MONADS AND ALGEBRAIC STRUCTURES

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ABSTRACT. This expository paper introduces the concept of monads and explores some of its connections to algebraic structures. With an emphasis on the adjoint functors that naturally participate in our conclusions, we justify how monads give us a distinct, 'categorical' way of discussing common structures such as groups and rings. In the final section, we consider a key example of how looking at structures this way is useful, by converting the problem of finding a natural way to combine algebraic structures to one of understanding a proper interaction between their relevant monads. We assume from the reader a familiarity with the basic notions of category theory.

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1. INTRODUCTION

In his classic introductory book on category theory [1], Saunders Mac Lane (who cofounded the field with Samuel Eilenberg) stressed the importance of commutative diagrams, writing that "a considerable part of the effectiveness of categorical methods rests on the fact that such diagrams in each situation vividly represent the actions of the arrows at hand". It is important, for example, that even a condition that doesn't appear to demand such a diagram, such as the associativity axiom for a category, can be described by one. If we take this emphasis on commutative diagrams seriously, then our first motivation for this paper is that we are not entirely satisfied with, for example, a description of **Grp** as a subcategory of **Set**, where the objects are groups and the morphisms are homomorphisms of such groups; while not at all incorrect, such descriptions appear to invoke conditions that seem to lie outside of purely categorical constraints. This limits the kind of conclusions that can be made about them within the sphere of category theory.

Through monads and the associated concept of algebras over a monad, we indeed obtain a categorical way to discuss **Grp** and categories of many other algebraic

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structures (specifically, any *variety* of algebras). We then illustrate one use of such a categorical description for the problem of combining algebraic structures; for example, what are some valid ways to combine an (additive) abelian group structure with a monoid structure, and why is a ring (with its particular distributive law) one such way? We gain much insight to these questions by converting them into a categorical problem concerning the relevant monads.

Our story begins by introducing and discussing the powerful concept of *adjoint functors* that is central to the overall picture. In fact, in the abstracted, purely categorical thread of this paper, our approach towards monads is much to view them as belonging to the theory of adjoint functors by considering them as "traces" of adjunctions.

A truly rigorous treatment of category theory always invokes considerations of size and set-theoretic conditions. This author has neither the background nor the space to take these considerations into account. A category is always "small enough" so that a statement we make holds true: sometimes this requires a category to be small, and at other times only requires it to be locally small.

2. Adjunctions

We define the key concept and immediately illustrate two typical, though very different instances of of its occurence; along the way, we pick up a few of its most crucial properties and also illustrate some of its power in category theory.

Definition 2.1. (Adjunction) Given categories $\mathcal{D}$ and $\mathcal{C}$, an adjunction between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that there is a natural isomorphism ψ between the bifunctors $\mathcal{C}(F-,-)$ and $\mathcal{D}(-,G-)$ from $\mathcal{D}^{op} \times \mathcal{C}$ to **Set**. In other words, for every pair of objects $x \in \mathcal{D}$ and $y \in \mathcal{C}$, there is a bijection of sets

$$\psi_{x,y} : \mathcal{C}(Fx, y) \cong \mathcal{D}(x, Gy) \tag{2.2}$$

which is natural in x and y. In this case we say F and G are an *adjoint pair of functors* where F is *left adjoint* to G and G is *right adjoint* to F.

Notations 2.3. The notation $(F, G, \psi) : \mathcal{D} \to \mathcal{C}$ in that particular order will unambiguously denote the adjunction

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

with F as a left adjoint and ψ the natural bijection. Also, a shorthand notation to indicate that F is a left adjoint in the adjoint pair is $F \dashv G$.

Example 2.4 (A "free-forgetful" pair of adjoint functors)**.** Some of the most common instances of adjunctions (and indeed, the type of adjunction we are most concerned with for this paper) involves a "free" left adjoint to a "forgetful" functor. A typical example of this is found in an adjunction between the categories $\mathbf{Vct_K}$ (vector spaces over a field K) and **Set**. In this situation we have a free functor $F : \mathbf{Set} \to \mathbf{Vct_K}$ which takes every set X to the K-vector space that has X as a basis (that is, the vectors are all formal linear combinations of the basis X), while the forgetful functor $U : \mathbf{Vct_K} \to \mathbf{Set}$ simply returns the underlying set of vectors (hence the notation U) of a vector space W, thus "forgetting" its vector properties.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] & \mathbf{Vct_K} \arrow[l, bend left, "U"] \end{tikzcd}$$

From elementary linear algebra we know that given K-vector spaces V and W, a linear transformation $T : V \to W$ is uniquely determined by its values on a basis B of V, and any choice of action on this basis B gives rise to a unique valid linear transformation. In particular, for any set X, any function (between sets) $f : X \to UW$ where W is a K-vector space can be extended to a unique linear transformation $T : FX \to W$, and conversely every such linear transformation restricts to a function between sets $f : X \to UW$. Thus for every pair $X \in \mathbf{Set}$ and $W \in \mathbf{Vct_K}$ we have a bijection of sets

$$\psi_{X,W} : \mathbf{Vct_K}(FX, W) \cong \mathbf{Set}(X, UW)$$

The fact that that this bijection is obtained through canonical extensions and restrictions from set functions to linear transformations and vice-versa for all X and W ensures that this bijection is indeed natural in X and W. Hence $F \dashv U$. This example is a prototype for a general situation. $\mathbf{Vct_K}$ is an instance of $R-\mathbf{Mod}$ for R a commutative unital ring, and exactly the same argument with "R-linear map" substituted for "linear transformation" gives a free-forgetful adjunction between $R - \mathbf{Mod}$ and $\mathbf{Set}$. Similar observations on the relations between functions (of sets) and homomorphisms of algebraic structures, gives, for example, free-forgetful adjunctions $\mathbf{Mon} \to \mathbf{Set}$ and $\mathbf{Grp} \to \mathbf{Set}$. In fact, we can treat all these cases together and obtain a free-forgetful adjunction for a more general algebraic construction; however, it is really only with the next theorem that this is both possible and desirable.

Proposition 2.5. *Any two left adjoints F and F' of a functor $G : \mathcal{C} \to \mathcal{D}$ are naturally isomorphic. Dually, any two right adjoints G and G' of a functor $F : \mathcal{D} \to \mathcal{C}$ are naturally isomorphic.*

Proof. Let $\psi_1 : \mathcal{C}(F-, -) \to \mathcal{D}(-, G-)$ be the natural isomorphism corresponding to $F \dashv G$ and let $\psi_2 : \mathcal{C}(F'-, -) \to \mathcal{D}(-, G-)$ that corresponding to $F' \dashv G$. Then $\psi_2^{-1}\psi_1 : \mathcal{C}(F-, -) \to \mathcal{C}(F'-, -)$ is a natural isomorphism of bifunctors. In particular, for any $x \in \mathcal{D}$, $\mathcal{C}(Fx, -) \cong \mathcal{C}(F'x, -)$. By the Yoneda Lemma, and in particular by the fact that the Yoneda functor is an embedding from $\mathcal{C}^{op}$ into $\mathbf{Set}^{\mathcal{C}}$, we must have $Fx \cong F'x$ via an isomorphism μ_x. It is easy to check that (μ_x) are the components of a natural transformation $\mu : F \dot{\to} F'$. □

Remark 2.6. As a consequence of this proposition, we can *define* a free functor to be the left adjoint (if it exists) of a forgetful functor $U : \mathcal{C} \to \mathcal{D}$, because any two candidates for a free functor are isomorphic in the category $\mathcal{D}^C$. This is in many ways a cleaner and more useful way to define freeness than "equivalence classes of strings modulo relations given by the axioms". In a few other instances in this section, we use the existence of an adjoint to a functor to define certain functors unambiguously up to isomorphism.

We now look to define the general notion of an algebraic systems of a certain 'type', so that we can obtain a category corresponding to a *variety of algebras.* For the way in which we will need this concept for this paper, which is for now just to obtain a "free-forgetful" type adjunction between **Set** and such a variety, and then later to understand this adjunction through monads, we will not need the most general definition of *variety* from universal algebra. Instead, we will borrow the notation and conceptualization in [1], which is well motivated by our intuition of what is common to many algebraic systems - namely, that it involves a set with operations that must satisfy certain identities, such that we can take operating preserving maps (homomorphisms) to others of its kind.

