HYPERBOLICITY AND THE WORD PROBLEM

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ABSTRACT. This paper addresses the word problem, i.e. the problem of determining which words in the generators of a group equal the identity. To do so, we will define the notion of hyperbolicity that requires geodesic triangles to be thin, and use this to define hyperbolic groups. We then present a proof of the fact that hyperbolic groups have solvable word problems through Dehn presentations, as well as the converse fact that groups admitting Dehn presentations are hyperbolic.

CONTENTS

1. INTRODUCTION

Definition 1.1. Let $\mathcal{A}$ be any set, which we will call our *alphabet*. A *word* is a finite sequence of elements of $\mathcal{A}$.

Consider, as our alphabet, the set of generators $\mathcal{A}$ of some finitely presented group Γ, and view concatenation of letters as multiplication. It is clear that every word is an element of Γ, and every element of Γ can be represented by many words: for instance, for an abelian group with generators a and b, the words aa and $abab^{-1}$ both represent the same element, but are distinct words, even when freely reduced. If two words v and w represent the same element of Γ, we will denote this by $v =_\Gamma w$. If a word $w =_\Gamma 1$, we will call such a word null homotopic. We will also denote the length of a word w by $|w|$. Given an arbitrary word w, is there an algorithm to determine whether it is null homotopic? We will refer to this question as the word problem, and if such an algorithm exists for a group, then we will say that the group has a solvable word problem.

We will discuss a particularly simple algorithm, known as a Dehn algorithm, that will efficiently solve a group's word problem when this algorithm exists. The groups for which we can find a Dehn algorithm turn out to be precisely those that

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are hyperbolic groups, where we define a metric space to be hyperbolic if all geodesic triangles in it are thin in a particular sense, and a group is hyperbolic if its Cayley graph is a hyperbolic metric space.

In this paper, we will start by introducing Dehn functions and the Dehn algorithm, our solution to the word problem. We will then shift gears to define our notion of hyperbolicity in metric spaces and groups. Finally, we will give a proof that a group has a solvable word problem via Dehn algorithm if and only if it is a hyperbolic group.

We assume the reader is familiar with basic group theory, but don't expect prior exposure to any of the geometric aspects of this paper. Armstrong's book [2] provides sufficient background.

The argument presented here is primarily adapted from Bridson's paper [4].

2. Dehn Functions

Perhaps the most obvious approach to attacking the word problem is to simply replace relators in a word w with the identity until the entire word is shown to be the identity. This solution will be effective if we can bound the number of relators we need to replace in order to determine whether or not $w =_\Gamma 1$ as a function of $|w|$. To quantify this, we will work in the free group generated by $\mathcal{A}$, denoted by $F(\mathcal{A})$, and we will denote equality in $F(\mathcal{A})$ with the symbol $\overset{\text{free}}{=}$.

Definition 2.1. Given a finite presentation $P \equiv \langle \mathcal{A} \mid \mathcal{R} \rangle$ for a group Γ and a null homotopic word w in the letters $\mathcal{A}^{\pm 1}$, we define the *algebraic area* of w to be

$$\text{Area}(w) := \min \left\{ N \mid w \overset{\text{free}}{=} \prod_{i=1}^{N} x_i^{-1} r_i x_i \text{ with } x_i \in F(\mathcal{A}), r_i \in \mathcal{R}^{\pm 1} \right\}$$

To get a sense for what this definition means, we'll show how to go begin calculating it in a specific case. Hopefully this example shows how we can think of Area as denoting the number of times we need to use information from our relators to arrive at the identity when testing if a word is null homotopic.

Example 2.2. Let Γ be a Baumslag-Solitar group: particularly,

$$\Gamma = \text{BS}(3,2) := \langle x, y \mid x^{-1}y^3xy^{-2} \rangle,$$

and define a word $w := xy^{-2}xy^2x^{-1}y^{-1}x^{-1}$. Perhaps you notice that we can rewrite this word as

$$w = (xy)(y^{-3}xy^2x^{-1})(xy)^{-1}.$$

We can remove the letters in the middle parentheses by applying our relator $r^{-1} = y^2x^{-1}y^{-3}x$ once, even though that word is not precisely our relator. Rewrite

$$r^{-1} = (y^2x^{-1})(y^{-3}xy^2x^{-1})(xy^{-2}),$$

where we will define $x_1 := y^2x^{-1}$ to match with our definition of Area. Thus, we have

$$w = (xy)(y^{-3}xy^2x^{-1})(xy)^{-1} \overset{\text{free}}{=} (xy)x_1^{-1}r^{-1}x_1(xy)^{-1} = 1,$$

and so the Area of this word is 1. In longer words, we would repeatedly attempt to reduce subwords and use the fact that

$$\text{Area}(ww') \leq \text{Area}(w) + \text{Area}(w').$$

It is also worth pointing out that this complexity is relative to the presentation we pick. If w is a null homotopic word, then there exist presentations in which w is a relation. Relative to such a presentation, $\text{Area}(w) = 1$.

