \Rightarrow y = y/x$

The third basic operation of fuzzy logic systems is negation, or “not”. Hájek calls these negations “precomplements” of the established t-norm. Corresponding examples of negation are as follows:

i) Lukasiewicz negation: $(-)x = 1 - x$

ii) Gödel negation: $(-)0 = 1, (-)x = 0$ for $x > 1$

iii) Goguen negation: $(-)0 = 1, (-)x = 0$ for $x > 1$

Evidently, two rather different t-norms can have the same precomplement as displayed by Gödel and Goguen negation.

4. MANY-VALUED PROPOSITIONAL LOGIC

For any given t-norm we may begin to form a method with which to create sentences and formulas and give meaning and purpose to our work thus far. Hájek outlines a propositional calculus as follows:

Definition 4.1. A propositional calculus given by a certain t-norm has $*$ taken as the truth function of strong conjunction, $\&$, and the residuum \Rightarrow of the t-norm is the truth function of the implication. Using only these two definitions we may begin to define other functions as the following:

$$\begin{aligned} \varphi \wedge \psi &\text{ is } \varphi \& (\varphi \rightarrow \psi) \\ \varphi \vee \psi &\text{ is } ((\varphi \rightarrow \psi) \rightarrow \psi) \& ((\psi \rightarrow \varphi) \rightarrow \psi) \\ \neg \varphi &\text{ is } \varphi \rightarrow 0 \\ \varphi \equiv \psi &\text{ is } (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi) \end{aligned}$$

Remark 4.2. These might seem like very complicated foundational ideas, but using as few concepts as possible in order to define more concepts will help us in the long run. The fewer things we have to define, the easier our system is to analyze.

We also have that, for any continuous t-norm and for any evaluation, the functions \wedge and \vee have the special properties of the following:

$$\begin{aligned} e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\ e(\varphi \vee \psi) &= \max(e(\varphi), e(\psi)) \end{aligned}$$

These facts lend names to their respective functions: min-conjunction and max-disjunction.

Remark 4.3. Now, however, we have two different conjunction operations. One important distinction to note is that $e(\varphi \& \psi) \leq e(\varphi \wedge \psi)$.

Since we have established the operations in our fuzzy system, we can start to establish some of the basic rules, called axioms. First, we must establish what an axiom is according to Hájek.

Definition 4.4. A formula φ is a tautology if $e(\varphi) = 1$ for every evaluation e within any possible t-norm.

Thus, as Hájek explains, a tautology is always absolutely true, regardless of how we evaluate the symbols comprising it. There is a group of tautologies to which we refer as axioms of fuzzy logic. The proof system of basic logic consists of the following axioms and one deduction rule.

Definition 4.5. Below are the axioms of basic logic according to Hájek:

- (A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (A2) $(\varphi \& \psi) \rightarrow \varphi$
- (A3) $(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
- (A4) $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi))$
- (A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
- (A5b) $((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
- (A7) $\bar{0} \rightarrow \varphi$ where $\bar{0}$ is the truth constant for 0.

The deduction rule of basic logic is modus ponens, meaning that, if given φ and $\varphi \rightarrow \psi$ one may deduce ψ .

As a quick proof, let us examine axiom 2.

Lemma 4.6. *A2 is a tautology.*

Proof. A2 states that $(\varphi \& \psi) \rightarrow \varphi$ so we want to show that this is always true, in order for it to be an axiom. We have defined that $e(\varphi \& \psi) = (e(\varphi) * e(\psi))$. We also know, from the definition of a t-norm, that $e(\varphi \& \psi) \leq e(\varphi)$ because a t-norm must be non-decreasing. Then, by the definition of a residuum, we have that $e((\varphi \& \psi) \rightarrow \varphi) = 1$ since the first term has smaller truth value than the larger term. \square

We can examine axiom 3 in a similar way.

Lemma 4.7. *A3 is a tautology.*

Proof. By the definition of the t-norm, we know that it is commutative. This means $e(\varphi \& \psi) = e(\psi \& \varphi)$. Thus we know that, by the definition of the residuum, $e((\varphi \& \psi) \rightarrow (\psi \& \varphi)) = 1$ since the first term is smaller than or equal to the second. \square

We also note that A7 is “vacuously true”, because the premise or assumption is false. Any implication with a false assumption is automatically vacuously true. Hájek provides some conceptual explanation for these explanations as well. (A1) displays that implication is transitive. (A2) illustrates that $\&$ -conjunction implies the first element. (A3) displays the commutativity of $\&$ -conjunction. (A4) shows that \wedge -conjunction is commutative. (A5) illustrates how chained residuation corresponds to conjunction. (A6) depicts a version of proof by cases, namely, if χ follows from $\varphi \rightarrow \psi$, then χ following from $\psi \rightarrow \varphi$ implies χ . Lastly, (A7) shows vacuous truth as above.