Definitions 2.7. (Preliminary definitions) An *algebraic system* of type τ is given by a graded set Ω of *operators* and and a set E of *identities.* The gradation in Ω is given by the function that assigns to each element $\omega \in \Omega$ an *arity* $n \in \mathbb{N}$. An *action* A of Ω on a set S that assigns to each operator ω of arity n an n-ary operation $\omega_A : S^n \to S$. The action of operators on sets suggests that we should be able to combine operators in several ways, and motivates the set Λ_Ω of *derived operators* of Ω: given ω of arity n and and n other operators $w_1, .., w_n$ of arities $m_1, ..., m_n$, we can consider first the evident composite $\omega(w_1, ..., w_n)$ to obtain a derived operator of arity $m_1 + ... + m_n$. In the preceeding sentence, we can in fact substitute the operators $w_1, ...w_n$ with derived operators $\lambda_1, ..., \lambda_n$. Finally, given $\lambda \in \Lambda_\Omega$ of arity n, and any function $f : 1, ..., n \to 1, ..., m$, we can obtain a derived operator θ by 'substitution' of λ, that is, by defining in terms of variables $\theta(x_1, ..., x_m) = \lambda(x_{f1}, ..., x_{fn})$. Through all such combinations, we obtain all derived operators Λ_Ω.

The set E of identities (for algebraic systems for type τ) is a set of pairs of derived operators (λ, μ) of the same arity n. An action A of Ω on a set S is said to *satisfy* the identity (λ, μ) if $\lambda_A = \mu_A : S^n \to S$

Definition 2.8. (Variety of algebras) An algebra A of type $\tau = (\Omega, E)$ is a set S together with an action A of Ω on S which satisfies all identities of E. We call S the underlying set and often consider it to be impliclitly specified by A, and for this reason often write $|A| = S$. A morphism $g : A \to A'$ of τ algebras is a function $g : S \to S'$ of the underlying sets which preserve operators of Ω in the sense that

$$g\omega_A(a_1, ..., a_n) = \omega_{A'}(ga_1, ..., ga_n) \tag{2.9}$$

where all $a_i \in |A|$. Since the composition of τ algebra morphisms is well defined and associative, and identity morphisms can be defined in the obvious way, we can form the category of τ algebras, denoted better as $(\Omega, E) - \mathbf{Alg}$, or $\mathbf{Alg}_\tau$. We call each such category a variety of algebras, or a τ-variety.

Remarks 2.10. Stemming from these definitions themselves, we can already guess that each of the previously mentioned categories **Mon**, **Grp**, $R - \mathbf{Mod}$ are simply distinct varieties of algebras. To see how this is true, for example, of **Grp**, consider Ω with three operations of arities 2, 1, 0, corresponding to the actions of product, inverse, and identity respectively. It is furthermore easy to see that we can construct E to encode exactly the group axioms (the axioms for associativity, identity, and inverse). Then the condition on the morphisms between (Ω, E) algebras is precisely the condition that the 'product' operator is a homorphism. Thus **Grp** is exactly the variety (Ω, E) (or more precisely the two categories are isomorphic).

Theorem 2.11. *The forgetful functor* $U : \mathbf{Alg_T} \to \mathbf{Set}$ *for any* τ*-variety has a left adjoint. In view of remark 2.6, we can call this the free functor.*

Proof. A standard proof of this uses Freyd's famous adjoint functor theorem, which specifies certain conditions for checking that a functor is a right adjoint without needing any knowledge of the left adjoint, where the explicit description can remain mysterious. We refer the reader to [1] for the proof of this theorem and its application here. □

Warning 2.12. Variety of algebras are indeed a general notion, but they do not include all possible algebraic structures. In fact it excludes a very familiar one: fields (specifically, the category **Field** of fields). The operators of arity n involved in any algebra of a certain variety must have an action on every n-tuple of the underlying set, but a 1-arity operator corresponding to the "mulitplicative inverse" of an element does not have an action on the element 0. Indeed, the forgetting functor on a field does not have a left adjoint; there is *no* free field. A small hint should convince the reader that there is no field with the requisite universal property: field homomorphisms have to be injective (i.e. *embeddings*), but a field of characteristic 0 obviously cannot be embedded in a field of characteristic p a prime.

Examples 2.13 (Adjunction for colimits)**.** The following is also a typical, though very different instance of an adjunction; though we do not strictly require it for this paper, we include it because of the wide range of concrete examples it encapsulates, and to illustrate how it can be applied to such an example to derive important results for it.

If a category $\mathcal{C}$ has all colimits of shape $\mathcal{J}$ then there is a special adjunction between $\mathcal{C}$ and $\mathcal{C}^{\mathcal{J}}$ that can in fact be used to define the colimits in $\mathcal{C}$. One functor in the adjoint pair is the *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ which sends each object of c to Δc, the functor which is c on all objects of $\mathcal{J}$ and the identity 1_c on all morphisms of $\mathcal{J}$. For $f : c \to c'$ a morphism in $\mathcal{C}$, $\Delta f : \Delta c \to \Delta c'$ is the natural transformation which simulates f exactly, that is, has f for every component.

The functor in the other direction is the colimit functor $\underline{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$, which on objects of $\mathcal{C}^{\mathcal{J}}$, that is functors $F : \mathcal{J} \to \mathcal{C}$, maps to $colim(F)$, which exists by assumption. On morphisms of $\mathcal{C}^{\mathcal{J}}$, that is natural transformations between functors in $\mathcal{C}^{\mathcal{J}}$, $\underline{colim}$ is defined as follows:

Consider $\tau : F_1 \dot{\to} F_2$ a morphism (natural transformation) in $\mathcal{C}^{\mathcal{J}}$. By definition of colimit, $colim(F_1)$ is accompanied by a 'universal cone' from F_1 to Δ, that is a universal natural transformation $\mu_1 : F_1 \dot{\to} \Delta colim(F_1)$. Likewise, $colim(F_2)$ has an associated universal cone $\mu_2 : F_2 \dot{\to} \Delta colim(F_2)$. Then $\mu_2 \cdot \tau$ is a natural transformation $F_1 \dot{\to} \Delta colim(F_2)$, that is a cone from F_1 to Δ. By the universal property of $colim(F_1)$ and its associated μ_1, there is a unique morphism $\alpha : colim(F_1) \to colim(F_2)$ such that $\Delta\alpha \cdot \mu_1 = \mu_2 \cdot \tau$ as natural transformations $F_1 \dot{\to} \Delta colim(F_2)$. We set $\underline{colim}(\tau)$ to be this α. Then it can be easily checked that this defines a valid functor $\underline{colim}$

So far we have defined a pair of functors $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ and $\underline{colim} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$.

$$\begin{tikzcd} \mathcal{C}^{\mathcal{J}} \arrow[r, bend left, "\underline{colim}"] & \mathcal{C} \arrow[l, bend left, "\Delta"] \end{tikzcd}$$

The universal property of colimits makes Δ and $\underline{colim}$ an adjoint pair of functors with $\underline{colim} \dashv \Delta$. In particular, the "unique arrow" part of the universal property of a colimit ensures that we have for all pairs c in C, F in C^J a bijection of sets

$$\psi_{c,F} : \mathcal{C}(colimF, c) \cong \mathcal{C}^{\mathcal{J}}(F, \Delta c)$$

while the commutativity condition of the universal property ensures that this bijection is natural in c and F.

Again, by remark 2.6, we can in fact *define* colimits of shape $\mathcal{J}$ as the (objects of) the functor left adjoint to the diagonal functor Δ, provided $\mathcal{C}$ has *all* colimits of shape $\mathcal{J}$ - which happens, for example, when $\mathcal{C}$ is cocomplete. Dually, if a category $\mathcal{C}$ is complete, we can define the limits of shape $\mathcal{J}$ as the (objects of) a right adjoint to Δ. In practice, many categories *do* have all limits or colimits of a certain shape (in fact, some crucial ones are moreover complete or co-complete), so it is very useful to think of limits or colimits as a part of adjunction this way. Let us illustrate this in a compact proof of an important result.

Definition 2.14. A functor $H : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of functors $F : \mathcal{J} \to \mathcal{C}$ for $\mathcal{J}$ an index category when for every universal cone $\nu : lim(F) \to F$ in $\mathcal{C}$, $H\nu : Hlim(F) \to HF$ is a universal cone for the functor HF. In particular, $lim(HF) = Hlim(F)$.

Proposition 2.15. *A right adjoint G between categories that have all limits of shape J preserves those limits.*

Proof. Let G be part of the adjunction:

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

We can now establish, for a functor $K \in \mathcal{C}^{\mathcal{J}}$, a chain of natural isomorphisms:

$$\begin{aligned} D(x, G(limK)) &\cong C(Fx, limK) \text{ (using } F \dashv G) \\ &\cong C^J(\Delta Fx, K) \text{ (using } \Delta \dashv \underline{lim}) \\ &\cong C^J(F\Delta x, K) \\ &\cong D^J(\Delta x, GK) \text{ (using } F \dashv G) \\ &\cong D(x, limGK) \text{ (using } \Delta \dashv \underline{lim}) \end{aligned}$$

where, in the penultimate statement we have implicitly used the fact that the adjunction $F \dashv G$ passes to the functor category. Finally, using as before that the Yoneda functor is an embedding, we have $G(limK) \cong lim(GK)$ as desired. $\square$

Remark 2.16. We could have omitted the condition that the categories have *all* limits of shape $\mathcal{J}$ (it is sufficient that $\mathcal{C}$ has the limit of K), but with it our proof takes a particularly compact and elegant form thanks to the use of adjunctions.