Definition 2.3. The *Dehn function* of a finite presentation P is the function $\delta_P : \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_P(n) := \max\{\text{Area}(w) \mid w =_\Gamma 1, |w| \leq n\}$$

The Dehn function essentially tells us how difficult it is to determine if words of length $\leq n$ are null homotopic. Analyzing this function will tell us how much harder this question becomes as you increase the length of w.

3. Dehn Algorithms

We will now describe one efficient algorithm for solving the word problem. Take a finitely presented group Γ with generators $\mathcal{A}$. Suppose we can pick a finite list of pairs of words $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ such that $u_i =_\Gamma v_i$, $|v_i| < |u_i|$, and if w is a freely-reduced word in the letters $\mathcal{A}^{\pm 1}$ that is null homotopic, then w contains at least one of the u_i as a subword. To determine if a word w represents the identity, we proceed using the following simple algorithm.

Definition 3.1. A *Dehn algorithm* for a group Γ solves the word problem using the following steps:

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while w is not the trivial word do
  | freely reduce w
  | if some u_i is a subword of w then
  |   | replace that instance of u_i with v_i
  | else
  |   | return w ≠_Γ 1
  | end
end
return w =_Γ 1
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This algorithm will terminate in at most $|w|$ steps, as every time we replace a subword u_i with v_i, the resulting w is at least one letter shorter. Note that this proves the following proposition:

Proposition 3.2. *If a Dehn algorithm exists for a group Γ, then Γ has a linear Dehn function.*

The existence of a Dehn algorithm gives rise to a natural presentation.

Definition 3.3. We call $\langle \mathcal{A} \mid u_1 v_1^{-1}, \ldots, u_n v_n^{-1} \rangle$ a *Dehn presentation.*

Of course, this algorithm only works if we can pick appropriate words u_i and v_i, i.e. Γ has a finite Dehn presentation. Perhaps surprisingly, it turns out that a group admits a Dehn presentation if and only if it is hyperbolic. Before we can define hyperbolic groups, however, we need to talk about hyperbolic metric spaces.

4. Hyperbolic Metric Spaces

Consider a metric space (M, d).

Definition 4.1. Let $I \subset \mathbb{R}$ be an interval, and let (M, d) be a metric space. A *geodesic* is a path $\gamma : I \to M$ such that $d(\gamma(a), \gamma(b)) = |a - b|$ for all $a, b \in I$.

We will also need a local version of this notion:

Definition 4.2. For an interval I and metric space (M, d), a *k-local geodesic* is a path $\gamma : I \to M$ such that $d(\gamma(a), \gamma(b)) = |a-b|$ for all $a, b \in I$ such that $|a-b| \leq k$.

What we are calling a geodesic is sometimes referred to as a *unit speed geodesic*. We could perhaps talk about constant speed geodesics by putting a scaling factor in front, but the following notion will be more useful for our purposes.

Definition 4.3. Again taking an interval I and metric space (M, d), a (λ, ε)-*quasigeodesic* is a path $\gamma : I \to M$ such that

$$\frac{1}{\lambda}|a-b| - \varepsilon \leq d(\gamma(a), \gamma(b)) \leq \lambda|a-b| + \varepsilon$$

We can think of quasigeodesics as acting on a large scale like geodesics with a constant scaling factor λ, while exhibiting potentially wild behavior on small scales less than ε.

Definition 4.4. A metric space (M, d) is a *geodesic space* if any two points are connected by a geodesic.

For example, any graph with edges of unit length is a geodesic space. $\mathbb{R}^n$ with the Euclidean metric is a geodesic space, as is $\mathbb{H}^n$. However, $\mathbb{R}^2 \setminus \{0\}$ is not a geodesic metric space, since there is no geodesic connecting x to $-x$.