Using the axioms, Hájek explains that there is a set of formulas that are provable from the axioms of basic logic. We will list some of these formulas below.

- (1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi(\varphi \rightarrow \chi))$
- (3) $\varphi \rightarrow \varphi$
- (4) $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow \psi$
- (5) $\varphi \rightarrow (\psi \rightarrow (\varphi \& \psi))$
- (6) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \& \chi) \rightarrow (\psi \& \chi))$
- (7) $((\varphi_1 \rightarrow \psi_1) \& (\varphi_2 \rightarrow \psi_2)) \rightarrow (\varphi_1 \& \varphi_2) \rightarrow (\psi_1 \& \psi_2)$
- (8) $(\varphi \& \psi) \& \chi \rightarrow \varphi \& (\psi \& \chi), (\varphi \& (\psi \& \chi)) \rightarrow ((\varphi \& \psi) \& \chi)$
- (9) $\varphi \rightarrow (\neg \varphi \rightarrow \psi)$, in particular, $\varphi \rightarrow \neg \neg \varphi$ and $(\varphi \& \neg \varphi) \rightarrow 0$.
- (10) $(\varphi \rightarrow (\psi \& \neg \psi)) \rightarrow \neg \varphi$
- (10') $(\varphi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \varphi)$
- (10'') $(\varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \neg \varphi)$

5. THEORIES, PROVABILITY, AND DEDUCTION

Now that we have the proper building blocks and baselines of fuzzy logic systems, we can finally start doing what we set out to do: proving things. Using operations,

axioms, and modus ponens, we can begin to create proofs in our different fuzzy systems. To do so, we must have a few more definitions.

Definition 5.1. Hájek defines a theory over basic logic as a set of formulas. A proof in a theory T is a sequence of formulas that are comprised of either axioms of basic logic, special (given or assumed) axioms of T , or follow from a preceding member using modus ponens. $T \vdash \varphi$ means that φ is provable in T (it is the last member of a proof in T).

Interestingly, some t-norms will believe formulas to be true even though they are not provable in basic logic. For instance, take the Gödel t-norm. Using the Gödel t-norm, every evaluation e satisfies $e(\varphi \rightarrow (\varphi \& \varphi)) = 1$ but we cannot prove these formulas in basic logic.

Hájek also explains a variation on the deduction theorem as the following:

Theorem 5.2. *Let T be a theory and φ, ψ be formulas. $T \cup \{\varphi\} \vdash \psi$ iff $\exists n$ s.t. $T \vdash \varphi^n \rightarrow \psi$ (where $\varphi^n = \varphi \& \dots \& \varphi$, n times)*

Proof. By commutativity and associativity of $\&$ we can place brackets in any way we want around $\varphi \& \dots \& \varphi$. If $n \geq 1$ and $T \vdash \varphi^n \rightarrow \psi$ then $T \vdash \varphi \& \varphi^{n-1} \rightarrow \psi$, and by (A5b) $T \vdash \varphi \rightarrow (\varphi^{n-1} \rightarrow \psi)$ so $T \cup \{\varphi\} \vdash \varphi^{n-1} \rightarrow \psi$. Using this we get $T \cup \{\varphi\} \vdash \varphi \rightarrow \psi$ and thus $T \cup \{\varphi\} \vdash \psi$.