This is a powerful proposition; for example, together with the fact that **Grp** is complete, we immediately obtain consequences such as:

Corollary 2.17. *The underlying set of a (categorical) product of two groups is the cartesian product of the groups' underlying sets.*

This conclusion is impressive considering that "product" is defined only as a certain limit, that is, only by its universal property. Finally, for this section, we state an alternate characterization of adjunctions that is particularly useful when we draw the connection between and adjunctions and monads.

Proposition 2.18 (Alternate characterization of adjunctions using unit and counit)**.** *Given categories $\mathcal{C}$ and $\mathcal{D}$, a pair of functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ are adjoint if and only if there are natural transformations $\eta : 1_d \dot{\to} GF$ and $\varepsilon : FG \dot{\to} 1_c$ (called the* unit *and* counit *respectively) such that the following "triangle identities" hold:*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1"'] & GFG \arrow[d, "G\varepsilon"] \\ & F \end{tikzcd}$$

where 1 is the identity natural transformation.

Notation 2.19. In view of the above proposition, $(F, G, \eta, \varepsilon) : D \to C$ will unambiguously denote the adjunction

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

with $F \dashv G$.

Remarks 2.20. This alternate characterization of adjunctions bears the following relationship with the original definition. The natural bijection ψ in the first definition can now be expressed as:

$$\begin{aligned} \psi(f) &= G(f) \circ \eta_X \text{ for } f : Fx \to y \\ \psi^{-1}(g) &= \eta_y \circ \ F(g) \text{ for } g : x \to Gy \end{aligned}$$

Also, given first the original bijection ψ, the unit and counit of the alternate characterization are realized on components as

$$\begin{aligned} \eta_x &= \psi(1_{Fx}) \text{ for } 1_{Fx} : Fx \to Fx \text{ the identity morphism} \\ \varepsilon_y &= \psi^{-1}(1_{Gy}) \text{ for } 1_{Gy} : Gy \to Gy \text{ the identity morphism} \end{aligned}$$

Our last remark is that the unit and counit have simple interpretations for a "free-forgetful" type adjunction. The unit can be considered an "insertion of generators" while the counit an "evaluation". The reader is encouraged to observe this in the example of the adjunction between $\mathbf{Vct_K}$ and **Set**. The arguments involved generalize readily.

3. Monads

There are many ways to approach a discussion of monads; for our purposes, since we are primarily concerned about the relationship between monads and algebraic structures, we motivate our discussion with the following. Given any adjoint pair of functors F and G with $F \vdash G$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

The functor GF is an endofunctor on $\mathcal{D}$. The information consisting of this endofunctor together with the unit $\eta : 1 \dot{\to} GF$ of the adjunction and the natural transformation $G\varepsilon F : (GF)^2 = GFGF \dot{\to} GF$, along with certain specified relationships between these three items, is a certain "trace" or "imprint" of the adjunction on $\mathcal{D}$ called a *monad*. The question we are primarily concerned with in this section is, suppose we were first only given this monad on $\mathcal{D}$, and knew it was an "imprint" of some adjunction, what, if anything, can we then say about the original adjunction involving $\mathcal{C}$? Could we, for example, "recover" this adjunction (the category $\mathcal{C}$ and the adjoint pair) completely from the monad? The question becomes more impressive when we consider the case of a "free-forgetting" pair of adjoint functors, such as that between **Set** and **Mon**. This is the example we want to keep in mind for much of the discussion in this section.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] & \mathbf{Mon} \arrow[l, bend left, "U"] \end{tikzcd}$$

The functor UF is an endofunctor on **Set** which returns, for each set X, the underlying set of the free monoid on X. This is $\sqcup_n X^n$, the disjoint union of n-length strings of X. Meanwhile, the unit $\eta : 1 \dot{\to} UF$ is the 'insertion of generators" while the counit $\varepsilon : FU \dot{\to} 1$ is the function that evaluates a formal product in a certain monoid. The natural transformation $U\varepsilon F : (UF)^2 \dot{\to} UF$ is a natural transformation that in each component takes a string of strings of a set X to a string of X in the easiest way (concatenation). From these three pieces of information $(UF, \eta, U\varepsilon F)$ and knowing that they satisfy certain specified relationships for a monad on **Set**, can we for example find all monoids on **Set**, or more strongly, recover the category **Mon** and its associated adjunction? As we shall see, the answer is "yes" and furthermore, the construction that recovers **Mon** from this monad, the "Eilenberg-Moore construction", also recovers any *variety of algebras* on **Set**, and the particular constructions involved in recovering these adjunctions from monads give us a distinct way of discussing these algebraic structures in category theory.

Without further ado, we supply the definition of a monad:

Definition 3.1. A *monad* on a category $\mathcal{X}$ is a triple (T, η, μ), where T is an endofunctor of $\mathcal{X}$, and η (the unit of the monad) and μ (the multiplication of the monad) are two natural transformations $\eta : 1 \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ satisfying the commutativity of following diagrams:

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "1"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1"] \\
& T &
\end{tikzcd}
$$

These are known respectively as the multiplication and unit axiom for a monad. Often we refer to monads by the endofunctor T itself. We remark that a monad is a monoid in the category of endofunctors of $\mathcal{X}$ (the functor category $\mathcal{X}^{\mathcal{X}}$), though we make no use of this fact in this paper.

As our discussion of monoids stated, every adjunction gives rise to monad.

Proposition 3.2. *Every adjunction $(F, G, \eta, \varepsilon) : \mathcal{X} \to \mathcal{C}$ gives rise to a monad $(GF, \eta, G\varepsilon F)$. We call this the monad* defined *on $\mathcal{X}$ by the adjunction.*

We leave the proof, which just amounts to a verification of the monad multiplication and unit axioms, as an exercise. The multiplication axiom will follow simply from an application of the interchange law for the horizontal composition of natural transformations $\varepsilon\varepsilon$, while the right and left unit law follow automatically from the two triangle identities (one of the identities giving the right unit law, the other one giving the left).

Recalling the motivation at the beginning of this section, we are interested in the following question: Given a monad T on $\mathcal{X}$, is there an adjunction from $\mathcal{X}$ (that is, a category $\mathcal{X}'$ an an adjoint pair $F, G : \mathcal{X} \to \mathcal{X}'$ such that the monad that this adjunction defines on $\mathcal{X}$ is exactly T? Indeed there is, but it is not in general unique. In fact, there is a *category* of adjunctions that gives rise to this monad. We are most interested in the canonical construction of the terminal object in this category due to Samuel Eilenberg and John C. Moore.

3.1. **The Eilenberg-Moore construction.** In this construction, the other category involved in an adjunction with $\mathcal{X}$ with a monad T is $\mathcal{X}^T$, the category of T-algebras over X. We now proceed to define what this is.

Definition 3.3 (Algebras for a monad)**.** Given a monad T on a category $\mathcal{X}$, a $T-$*algebra* for $\mathcal{X}$ is a pair (x, θ), where x is an object of $\mathcal{X}$, and θ a morphism $Tx \to x$ known as a *structure map* which 'interacts well' with the unit and multiplication of the monad in the sense that the following diagrams commute:

$$
\begin{tikzcd}
T^2x \arrow[r, "T\theta"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "\theta"] \\
Tx \arrow[r, "\theta"'] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, "1"'] & Tx \arrow[d, "\theta"] \\
& x
\end{tikzcd}
$$

These are known respectively as the associative and unit law for the algebra. Furthermore, a morphism of T-algebras $f : (x, \theta) \to (x', \theta')$ is a morphism $f : x \to x'$ which 'interacts well' with the structure map in the sense that the following diagram commutes:

$$\begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "\theta"'] & Tx' \arrow[d, "\theta'"] \\ x \arrow[r, "f"] & x' \end{tikzcd}$$

Since composition of morphisms is easily seen to be well defined and associative, and since identity morphisms for T-algebras are obviously those ones in $\mathcal{X}$ (ignoring the structure maps), the T-algebras of $\mathcal{X}$ form a category $\mathcal{X}^T$. We also call it the *Eilenberg-Moore category*, or the *category of algebras* for the monad T.