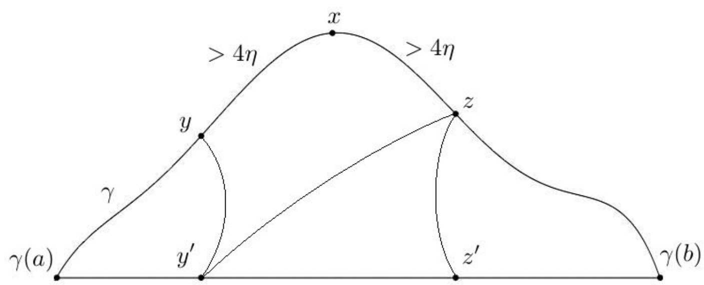
We can now define what it means for a metric space to be hyperbolic. Our condition involves geodesic triangles, which are simply triangles for which every edge is given by a geodesic.

Definition 4.5. A geodesic metric space M is *η-hyperbolic* if there exists a constant $\eta > 0$ such that for every geodesic triangle $\Delta \subset M$, each edge of Δ lies in the η-neighborhood of the union of the other two edges. We say M is hyperbolic if it is η-hyperbolic for some η.

Essentially, the requirement is that geodesic triangles are thin in the metric space. As we would hope, the hyperbolic plane $\mathbb{H}^2$ is $\log(\sqrt{2}+1)$-hyperbolic [1]. As another example, trees are 0-hyperbolic.

In general, local geodesics may not behave like geodesics globally. For instance, a long local geodesic might have close endpoints. However, in a hyperbolic metric space, local geodesics are close to geodesics.

Lemma 4.6. *Let M be an η-hyperbolic geodesic space and let $\gamma : [a, b] \to M$ be a k-local geodesic, where $k > 8\eta$. Then $\operatorname{im}(\gamma)$ is contained in the 2η-neighborhood of any geodesic segment $[\gamma(a), \gamma(b)]$.*



Proof. Pick a point $x = \gamma(t) \in \mathrm{im}(\gamma)$ that is at maximal distance from $[\gamma(a), \gamma(b)]$. First, suppose $(t-a) > 4\eta$ and $(b-t) > 4\eta$. Then, we can pick a subarc of γ with midpoint x and length strictly greater than 8η but less than k. Label the endpoints of this subarc as y and z, and let y' and z' be the points on $[\gamma(a), \gamma(b)]$ that are closest to y and z respectively. Now, consider a geodesic quadrilateral with vertices y, z, z', and y'. Since γ is a k-local geodesic, we know that $[y, z]$ is a geodesic, so we can use that as one side of our quadrilateral. Since $[\gamma(a), \gamma(b)]$ is a geodesic, we can obviously do the same for $[y', z']$.

Now, draw in a geodesic diagonal $[y', z]$. We now have a geodesic triangle $[y, y', z]$. Since M is η-hyperbolic, we know that there exists a point w on either $[y', z]$ or $[y', y]$ such that $d(w, x) \leq \eta$. Suppose that $w \in [y, y']$. Then, we have a path from x to y' through w that is shorter than $d(y, y')$:

$$\begin{aligned} d(x, y') - d(y, y') &\leq (d(x, w) + d(w, y')) - (d(y, w) + d(w, y')) \\ &= d(x, w) - d(y, w) \\ &\leq d(x, w) - (d(y, x) - d(x, w)) \\ &= 2d(x, w) - d(x, y) \\ &< 4\delta - 4\delta = 0 \end{aligned}$$

However, this contradicts our choice of x. Thus, $w \in [y', z]$.

Again, since M is η-hyperbolic, we know there exists a point w' on either $[z, z']$ or $[y', z']$ such that $d(w, w') \leq \eta$. By a similar argument, we can show that $w' \in [y', z']$, and so by the triangle inequality, we have that the distance from x to $[\gamma(a), \gamma(b)]$ is at most 2η.