Conversely, Hájek shows if we assume $T \cup \{\varphi\} \vdash \psi$ and $\gamma_1, \dots, \gamma_k$ are a $T \cup \{\varphi\}$ -proof of ψ , we can prove by induction that, for each $j = 1, \dots, k$, there is an n_j s.t. $T \vdash \varphi^{n_j} \rightarrow \gamma_j$. This clearly holds if γ_j is an axiom of basic logic or of our theory $T \cup \{\varphi\}$. γ_j could also be a modus ponens result from previous members $\gamma_i, \gamma_i \rightarrow \gamma_j$, which means that, by the induction hypothesis, we can assume $T \vdash \varphi^{n_i} \rightarrow \gamma_i$, $T \vdash \varphi^{n_j} \rightarrow (\gamma_i \rightarrow \gamma_j)$. So by formula 7 above of strong-conjunction, we know $T \vdash \varphi^{n_i \& n_j} \rightarrow (\gamma_i \& (\gamma_i \rightarrow \gamma_j))$, which means $T \vdash \varphi^{n_i + n_j} \rightarrow \gamma_j$, completing the proof. \square

Remark 5.3. Here with the deduction theorem is where fuzzy logic encounters its first major difference with traditional logic. The traditional deduction theorem is similar, but the small difference is a very important one. The traditional theorem states $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \varphi \rightarrow \psi$. Notice that the multiple strong-conjunction is absent in this case (the fuzzy deduction theorem could also employ implication instead of strong-conjunction, giving an analog to the deduction theorem because this proof would provide a series of implications finally leading to a conclusion. This is still different, however, from the traditional deduction theorem). While, beforehand, operations and definitions have only been slightly different, but still functional, this is one concept that does not hold without considerable variation. Essentially, we have changed very few things from traditional logic systems in order to create a fuzzy logic system. In spite of everything that is still in common, the lack of a fuzzy deduction theorem shows that fuzzy logic can lack some of the power that traditional logic holds. It is true, however, that any proof in our basic logic system does hold in traditional propositional logic.

One interesting result of the above theorem is that some fuzzy systems do have their own version of the deduction theorem. For instance, looking at the Gödel t-norm, if we add an axiom saying $\varphi \rightarrow (\varphi \& \varphi)$, then this system does have a deduction theorem.

Following the idea of the deduction theorem, Hájek defines a property of a theory called consistency as follows:

Definition 5.4. Hájek defines a theory T to be contradictory (or inconsistent) if $T \vdash 0$, otherwise it is consistent.

This definition reads much as one would expect. For a theory to be consistent it must not prove opposite things, nor things known to be completely false.

As a consequence of this, T is inconsistent iff $T \vdash \varphi$ for each φ (because $\exists \varphi \equiv 0$). Finally, if $T \cup \{\varphi\}$ is inconsistent, then there is an n s.t. $T \vdash \neg(\varphi^n)$.

6. COMPLETENESS

We will very briefly sample Hájek's work on an integral concept in this section. Completeness is essentially what makes a proof system or a theory usable. It is defined quite simply, but its impact is vast. Furthermore, while the idea is simple, the proof is not. Thus we will not spend time examining these ideas closely. Completeness shows that a proof system is as strong as possible. It is obviously desirable that anything true should be provable, and that everything provable should be true. Take, for instance, the tautologies. Given our proof system, we would like every tautology to be provable, and we would like that nothing else is provable. Following this, Hájek explains:

Definition 6.1. A logical system L is complete if every tautology of L can be proven in L .

Thus we have:

Theorem 6.2. *Basic logic is complete, or for each tautology φ of basic logic, φ is provable in basic logic.*

This explains that the collection of everything provable in basic logic is precisely the tautologies of basic logic. While this may seem incredibly simple, it is very difficult to prove, and will be left for another paper.

Along this vein, an additional Gödel-tautology is the formula $\varphi \rightarrow (\varphi \& \varphi)$ for any formula φ . Those formulas, though, are not provable in basic logic since they are not tautologies in basic logic. If we add these formulas as axioms to our system, then we have a new system with which every Gödel tautology can be proved. Otherwise, the tautologies in the Gödel logic system under the Gödel t-norm without the added axioms are not all provable in basic logic.

7. CONCLUSION

In this paper we have established that a fuzzy logic system is one with truth values between 0 and 1, not solely on the extremes. In order to work with this, we have shown that t-norms, corresponding residuations, and negation must be defined in such a way as to work with these values. With abundant help from Hájek, we have shown that any axiom or proof that holds in a traditional binary logic system also holds in a fuzzy system. It must for the system to make any sense. Fuzzy logic systems have their own axioms, some have more than others but all have the basic ones defined above. Because of these differences, we must make variations to the deduction theorem which claims that, if $T \vdash \varphi \rightarrow \psi$ then $T \cup \{\varphi\} \vdash \psi$. We have also referenced consistency and completeness in fuzzy systems, concluding our introduction to the topic.

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