Definition 3.4 (Eilenberg-Moore construction)**.** Given a monad (T, η, μ) on a category $\mathcal{X}$, the Eilenberg-Moore construction is the 'free-forgetful' adjunction between $\mathcal{X}$ and its Eilenberg-Moore category $\mathcal{X}^T$:

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F^T"] & \mathcal{X}^T \arrow[l, bend left, "U^T"] \end{tikzcd}$$

where F^T and U^T are defined as follows:

$$F^T : x \xrightarrow{\quad f \quad} y \dashrightarrow (Tx, \mu_x) \xrightarrow{Tf} (Ty, \mu_y)$$

$$U^T : (x, \theta) \xrightarrow{f} (x', \theta') \dashrightarrow x \xrightarrow{\quad f \quad} x'$$

We now justify the claims implicit in this definition. First, the monad axioms ensure that F^T maps to T-algebras and algebra morphisms and F^T and U^T are indeed functors; the reader can check the details. Next, we verify that these functors are adjoint. Clearly $U^T F^T = T$. The candidate for the unit for the adjunction η^T is thus the unit η for the monad. Also, $F^T U^T(x, \theta) = (Tx, \mu_x)$, and since $\theta : Tx \to x$, on components the candidate for the counit is:

$$\varepsilon^T_{x,\theta} = \theta : F^T U^T(x, \theta) \to (x, \theta)$$

The axioms satisfied by T-algebras ensure that this is a valid T-algebra morphism and that the overall transformation $\varepsilon = (\varepsilon_{x,\theta})$ is in fact natural. Checking the triangle identities requires in the one case quoting the unit law for T, in the other the unit law for a T-algebra. Thus $(F^T, U^T, \eta^T = \eta, \varepsilon^T) : \mathcal{X} \to \mathcal{X}^T$ is an adjunction. That this adjunction is "free-forgetful" is justified as follows: U^T 'forgets' structure maps, and F^T is thus free, in view of remark 2.6, as its left adjoint.

We motivated the Eilenberg-Moore construction in part because we expect that:

Proposition 3.5. *The Eilenberg-Moore adjunction formed from a monad T on $\mathcal{X}$ defines on $\mathcal{X}$ this same monad.*

Proof. We have already observed that $U^T F^T = T$ and $\eta^T = \eta$. It remains to check that $\mu^T = \mu$. This follows from:

$$\mu^T_x = U^T \varepsilon^T F^T x = U^T \varepsilon^T (Tx, \mu_x) = U^T \mu_x = \mu_x$$

$\square$

Summarizing our position thus far, given an adjunction $(F, G, \eta, \varepsilon) : \mathcal{D} \to \mathcal{C}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

It defines a monad $(GF, \eta, G\varepsilon F)$ on $\mathcal{D}$. From this monad, the Eilenberg-Moore adjunction with the category of GF-algebras defines the same monad on $\mathcal{D}$.

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F^{GF}"] & \mathcal{C} \arrow[l, bend left, "G"] \\ & \mathcal{D}^{\mathcal{GF}} \arrow[ul, bend left, "G^{GF}"] \end{tikzcd}$$

Recalling again the discussion that motivated the Eilenberg-Moore construction (trying to recover an adjunction from its monad), our first question, cast most naively, is whether this "new" Eilenberg-Moore adjunction is not really exactly the same as the original adjunction. It certainly doesn't seem like it - what does an elaborate construction with GF-algebras have to do with $\mathcal{D}$ and its associated adjunction, which we describe very differently (and often more simply, recalling the monoid example!)? In category theory however, this notion of *sameness* has to be made more precise. First of all, categories are only themsleves defined up to an isomorphism of categories. Thus, for example, when we speak about the category **Mon**, we only ever understand it uniquely up to its isomorphism class. Thus, we can ask, for example, if $\mathcal{C}$ and $\mathcal{D}^{GF}$ are isomorphic. But we are interested in not only comparing these two categories but their adjunctions as well. We now define a notion of maps between categories involved in an adjunction that preserve those adjunctions:

Definition 3.6 (Map of Adjunctions)**.** Given adjunctions $(F, G, \psi, \eta, \varepsilon) : \mathcal{D} \to \mathcal{C}$ and $(F', G', \psi', \eta', \varepsilon') : \mathcal{D}' \to \mathcal{C}'$ a *map of adjunctions* is a pair of functors $K : \mathcal{C} \to \mathcal{C}'$ and $L : \mathcal{D} \to \mathcal{D}'$

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[d, "L"] & \mathcal{C} \arrow[l, bend left, "G"] \arrow[d, "K"] \\ \mathcal{D}' \arrow[r, bend left, "F'"] & \mathcal{C}' \arrow[l, bend left, "G'"] \end{tikzcd}$$

that preserves the adjunction structure on two levels. First of all, the above diagram must commute *serially* (that is, commute if we only took the diagram with the topmost arrows F and F', and then commutes once again if we took the bottom arrows G and G'). Secondly, this pair (K, L) should interact properly with ψ and ψ' in the sense that the following diagram of hom-sets commutes for all objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$

$$
\begin{tikzcd}
\mathcal{C}(Fx, y) \arrow[r, "\psi"] \arrow[d, "K"'] & \mathcal{D}(x, Gy) \arrow[d, "L"] \\
\mathcal{C}(KFx, Ky) \arrow[d, equal] & \mathcal{D}'(Lx, LGy) \arrow[d, equal] \\
\mathcal{C}'(F'Lx, Ky) \arrow[r, "\psi'"] & \mathcal{D}'(Lx, G'Ky)
\end{tikzcd}
\tag{3.7}
$$

It is easy to check that we can form a category of adjunctions $\tilde{\mathcal{A}}$ this way with morphisms the maps of adjunctions.

Lemma 3.8. *The condition 3.7 in the above definition is equivalent to $L\eta = \eta' L$ and $\varepsilon' K = K\varepsilon$.*

Proof. The proof is a simple application of the Yoneda lemma. See [1]. □

Having defined $\tilde{\mathcal{A}}$, we can state precisely what we mean by the category of adjunctions giving rise to a monad T on $\mathcal{X}$. We mean the subcategory Γ_T of $\tilde{\mathcal{A}}$ where (1) the objects are those adjunctions $F, G : \mathcal{X} \to \mathcal{C}$ for various categories C which define the monad T on $\mathcal{X}$, and (2) the morphisms are maps (K, L) of adjunctions which are the identity on $\mathcal{X}$, that is, $L = 1$. Note that since all adjunctions in Γ_T define the same monad on $\mathcal{X}$, proposition 3.2 implies that they have the same unit of adjunction; together with $L = 1$, this implies $L\eta = \eta' L$. Because of lemma 3.8, we can consider morphisms in Γ_T to be functors $K : \mathcal{C} \to \mathcal{C}'$ as below

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F'"] & \mathcal{C} \arrow[l, bend left, "G"] \arrow[d, dotted, "K"] \\
& \mathcal{C}' \arrow[ul, bend left, "G'"]
\end{tikzcd}
$$

such that the above diagram commutes serially, that is $KF = F'$ and $G'K = G$. We now arrive at a key theorem for this secton.

Theorem 3.9. *The Eilenberg-Moore adjunction is terminal in Γ_T.*

Proof. We need to show that given any other adjunction as below:

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F^T"] & \mathcal{C} \arrow[l, bend left, "G"] \arrow[d, dotted, "K"] \\
& \mathcal{D}^T \arrow[ul, bend left, "U^T"]
\end{tikzcd}
$$

there is a unique functor K satisfying serial commutativity. We sketch the proof of the existence of K but fully prove its uniqueness. Our candidate for K can be described explicitly:

$$K : c \xrightarrow{f} c' \dashrightarrow (Gc, G\varepsilon_c) \xrightarrow{Gf} (Gc', G\varepsilon_{c'})$$

The reader has to check that this definition is valid and makes K a functor; the hardest part is checking that $G\varepsilon_c$ is a valid structure map, where one of the required commutativity diagrams is met by the definition of $G\varepsilon\varepsilon$ and the other is met by one of the triangular identities for the given adjunction. It is furthermore easy to check that K satisfies the serial commutativity conditions. Now we prove uniqueness. Given any valid such K, the commutativity condition $U^T K = G$ implies that on objects $Kc = (Gc, \theta)$ for some structure map θ and on morphisms $Kf = Gf$. Thus it remains only to show $\theta = G\varepsilon_c$. Since K is part of a map of adjunctions, lemma 3.8 gives that, on components $K\varepsilon_c = \varepsilon^T Kc$. On the left hand side, this is just $G\varepsilon_c$, and on the right $\varepsilon^T(Ga, \theta) = \theta$. Together we have $\theta = G\varepsilon_c$, so K is indeed unique. □

The unique functor K above is known as the *comparison functor*. We want to know under what general conditions K is, for example, an isomorphism. The most well known result in this regard is due to Beck. We will define and state a version of his theorem that can be applied to the case where we are comparing the adjunction associated with a variety of algebras with the corresponding Eilenberg-Moore adjunction. Unfortunately, in the interest of space, we will not have the opportunity to provide the substantial proof of Beck's Theorem nor, frankly, even state it in an enlightening form; we recommend [1] and [2] for a far more substantive treatment of his theorem.