Next, suppose that $(t-a) < 4\eta$, and consider the geodesic triangle $[\gamma(a), z, z']$, where we have again picked z such that $d(x, z) > 4\eta$. We know that $d(x, m) < \eta$ for $m \in [z, z']$ or $m \in [\gamma(a), z']$. Suppose $m \in [z, z']$. By the triangle inequality, we have that $d(m, z) > 3\eta$. Thus, we have

$$\begin{aligned} d(x, z') - d(z, z') &\leq (d(x, m) + d(m, z')) - (d(z, m) - d(m, z')) \\ &= d(x, m) - d(z, m) \\ &< \eta - 3\eta < 0 \end{aligned}$$

which contradicts our choices of x and z'. Thus, $m \in [y, z']$, and we're done with this case. The same argument works if we have $(b-t) < 4\eta$. Finally, if both $(t-a)$ and $(b-t)$ are less than 4η, then γ is a geodesic. □

Conceptually, it seems that local geodesics and quasigeodesics are almost opposites: having a local geodesic tells us that it behaves well locally but tells us nothing about how it behaves globally, while having a quasigeodesic describes the global behavior while ignoring the local behavior. Perhaps surprisingly, it turns out that in a hyperbolic metric space, many local geodesics are quasigeodesics.

Theorem 4.7. *If X is η-hyperbolic then every k-local geodesic in X for $k > 8\eta$ is a (λ, ε)-quasi-geodesic, where the constant $\lambda > 0$ depends only on η, and $\varepsilon < 8\eta$.*

Proof. Let $\gamma : [a, b] \to X$ be an 8η-local geodesic, where $a, b \in \mathbb{R}$. If a and b are within 8η of each other, then we can take $\lambda = 1$ and $\varepsilon = 0$, since γ is an 8η-local geodesic.

Now suppose that $|a - b| > 8\eta$. We will deal with the upper bound first. If we divide up the interval $[a, b] \subset \mathbb{R}$ into intervals of length 4η, the images under γ of these intervals also clearly have length 4η since γ is an 8η geodesic. By the triangle inequality, we can thus say that

$$d(\gamma(a), \gamma(b)) \leq \left\lfloor \frac{a-b}{4\eta} \right\rfloor 4\eta + \varepsilon \leq |a - b| + \varepsilon$$

for some $\varepsilon < 4\eta$.

Next, we deal with setting the lower bound. Let $\xi = [\gamma(a), \gamma(b)] \subset X$, i.e. a geodesic between the endpoints of γ, and ket $k' = k/2 + 2\eta$. Express γ as a concatenation of $M = \lfloor (b-a)/k' \rfloor$ geodesic segments of length k', perhaps the final segment of length l if $(b-a)/k'$ is not an integer; i.e.,

$$b - a = Mk' + l.$$

By Lemma 4.6 above, we can project each endpoint of each of these geodesic segments to points on ξ within 2η of the original endpoint. For now, assume that these projections form a monotone sequence. We will prove this shortly. The distance between successive projections must be at least $k' - 4\eta$, and the distance from the last projection point to $\gamma(b)$ is at least $l - 2\eta$. Thus,

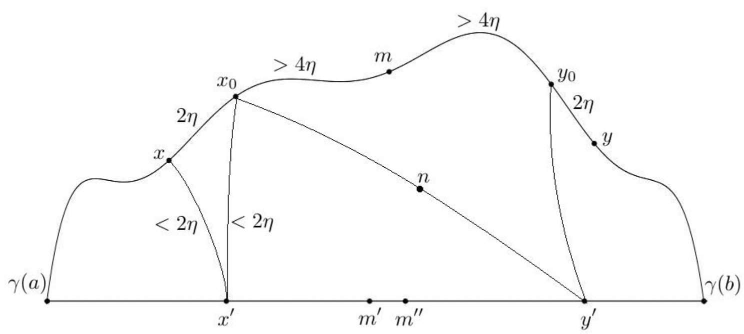
$$d(\gamma(a), \gamma(b)) \geq M(k' - 4\eta) + l - 2\eta = (b - a) - 4\eta M - 2\eta.$$

Since $M \leq (b - a)/k'$, we conclude

$$d(\gamma(a), \gamma(b)) \geq \frac{k' - 4\eta}{k'}(b - a) - 2\eta,$$

completing the proof.

Now, we will prove that the projections used above form a monotone sequence.



Consider a subarc of γ of length $2k'$, denoting its endpoints by x and y and its midpoint by m. Pick points x', y', and m' on ξ that are a distance at most 2η from x, y and m respectively, which we can pick by the above lemma. We will now show that m' must lie between x' and y' in order to show that the points we pick on ξ appear in the same order as they do on γ.

Let x_0 be the point on $\operatorname{im}(\gamma)$ that is 2η away from x in the direction of m, and pick y_0 analogously. Consider a geodesic triangle $[x, x', x_0]$. By definition, both x_0 and x' have distance at most 2η from x. Since this is a geodesic triangle in an η-hyperbolic space, we know that any point $p \in [x_0, x']$ is a distance at most η from some point on $[x, x']$, and so by the triangle inequality, every point on $[x_0, x']$ is 3η-close to x. We have thus shown that the entire triangle $[x, x', x_0]$ is in the 3η-neighborhood of x. Since $d(x, m) = k' > 6\eta$, we know that this triangle is also entirely outside of the 3η-neighborhood of m. We can get the same results for the geodesic triangle $[y, y', y_0]$.

Next, consider a geodesic triangle $[x_0, y_0, y']$. By hyperbolicity, we know that m is at most η away from some point n on either $[x_0, y']$ or $[y_0, y']$. Suppose $n \in [y_0, y']$. By the triangle inequality, we get that

$$d(m, y_0) \leq d(m, n) + d(n, y_0) \leq 3\eta.$$

However, we also know that along the 8η-local geodesic γ, a path of length at least 4η separates m and y_0, which is a contradiction. Thus, $n \in [x_0, y']$. We make a similar argument in a triangle $[x_0, y', x']$ to determine that a point m'' such that $d(n, m'') < \eta$ must lie on $[y', x']$, and thus we have that m lies within a distance of 2η from some point $m'' \in [x', y'] \subset [\gamma(a), \gamma(b)]$.

Finally, consider the triangle $[m, m', m'']$. Any point on $[m', m'']$ is a distance at most η from some point on either $[m, m']$ or $[m, m'']$, both of which have length at most 2η. Thus, any point between m' and m'' is 3η-close to m, which means that neither x' nor y' can lie between m' and m''. Thus, $m' \in [x', y']$.

This is sufficient to show that the projection from γ to ξ is monotone. Label $\gamma(a + nk') = p_n$. Since $\gamma(a) = \xi(a)$, we know that p_0, p_1, and p_2 will be projected in the correct order. Considering p_2 as the midpoint between p_1 and p_3, we know that given that p_1 and p_2 were projected in the correct order, p_3 must be projected correctly as well. We can proceed by simple induction to show that this projection is monotone. $\square$

5. Hyperbolic Groups

To apply our notion of hyperbolicity to groups, we first need to define a metric on groups. Let $\Gamma = \langle \mathcal{A}, \mathcal{R} \rangle$ be a finitely generated group. We will construct a graph $\Delta = \Delta(\Gamma, \mathcal{A})$ as follows. Let each element of Γ be a vertex of Δ. For each $g \in \Gamma$ and $a \in \mathcal{A}$, form a directed edge from g to $g \cdot a$. Thus, edges of the graph correspond to right multiplication by elements of the generating set.

We can define a metric on this graph Δ. We can view each edge as a copy of the unit interval. If we fix a parametrization for each edge, we can then define a length for each interval in an edge, where the length of the entire edge is 1. With this parametrization, we have a generalized notion of a path that can start and finish both in the interior of edges and at vertices. Thus, such paths have well-defined length. Given $x, y \in \Delta$, we define $d(x, y) \in [0, \infty)$ to be the minimum length of a path connecting x to y.

It is clear that a graph constructed in this fashion is connected and locally finite.

Theorem 5.1 (Cayley's Theorem). *Every finitely generated group can be faithfully represented as a symmetry group of a connected, locally finite graph. We will refer to such a graph as a Cayley graph.*

Now that we have a metric space, we can check if it's hyperbolic.

Definition 5.2. A group is hyperbolic if it is finitely generated and its Cayley graph is hyperbolic.

A group's hyperbolicity is independent of our choice of generating set. To show this, we need to show that all Cayley graphs for a group are the same in some sense, regardless of generating set. The sense that we're looking for is quasi-isometry.

Definition 5.3. Let X be a metric space. A map $\phi : X \to X'$ is a *quasi-isometric embedding* if there are constants $k_1 \geq 1$, $k_2 \geq 0$ such that for all $x, y \in X$,

$$\frac{1}{k_1} d(x, y) - k_2 \leq d'(\phi(x), \phi(y)) \leq k_1 d(x, y) + k_2.$$

A quasi-isometric embedding ϕ is a *quasi-isometry* if, in addition, there is a constant $k_3 \geq 0$ such that for all $y \in X'$ there exists some $x \in X$ such that

$$d(y, \phi(x)) \leq k_3.$$

A quasi-isometry essentially preserves distances within fixed linear bounds, and its image is cobounded.