Definitions 3.10 (Fork and absolute coequalizer)**.** In a category $\mathcal{C}$, a *fork* is the basic template for a coequalizer; namely, a diagram

$$\begin{tikzcd} a \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_2"'] & b \arrow[r, "e"] & c \end{tikzcd}$$

such that $ed_1 = ed_2$. A coequalizer can therefore be considered a universal fork. Note that if $T : \mathcal{C} \to \mathcal{X}$ is any functor to any category $\mathcal{X}$, it takes forks (like the one above) to forks in $\mathcal{X}$:

$$\begin{tikzcd} Ta \arrow[r, bend left, "Td_1"] \arrow[r, bend right, "Td_2"'] & Tb \arrow[r, "Te"] & Tc \end{tikzcd}$$

But it may not necessarily take coequalizers in $\mathcal{C}$ to coequalizers in $\mathcal{X}$. We call e an *absolute coequalizer* (of d_1 and d_2) if any such T does exactly that: take coequalizers to coequalizers. By taking $T = 1_{\mathcal{C}}$, we see that absolute coequalizers are trivially coequalizers.

Definition 3.11 (Creating coequalizers)**.** Creating coequalizers is a special case of creating limits. In particular, a functor $G : \mathcal{C} \to \mathcal{D}$ *creates coequalizers* for a parallel pair of arrows $f, g : a \to b$ in $\mathcal{C}$ if for each coequalizer $u : Gb \to z$ of Gf, Gg in $\mathcal{D}$, u has a "preimage coequalizer", that is a coequalizer $e : b \to c$ of f and g such that $Gc = z$ and $Ge = u$.

Theorem 3.12 (Beck's Precise Monadicity Theorem). *Let $(F, G, \eta, \varepsilon) : \mathcal{X} \to \mathcal{C}$ be an adjunction which defines a monad T on $\mathcal{X}$. Let $(F^T, G^T, \eta^T, \varepsilon^T) : \mathcal{X} \to \mathcal{X}^T$ be the Eilenberg-Moore adjunction:*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F^T"] & \mathcal{C} \arrow[l, bend left, "G"] \arrow[d, dotted, "K"] \\
 & \mathcal{X}^T \arrow[ul, bend left, "U^T"]
\end{tikzcd}$$

Let K be the comparison functor. Then K is an isomorphism if and only if G creates coequalizers in X for those parallel pairs f, g in A for which Gf, Gg has an absolute coequalizer in X.

We apply Beck's Theorem to prove our key result.

Theorem 3.13. *Consider:*

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F^T"] & \mathbf{Alg}_\tau \arrow[l, bend left, "U"] \arrow[d, dotted, "K"] \\
 & \mathbf{Set}^\mathbf{T} \arrow[ul, bend left, "U^T"]
\end{tikzcd}$$

Where at the top we have the free-forgetting adjunction associated with any variety of algebras, and on the bottom the Eilenberg-Moore adjunction. Then K is an isomorphism.

Proof. Consider any parallel pair $f, g : A \to B$ of morphisms in $\mathbf{Alg}_\tau$ (that is, homomorphisms for the specified variety) for which Uf, Ug (that is, the underlying functions between sets) have an absolute coequalizer $e : UB \to X$

$$\begin{tikzcd}
UA \arrow[r, bend left, "Uf"] \arrow[r, bend right, "Ug"'] & UB \arrow[r, "e"] & X
\end{tikzcd}$$

To apply Beck's Theorem, we need to prove that (1) X is (the underlying set of) an algebra of type τ, (2) e is a homomorphism for this type, and (3) it is in fact the coequalizer of f and g as homomorphisms. We show that these facts hold true "for each" n-ary operator $\omega \in \Omega$, and consequently get that they are true in general. Borrowing the key diagram from [1],

$$\begin{tikzcd}
A^n \arrow[r, bend left, "f^n"] \arrow[r, bend right, "g^n"'] \arrow[d, "\omega_A"'] & B^n \arrow[r, "e^n"] \arrow[d, "\omega_B"'] & X^n \arrow[r, dotted, "{h'}^n"] \arrow[d, dotted, "\omega_X"] & C^n \arrow[d, dotted, "\omega_C"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "e"] & X \arrow[r, dotted, "{h'}^n"] & C
\end{tikzcd}$$

The left most square commutes because f and g are homomorphisms in the specified variety. e^n is furthermore a coequalizer (as a function of sets) because e is one (and in fact an absolute coequalizer by assumption). Now we prove (1) and (2) simulatenously. One can easily check that with the commutativity of the left square that $e\omega_B f^n = e\omega_B g^n$. Thus there is a unique factorization ω_X that makes the second to leftmost square commute. If X were the underlying set of a variety with action of ω as ω_X, e would be a homomorphism as far as ω is concerned. Well, since this factorization is unique, any identities for the operators $\lambda = \lambda'$ satisfied in B will also be satisfied in X by the induced factorization. Thus, if we consider that this is true for all operators ω, then we indeed have (1) and (2) together.

Now we have to show (3). We consider any other homomorphism $h : B \to C$ with $hf = hg$. This must have a factorization in **Set** as $h = h'e$ for a unique function (between sets) h'. Now, all we have to prove is that the rightmost square commutes, for then h' would be a homomorphism. Using that h is a homomorphism and the commutativity of the middle diagram:

$$h'\omega_X e^n = h'e\omega_B = h\omega_B = \omega_C h^n = \omega_C {h'}^n e^n$$

And finally, using the almost trivial fact that the coequalizer e^n is epic (right cancellable), we indeed obtain the commutativity of the rightmost square. □

Remarks 3.14. The above theorem thus verifies what we claimed in our original discussion motivating monads, namely that we can recover the "free-forgetful" adjunction between **Set** and any variety of algebras just by the information contained in the monad "imprint" structure it leaves on **Set**. Note how this gives us a distinct, 'categorical' way of talking about **Mon**, **Rng**, **Grp**, etc; in each case these are just algebras over their corresponding monads. For this reason, we often name the monad by the category constructed out of its algebras, for example the 'monoid monad', or $T_{\mathbf{Mon}}$. For a τ-variety, these monads, or at least the endofunctors of these monads, are often easy to describe: they are just the underlying sets of the free objects in that variety.

Beck's Theorem allows us to assert that K is an isomorphism for any τ-variety. However, in specific cases of simple varieties such as **Mon**, it is possible and illuminating to exhibit a direct proof without Beck's Theorem. The proof in such a case is based on a simple idea: the structure map of an algebra over the monad can be interpreted as "n-fold products on n length strings", where associativity, for example, is built into the strucutre of the monad. One can then directly exhibit an isomorphism with the monoid one gets if he or she considers only the binary product in this n-fold product. See [1] for an example of this

Finally, note that we are not only interested in when the comparison functor K is an isomorphism. The question of the *monadicity* of an adjunction asks only that K be an equivalence of categories; that is, for $K : \mathcal{C} \to \mathcal{X}^T$ there is a functor $k : \mathcal{X}^T \to \mathcal{C}$ which also follows serial commutativity (i.e. is part of a map of adjunctions (k, L) where $L = 1$) such that $Kk \cong 1_{\mathcal{X}^T}$ and $kK \cong 1_{\mathcal{C}}$; in analogy with homotopy in topology, an equivalence of categories is often sufficient to capture invariant categorical information. To the end of discovering when such an equivalence occurs, one can apply a version of Beck's Theorem that identifies an equivalence K when an adjunction satisfies certain criteria (obviously, less stringent criteria than the version we offered). One can distinguish subtler features of the comparison K still:

the original adjunction is of *descent type* if K is full and faithful, and of *effective descent type* if it is futhermore an equivalence.

3.2. **The Kleisli construction.** We have seen that for any variety of algebras, the category and adjunction associated with it is, for all categorical considerations, just the Eilenberg-Moore adjunction. Γ_T, therefore, hardly seems like much of a category in this case - is there an adjunction that is neither isomorphic nor equivalent to the Eilenberg-Moore construction that gives rise to the same monad T? We start to understand Γ_T much better once we consider a canonical construction of the *initial* object in it, known as the *Kleisli adjunction.* Corresponding to our intuition of what an initial object is in familiar categories like **Top** or **Set**, the Kleisli adjunction can be considered to be the smallest adjunction giving rise to the monad T. Its associated category (known as the Kleisli category) can furthermore be considered the category of "free algebras for the monad". As we would then expect, its comparison functor is rarely an isomorphism or an equivalence of categories (although, as we will see, it is an equivalence with a certain subcategory). We will explore this idea and make them more precise in the following, leaving some proofs as simple exercises.

Definition 3.15 (Kleisli category)**.** Given a monad T on $\mathcal{X}$, the Kleisli category is denoted $\mathcal{X}_T$. Its objects are in bijection with those of $\mathcal{X}$: for every object $x \in \mathcal{X}$, we associate an object $x_T \in \mathcal{X}_T$. Meanwhile, the morphisms of $\mathcal{X}_T$ are such that $\mathcal{X}_T(x_T, y_T)$ is in bijection with $\mathcal{X}(x, Ty)$; in other words for every $f : x \to Ty$ we associate an arrow $f' : x_T \to y_T$. Composition is defined as follows: for $f' : x_T \to y_T$, $g : y_T \to z_T$,

$$g' \circ f' := (\mu_z \circ Tg \circ f)'$$

And the identity morphism $1 : x_T \to x_T$ is just $(n_x)'$. With these definitions it easy is simple (but tedious) to check that X_T is a category; the conditions of associativity and identity are fullfilled by the axioms of the monad.

Definition 3.16 (Kleisli adjunction)**.** Given a monad T on $\mathcal{X}$, the Kleisli construction is the adjunction $(F_T, U_T, \eta, \mu) : \mathcal{X} \to \mathcal{X}_T$ in Γ_T, where F_T and U_T are defined in the following way:

$$F_T : x \xrightarrow{f} y \dashrightarrow x_T \xrightarrow{\eta_y \circ f} y_T$$

$$U_T : x_T \xrightarrow{f'} y_T \dashrightarrow Tx \xrightarrow{\mu_z \circ Tf} Ty$$

We must again check the claims implicit in this definition. Verifying that F_T and U_T are in fact functors is an easy exercise involving quoting the monad axioms. It is furthermore easy to see, again using the monad axioms, that $T = U_T F_T$. The candidate for the unit for the adjunction η_T is, as it should be, the unit for the monad. Meanwhile, the components of the counit ε_{x_T} should be morphisms $F_T U_T x_T = Tx_T \to x_T$, and we choose the most obvious candidate $1'_{Tx}$. Checking that the triangle identies hold is yet another straightforward exercise involving the monad axioms. Thus, $(F_T, U_T, \eta, \mu) : \mathcal{X} \to \mathcal{X}_T$ is an adjunction. It is furthermore easy to check that $U_T \varepsilon_T F_T$ is the multiplication μ of the monad, thus completing the proof that this adjunction in fact defines the original monad T on $\mathcal{X}$.

Theorem 3.17. *The Kleisli adjunction $(F_T, U_T, \eta, \mu) : \mathcal{X} \to \mathcal{X}_T$ is initial in Γ_T.*

Proof. We again sketch the proof. Consider another adjunction $(F, G, \psi, \eta, \mu) : \mathcal{X} \to \mathcal{C}$ in Γ_T.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "F_T"] \arrow[dr, bend left, "F"] & \mathcal{X}_T \arrow[l, bend left, "U_T"] \arrow[d, dotted, "K"] \\
& \mathcal{C} \arrow[ul, bend left, "G"]
\end{tikzcd}$$

We need to prove, as in Theorem 3.9, that there is a unique functor K such that the above diagram commutes serially; this ensures that this is a map of adjunctions. We give the following candidate for K:

$$K : x_T \xrightarrow{f'} y_T \dashrightarrow Fx \xrightarrow{\psi^{-1}_{x,Fx}(f) = \varepsilon_{Fx} \circ F(f)} Fy$$

We leave it to the reader to check the following steps that would complete the proof:

1. The definition of K makes sense, and it is indeed a functor.
2. K satisfies the serial commutativity conditions $K \circ F_T = F$ and $G_T = G \circ K$, and is therefore a map of adjunctions.
3. K is unique. This is again an application of Lemma 3.8.

□

Consider the image of $\mathcal{X}_T$ under the comparison functor K: on objects it yields the "free" objects Fx. Indeed, together with the fact that the image of morphisms refers to the bijection ψ, it is not hard to see that the image of $\mathcal{X}_T$ under K is exactly the *full* subcategory $F\mathcal{X}$ of $\mathcal{C}$. This is not a coincidence; the definition of the Kleisli can be considered to have been engineered specifically to capture this. This is because if $(F, G, \phi) : \mathcal{X} \to \mathcal{C}$ is any adjunction then

$$Hom_{\mathcal{X}}(A, TB) = Hom_{\mathcal{X}}(A, GFB) \cong Hom_{\mathcal{C}}(FA, FB)$$

In fact, it is an elementary exercise in first reducing $\mathcal{X}_T$ to its skeleton to prove that:

Exercise 3.18. The restriction of K gives an equivalence of categories $\mathcal{X}_T \to F\mathcal{X}$

This explains why the Kleisli category is also known as the "free algebras for the monad". With this interpretation, U_T is indeed a forgetful functor in that it retrieves the underlying set of free objects, thus justifying the notion. Clearly, if we are comparing the Kleisli adjunction to the adjunction involving any typical variety of algebras such as **Grp**, K is not going to be an isomorphism or even an equivalence. However, consider another easy exercise:

Exercise 3.19. The comparison functor $K : \mathcal{X}_T \to \mathcal{X}^T$ is an embedding (that is, a faithful functor that takes non-isomorphic objects to non-isomorphic objects).

Thus, a K to **Grp** embeds the Kleisli category as the subcategory of free groups, while **Grp** are just the algebras from this category. All told, any adjunction can be thought to "lie between" the ("free-forgetful" adjunction correspondeing to the) free algebras over a monad and all the algebras over a monad. Under this light, the theory of monads can be rightfully considered to belong to the theory of adjoint functors.

4. Composition of Monads and Distributive Laws

Having seen that a whole collection of algebraic structures (in particular, any variety of algebras) can be viewed as algebras over monads on **Set**, our next question is what can be learned about them by viewing them this way? One answer concerns the interaction of such algebraic structures; we can often learn about them by studying the interaction between their monads. In this section, we are interested in a particular type of interaction: a composite. We can turn the problem of having a sensible composition of algebraic structures into a problem concerning a proper interaction between respective monads and their algebras (including a proper notion of a composition of monads).

The motivating example for this section is the case of rings. Rings involve an abelian group structure for addition, a monoid structure for multiplication, and a distributive law that tells us how the two are supposed to interact. We may ask: why does the distributive law take its particular form? Well, consider the monads $T_{\mathbf{Rng}}$, $T_{\mathbf{Mon}}$, $T_{\mathbf{Ab}}$ on **Set**. It is easy to see that as endofunctors, $T_{\mathbf{Rng}} = T_{\mathbf{Ab}}T_{\mathbf{Mon}}$. It is less clear that $T_{\mathbf{Ab}}$ and $T_{\mathbf{Mon}}$ can be composed as monads to give the ring monad. Nevertheless, they can, and each 'compatible' composite of these two monads, we will see, is equivalent to a natural transformation $\lambda : T_{\mathbf{MonAb}} \to T_{\mathbf{AbMon}}$ satisfying certain conditions. On the left, we have (for a given set X) formal products of formal sums, and on the right we have formal sums of formal products. Clearly, on components, λ_X is some sort of distributive law. For the composite of monads $T_{\mathbf{Ab}}T_{\mathbf{Mon}}$ that is exactly the ring monad, the unique distributive law to which it is equivalent is exactly the one we would expect for rings!

We will come back to this example. Monads allow us to speak more generally about 'distributive laws', and in this section we will explore notions and two key theorems that were discovered by Jon Beck and independently by J.P. May. The setup for this entire section is the following: we have two monads $(T_1, \eta_\otimes, \mu_\otimes)$ and $(T_2, \eta_\oplus, \mu_\oplus)$ on a category $\mathcal{C}$, where the notation (borrowed from May) is to indicate that we think of T_1 as *multiplicative* and T_2 as *additive.*

The first result is that three seemingly seperate notions concerning the interactions between these monads - 'distributive law', 'compatible composite', and 'lift' - are in fact equivalent.