Let S and S' be finite generating sets for some group Γ, and let $\Delta = \Delta(\Gamma, S)$ and $\Delta' = \Delta(\Gamma, S')$ be the corresponding Cayley graphs. We have that $V(\Delta) = \Gamma = V(\Delta')$. We can extend the identity map $V(\Delta) \to V(\Delta')$ to a map $\phi : \Delta \to \Delta'$ by sending an edge of Δ linearly to a geodesic in Δ' with the same endpoints. By choosing our geodesics carefully, it is always possible to ensure that the map ϕ is equivariant, i.e. $g\phi(x) = \phi(gx)$ for all $x \in \delta$ and $g \in \Gamma$. Letting

$$r = \max\{d'(1, a) \mid a \in S\},$$

it is apparent that each edge of Δ gets mapped to a path of length at most $r \in \Delta'$, i.e.

$$d'(\phi(x), \phi(y)) \leq r d(x, y)$$

for all $x, y \in \Delta$. Applying the same construction in the reverse direction gives us an equivariant map $\psi : \Delta' \to \Delta$. It is simple to check that these are quasi-isometries.

Lemma 5.4. *Suppose that S and S' are finite generating sets for a group Γ. Then there is an equivariant quasi-isometry from $\Delta(\Gamma, S)$ to $\Delta(\Gamma, S')$.*

Thus, we have that the Cayley graphs generated by a group are the same up to quasi-isometry. Now, we need to show that hyperbolicity is preserved under quasi-isometry.

Lemma 5.5. *Suppose X and X' are quasi-isometric geodesic spaces. Then, X is hyperbolic if and only if X' is.*

This proof is somewhat technical, so to preserve the focus of this paper it is omitted. It can be found as 6.19 in [3].

With these two results, it is apparent that our definition of a hyperbolic group is well-defined.

6. Hyperbolic Groups have Solvable Word Problem

The key insight here is that hyperbolic metric spaces allow us to get global information (quasigeodesics) from local phenomena (local geodesics). By simply applying (4.7) to the paths induced by words in our group, we get that hyperbolic groups admit Dehn algorithms, and thus have solvable word problems.

Theorem 6.1. *If a finitely presented group $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$ is hyperbolic, it admits a Dehn presentation.*

Proof. Define

$$S := \left\{ u \in F(\mathcal{A}) \;\middle|\; \begin{array}{l} |u| \leq 8\eta \max\{1, \lambda\} \\ \exists\, v \text{ s.t. } |v| < |u| \text{ and } u =_\Gamma v \end{array} \right\}$$

We know that this set is finite since the alphabet $\mathcal{A}$ is finite, and the words u have bounded length. Now, consider some $w \in F(\mathcal{A})$ of length l such that $w =_\Gamma 1$, and let $\gamma_w : [0, l] \to \Gamma$ denote the path induced by w. If γ_w is not an 8η-local geodesic, then there exist two points on γ_w less than 8η apart such that there exists a shorter path between them than γ_w. Thus, we can let the word that induces this subarc be an element of S as one of our u_i, let the shortcut be the corresponding v_i, and we're done.

Now, suppose w is an 8η-local geodesic. By the above lemma, w is a (λ, ε)-quasigeodesic. Thus, we have

$$\frac{1}{\lambda} - \varepsilon \leq d_\Gamma(\gamma(0), \gamma(l)) \leq \lambda l + \varepsilon$$

Since w is null homotopic, we have that $d_\Gamma(\gamma(0), \gamma(l)) = 0$, so this implies

$$8\eta > \varepsilon \geq \frac{l}{\lambda}$$
$$l < 8\lambda\eta$$

Hence, we know v exists because $w =_\Gamma 1$: namely, we can always pick v as just the identity element. Thus, we have that $w \in S$. □

The converse is also true. To prove this, we will need another characteristic of hyperbolic space.

If you consider a circle in $\mathbb{R}^2$ with the Euclidean metric, it's well known that its area will grow at a rate proportional to the square of its circumference. In hyperbolic space, the analogous statement would have the area growing at a rate directly proportional to its circumference. Moreover, having this linear relationship is unique to hyperbolic spaces.

In order to talk about this relationship, which we will soon define precisely as an isoperimetric inequality, we first need a notion of area. If you recall the definition of algebraic Area defined in (2.1), it is clear that for a word w,

$$\text{Area}(w) \leq \delta(l(w)).$$

We will be using a similar idea to define a coarse geometric notion of Area. Essentially, we will be taking loops in our space, filling them with triangles, and counting the triangles.