Definition 4.1 (Distributive Law)**.** A distributive law of T_1 over T_2 is a natural transformation $\lambda : T_1T_2 \to T_2T_1$, that interacts properly with the unit and multiplication of each monad in the sense that the following diagrams commute:

$$
\begin{tikzcd}
 & T_1 \arrow[dl, "T_1\eta_\oplus"'] \arrow[dr, "\eta_\oplus T_1"] & \\
T_1T_2 \arrow[rr, "\lambda"] & & T_2T_1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T_2 \arrow[dl, "\eta_\otimes T_2"'] \arrow[dr, "T_2\eta_\otimes"] & \\
T_1T_2 \arrow[rr, "\lambda"] & & T_2T_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
T_1T_2^2 \arrow[r, "\lambda T_2"] \arrow[d, "T_1\mu_\oplus"'] & T_2T_1T_2 \arrow[r, "T_2\lambda"] & T_2^2T_1 \arrow[d, "\mu_\oplus T_1"] \\
T_1T_2 \arrow[rr, "\lambda"] & & T_2T_1
\end{tikzcd}
\qquad
\begin{tikzcd}
T_1^2T_2 \arrow[r, "T_1\lambda"] \arrow[d, "\mu_\otimes T_2"'] & T_1T_2T_1 \arrow[r, "\lambda T_1"] & T_2T_1^2 \arrow[d, "T_2\mu_\otimes"] \\
T_1T_2 \arrow[rr, "\lambda"] & & T_2T_1
\end{tikzcd}
$$

Definition 4.2 (Lift)**.** A *lift* of the monad T_2 is a monad T_2^* on $\mathcal{C}^{T_1}$ induced by the monad T_2. In detail, $(T_2^*, \eta_\oplus^*, \mu_\oplus^*)$ is a monad on C^{T_1} which satisfies the following relationships with the original monad T_2:

(1) $U^{T_1}T_2^* = T_2U^{T_1}$
(2) $U^{T_1}\eta_\oplus^* = \eta_\oplus U^{T_1} : U^{T_1}\dot{\mapsto}U^{T_1}T_2^*$
(3) $U^{T_1}\mu_\oplus^* = \mu_\oplus U^{T_1} : U^{T_1}(T_2^*)^2\dot{\mapsto}U^{T_1}T_2^*$

In the above definition, (1) says exactly that T_2^* sends a T_1 -algebra with underlying set X to one with underlying set T_2X (and is furthermore exactly T_2 on morphisms; but now these morphisms are interpreted as T_1-algebra morphisms). Thus, requirements (2) and (3) make sense. They furthermore say that the components of $\eta_\oplus^*$ and $\mu_\oplus^*$ on T_1-algebras with underlying sets X and T_2^2X respectively are simply $\eta_\oplus X$ and $\mu_\oplus X$; or rather, that the latter are now algebra morphisms wherever X is the underlying set of a T_1-algebra. Note that, therefore, where (1) is satisfied by a functor T_2^*, (2) and (3) determine $\eta_\oplus^*$ and $\mu_\oplus^*$ completely. All of these explain the terminology of a "lift". It is then no work to check that $(T_2^*, \eta_\oplus^*, \mu_\oplus^*)$ is a monad - all the identities hold if we just consider the underlying sets, and therefore, by (1)-(3), hold trivially as well when "lifted".

Finally we have to define the notion of a natural or compatible composite of monads. One of the conditions requires the notion of a map of monads. This is an important definition in the formal theory of monads; however, we will have no opportunity to use it more deeply than in definition 4.4.

Definition 4.3. A map of monads from T_1 to T_2 is a natural transformation $\alpha : T_1 \mapsto T_2$ such that the following diagrams commute:

$$\begin{tikzcd} & 1 \arrow[dl, "\eta_\otimes"'] \arrow[dr, "\eta_\oplus"] & \\ T_1 \arrow[rr, "\alpha"] & & T_2 \end{tikzcd} \qquad \begin{tikzcd} T_1^2 \arrow[r, "\alpha^2"] \arrow[d, "\mu_\otimes"'] & T_2^2 \arrow[d, "\mu_\oplus"] \\ T_1 \arrow[r, "\alpha"] & T_2 \end{tikzcd}$$

Definition 4.4. A monad (T_2T_1, η, μ) is a *compatible composite* with the monads T_1 and T_2 if:

(1) $\eta = \eta_\oplus\eta_\otimes (= \eta_\oplus T_1 \cdot \eta_\otimes)$
(2) $T_2\eta_\otimes : T_2\dot{\mapsto}T_2T_1$ and $\eta_\oplus T_1 : T_1\dot{\mapsto}T_2T_1$ are morphisms of monads
(3) The following diagram commutes:

$$\begin{tikzcd} & T_2T_1 \arrow[dl, "\eta_\oplus T_1T_2\eta_\otimes"'] \arrow[dr, "1"] & \\ (T_2T_1)^2 \arrow[rr, "\mu"] & & T_2T_1 \end{tikzcd}$$

Theorem 4.5. *For the given setup, the following data are equivalent:*

(1) A distributive law of T_1 over T_2
(2) A natural transformation $\mu : (T_2T_1)^2\dot{\mapsto}T_2T_1$ such that (T_2T_1, η, μ) is a compatible composite of T_2 and T_1
(3) A lifting T_2^* of T_2 to C^{T_1}

By equivalent we mean that (a) we can define processes to construct the data of one out of the other and (b) if we apply processes consecutively to return to a starting *point, we will have the identity. In other words, the data are in a state of mutual determination.*

Proof. In the interests of space, we will not have the opportunity here to provide a full proof, which is a substantial part of Beck's paper [4]. Here we will give all the explicit descriptions of the constructive processes in terms of (a), prove one case of the validity of such a process, and finally prove one case of mutual determination in the sense of (b); these parts of the full proof are intended to serve as an example of how we can combine various axioms (monad axioms, algebras for the monad axioms, etc) at once by "attaching adjacent commutative diagrams" in order to prove the desired commutativity of another diagram. Here are the constructive processes:

- Given (1) a distributive law λ, we obtain (3) a lift as follows:
$$T_2^*(A,\theta) = T_1T_2A \xrightarrow{\lambda} T_2T_1A \xrightarrow{T_2\theta} T_2A$$
- Given (3) a lift, we obtain (1) a distributive law λ as follows:
$$\lambda_A = T_1T_2A \xrightarrow{T_1T_2\eta_\otimes} T_1T_2T_1A \xrightarrow{T_2^*(\mu_\otimes)} T_1T_2A$$
- Given (1) a distributive law λ, we obtain (2) a multiplication μ as follows:
$$\mu = T_2T_1T_2T_1 \xrightarrow{T_2\lambda T_1} T_2T_2T_1T_1 \xrightarrow{T_2T_2\mu_\otimes} T_2T_2T_1 \xrightarrow{\mu_\oplus T_1} T_2T_1$$
- Given (2) a multiplication μ, we obtain (1) a distributive law λ as follows:
$$\lambda = T_1T_2 \xrightarrow{T_1T_2\eta_\otimes} T_1T_2T_1 \xrightarrow{\eta_\oplus T_1T_2T_1} T_1T_2T_1T_2 \xrightarrow{\mu} T_2T_1$$

We now prove that the process (1) $\to$ (3) is valid. All the conditions that must be met can be expressed by the commutativity of certain diagrams, so accordingly, we subject each such diagram to analysis to prove that it does in fact commute. We first prove that $T_2^*(A,\theta)$ satisfies the unit law for a T_1 algebra.

$$\begin{tikzcd}
T_2A \arrow[r, "\eta_\otimes T_2"] \arrow[d, "1"'] \arrow[dr, dotted, "T_2\eta_\otimes" description] \arrow[dr, phantom, "(2)", pos=0.3] \arrow[dr, phantom, "(1)"', pos=0.3] & T_1T_2A \arrow[d, "\lambda"] \\
T_2A & T_2T_1A \arrow[l, "T_2\theta"]
\end{tikzcd}$$

The diagram in bold is exactly the unit law when it commutes. In the dashed arrow we wrote a component of a valid natural transformation. Now consider the two triangles (1) and (2). They commute: (1) commutes because of a distributive law axiom, and (2) commutes because of the unit law for the structure map θ. It is clear that the commutativity of (1) and (2) imply the commutativity of the square in bold, and thus we have verified that $T_2^*(A,\theta)$ satisfies the unit law. This is a general technique: locate a desired commutativity diagram in an adjacent attachment of commutative diagrams to prove it is commutative. Next we show that $T_2^*(A,\theta)$

satisfies the associative law:

$$\begin{tikzcd}
T_1^2T_2A \arrow[r, "T_1\lambda"] \arrow[dd, "\mu_\otimes T_2"'] \arrow[ddr, phantom, "(1)"] & T_1T_2T_1A \arrow[r, "T_1T_2\theta"] \arrow[d, dotted, "\lambda T_1"'] \arrow[dr, phantom, "(2)"] & T_1T_2A \arrow[d, "\lambda"] \\
 & T_2T_1^2A \arrow[r, dotted, "T_2T_1\theta"] \arrow[d, dotted, "T_2\mu_\otimes"'] \arrow[dr, phantom, "(3)"] & T_2T_1A \arrow[d, "T_2\theta"] \\
T_1T_2A \arrow[r, "\lambda"'] & T_2T_1A \arrow[r, "T_2\theta"'] & T_2A
\end{tikzcd}$$

(1) commutes due to a distributivity law axiom, (2) commutes due to the naturality of λ, and (3) commutes because of the associative law for the structure map θ. Hence the diagram in bold commutes, verifying the associative law. $\eta_\oplus^*$ must furthermore be an algebra morphism:

$$\begin{tikzcd}
T_1A \arrow[r, "T_1\eta_\oplus"] \arrow[dr, dotted, "\eta_\oplus T_1"] \arrow[dd, "\theta"'] & T_1T_2A \arrow[d, "\lambda"] \\
(1) & T_2T_1A \arrow[d, "T_2\theta"] \\
A \arrow[r, "\eta_\oplus"'] & T_2A
\end{tikzcd}$$

The trapezoid (1) on the bottom commutes due to the naturality of $\eta_\oplus$, and the upper triangle commutes due to a distributive law axiom. That $\mu_\oplus^*$ is an algebra morphism is proved similarily, using naturality and a square distributive law axiom. Thus the process (1) $\to$ (3) indeed obtains a lift. The other cases of valid constructive processes are proved similarily.