Definition 6.2. Let D^2 denote the unit disc in the Euclidean plane, i.e. $\partial D^2 = \mathbb{S}^1$. A *triangulation* of D^2 is a homeomorphism P from D^2 to a combinatorial 2-complex in which every 2-cell is a 3-gon. We endow D^2 with the induced cell structure and refer to the preimages under P of 0-cells, 1-cells, and 2-cells as, respectively, the vertices, edges, and faces of P.

Definition 6.3. Let M be a metric space, and $\gamma : \mathbb{S}^1 \to M$ be a loop of finite length in M. An *ε-filling* (P, Φ) of γ consists of a triangulation P of D^2 and a map $\Phi : D^2 \to M$ such that $\Phi |_{\mathbb{S}^1} = \gamma$ and the image under Φ of each face of P is a set of diameter at most ε. We will denote the number of faces of P by $|\Phi|$ and call this the *area of the filling*.

Our notion of area will be simply the smallest number of triangles needed to get an ε-filling of γ.

Definition 6.4. Let γ be as in Definition 6.3. The *ε-area* of γ is defined to be

$$\text{Area}_\varepsilon(c) := \min\{|\Phi| \mid \Phi \text{ is an } \varepsilon\text{−filling of } \gamma\}.$$

If no filling exists, then we define $\text{Area}_\varepsilon(c) := \infty$.

Armed with this notion of area, we can examine the relationship between area and perimeter in a space.

Definition 6.5. A function $f : [0, \infty) \to [0, \infty)$ is called a *coarse isoperimetric bound* for M if there exists some $\varepsilon > 0$ such that every loop $\gamma \in M$ of finite length has an ε-filling and $\text{Area}_\varepsilon(\gamma) \leq f(l(\gamma))$. We say that M satisfies a *linear isoperimetric inequality* if f is linear.

And so we can finally state the theorem that if a metric space satisfies a linear isoperimetric inequality, then it is hyperbolic. The converse is true as well (see H.2.7 from [5]), but for our purposes it will not be necessary.

Theorem 6.6. *Let M be a geodesic metric space. If there exist constants $K, N > 0$ such that $\text{Area}_N(c) \leq Kl(c) + K$ for every piecewise geodesic loop $\gamma \in M$, then M is η-hyperbolic, where η depends only on K and N.*

A proof is not included due to space constraints, but can be found in H.2.9 of [5].

Theorem 6.7. *If $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is a Dehn presentation for a group Γ, then Γ is hyperbolic.*

Proof. Let ρ be the length of the longest word in $\mathcal{R}$. For each $u_i \in \mathcal{R}$, we have a path in $\Delta(\Gamma, \mathcal{A})$ that starts at some element $g \in \Gamma$, follows the letters of u_i, then comes back to g via v_i^{-1}. Consider one such loop, and act on it by g^{-1} so that it starts at the identity. We can choose some ρ-filling of this loop using a finite number of triangles M_i. Take such a filling for each $u_i \in \mathcal{R}$, and let $M = \max\{M_i\}$. We will now show that any null homotopic loop γ can be filled with at most $Ml(\gamma)$ triangles.

Now, consider a loop γ of length $l(\gamma)$ in the Cayley graph $\Delta(\Gamma, \mathcal{A})$, labeled by a word w in the letters of $\mathcal{A}$. Since γ is a loop, the word w is null homotopic. Applying the Dehn algorithm, we can find a subword u_i of w that corresponds to a subpath of γ that is not geodesic, as we can find a shorter path with the same endpoints by following the letters of v_i.

Let γ' be the loop γ with the subpath labeled by u_i replaced by the path labeled by v_i. As an inductive hypothesis, suppose we have a standard ρ-filling $D^2 \to \Delta(\Gamma, \mathcal{A})$ of γ'. We can get a standard ρ-filling of γ by adding a polyhedral face to the filling at the point where u_i was replaced by v_i, and adding edges to divide this face into at most M triangles.

We can thus say that the ρ-Area of γ is at most $M + Ml(\gamma')$. Since $l(\gamma')+1 \leq l(\gamma)$, we have that the Area of γ is at most $Ml(\gamma)$. Thus, $\Delta(\Gamma, \mathcal{A})$ is hyperbolic. □

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