Now, assuming all such processes are valid, we prove an instance of mutual determination of the data above by showing that applying consecutive processes (1) $\to$ (3) $\to$ (1) and (3) $\to$ (1) $\to$ (3) retrieves the identity. In the former case, we begin with a distributive property λ. Following (1) $\to$ (3), the possible variation in the action of T_2^* can only be in its action on structure maps, therefore, for a cleaner notation, we consider the underlying set implied and consider T_2^* a functor in structure maps. As such its action on $\mu_\otimes$ is given by:

$$T_2^*(\mu A) = T_1T_2T_1A \xrightarrow{\lambda T_1} T_2T_1T_1A \xrightarrow{T_2\mu_\otimes} T_2T_1A$$

And hence following (3) $\to$ (1) we obtain the following distributive law on components:

$$\tilde{\lambda}A = T_1T_2A \xrightarrow{T_1T_2\eta_\otimes} T_1T_2T_1A \xrightarrow{\lambda T_1} T_2T_1T_1 \xrightarrow{T_2\mu_\otimes} T_2T_1A$$

We need to show $\tilde{\lambda} = \lambda$. We do this by locating the composition for $\tilde{\lambda}$ in an adjacent attachment of known commutative diagrams:

$$\begin{tikzcd}
& T_1T_2A \arrow[dl, dotted, "1"'] \arrow[drrr, dotted, "\lambda"] & & \\
& T_1T_1T_2A \arrow[u, dotted, "\mu_{\otimes}T_2"'] \arrow[d, dotted, "T_1\lambda"] & & \\
T_1T_2A \arrow[ur, dotted, "T_1\eta_{\otimes}T_2"] \arrow[r, "T_1T_2\eta_{\otimes}"] & T_1T_2T_1A \arrow[r, "\lambda T_1"] & T_2T_1T_1A \arrow[r, "T_2\mu_{\otimes}"] & T_2T_1A
\end{tikzcd}$$

The commutativity of the individual pieces together gives that the composition (in bold) for $\tilde{\lambda}A$ is the same as the composition given by the topmost 'route' so that indeed $\tilde{\lambda} = \lambda$. Now we examine (3) $\to$ (1) $\to$ (3). Given a lift T_2^*, following these consecutive processes prescribes the following lift:

$$\tilde{T_2}^*(\theta) : T_1T_2A \xrightarrow{T_1T_2\eta_{\otimes}} T_1T_2T_1A \xrightarrow{T_2^*\mu_{\otimes}} T_2T_1A \xrightarrow{T_2\theta} T_2A$$

for a T_1-structure map θ. Much like before, we can prove $\tilde{T_2}^* = T_2^*$ by locating it in an adjacent attachment of known commutative diagrams:

$$\begin{tikzcd}
& T_1T_2A \arrow[drr, dotted, "T_2^*(\theta)"] & & \\
T_1T_2A \arrow[ur, dotted, "1"] \arrow[r, "T_1T_2\eta_{\otimes}"] & T_1T_2T_1A \arrow[u, dotted, "T_1T_2\theta"'] \arrow[r, "T_2^*\mu_{\otimes}"] & T_2T_1A \arrow[r, "T_2\theta"] & T_2A
\end{tikzcd}$$

And again, since the composition defining $\tilde{T_2}^*$ is the same as the topmost 'route', $\tilde{T_2}^* = T_2^*$. The proof of the mutual determination of the data in (1) and (3) is complete, and the other cases are proved similarily. $\square$

With this theorem, we can understand better the motivating example of rings. The known distributive law for rings determines, for each set X, a map $\lambda_X : T_{\mathbf{Mon}}T_{\mathbf{Ab}} \to T_{\mathbf{Ab}}T_{\mathbf{Mon}}$. It can be checked that (λ_X) are the components of a natural transformation λ, which is a distributive law (in the sense of definition 4.1) for $T_{\mathbf{Mon}}$ over $T_{\mathbf{Ab}}$. By the above theorem, this is equivalent to a compatible composite monad $T_{\mathbf{Ab}}T_{\mathbf{Mon}}$. The reader is invited to verify that this is exactly the ring monad.

So what about the lift determined (again by 4.5) by this ring monad? In this case it would be an induced monad $T^*_{\mathbf{Ab}}$ on $\mathcal{C}^{T_{\mathbf{Mon}}}$. But, as we have seen, $C^{T_{\mathbf{Mon}}}$ is simply the category of monoids **Mon**. Thus, a lift here is an (induced) 'abelian group monad' on monoids. Our intuition that rings are abelian groups on monoids can be made precise: taking algebras of this induced monad on monoids will give us exactly rings. We will see how this happens by deducing an isomorphism between the relevant categories of algebras. However, as always we are interested in an isomorphism that also preserves the adjunction structures involved (in this case, the Eilenberg-Moore adjunctions that naturally participate). We will first need to define a certain 'horizontal' composite of adjunctions (the terminology 'horizontal' is justified in a certain 2-category of adjunctions, but we need not explore this further currently).

Lemma 4.6 ('Horizontal' Composition of Adjunctions)**.** *Given two adjunctions $(F, G, \eta, \varepsilon) : \mathcal{A} \to \mathcal{B}$ and $(F', G', \eta', \varepsilon') : \mathcal{B} \to \mathcal{C}$*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \arrow[r, bend left, "F'"] & \mathcal{C} \arrow[l, bend left, "G'"] \end{tikzcd}$$

they can be composed horizontally, that is, their composite functors $F'F$, $G'G$ yield an adjunction

$$(F'F, G'G, G\eta'F \cdot \eta, \varepsilon' \cdot F'\varepsilon G') : \mathcal{A} \to \mathcal{C} \tag{4.7}$$

Proof. We invoke the isomorphisms of bifunctors involved in each adjunction to retrieve the following chain of natural bijections, for objects $a \in \mathcal{A}$ and $c \in C$:

$$C(F'Fa, c) \cong B(Fa, G'c) \cong (a, GG'c)$$

and so $F'F \dashv GG'$. By considering the composite natural bijection on the identity $1 : F'Fa \to F'Fa$, we retrieve the unit of the composite adjunction: $\eta = G\eta'F \cdot \eta$. Dually, we get the desired formula for counit. □

Now consider monads T_1 and T_2 on $\mathcal{C}$ as before. A lift T_2^* of T_2 enables the horizontal composite of the following adjunctions:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F^{T_1}"] & \mathcal{C}^{T_1} \arrow[l, bend left, "U^{T_1}"] \arrow[r, shift left, "F^{T_2^*}"] & (\mathcal{C}^{T_1})^{T_2^*} \arrow[l, shift left, "U^{T_2^*}"] \end{tikzcd}$$

Lemma 4.8. *The horizontal composite of adjunctions for the above situation defines on $\mathcal{C}$ the 'compatible composite' monad determined by (and equivalent to) the lift as per theorem 4.5*

Proof. We just need to verify that the horizontal composite defines in $\mathcal{C}$ a monad with the correct endofunctor and unit as per definition 4.4, with multiplication determined by the lift in 4.5. Applying formula 4.7, the horizontal composition of adjunctions above is the adjunction

$$(F^{T_2^*}F^{T_1}, U^{T_1}U^{T_2^*}, U^{T_1}\eta^*_{\oplus}F^{T_1} \cdot \eta_{\otimes}, \varepsilon^*_{\oplus} \cdot F^{T_2^*}\varepsilon_{\otimes}U^{T_2^*}) : \mathcal{C} \to (\mathcal{C}^{T_1})^{T_2^*} \tag{4.9}$$

We label this adjunction $(F, U, \eta, \varepsilon)$. The monad it defines in $\mathcal{C}$ has endofunctor

$$U^{T_1}U^{T_2^*}F^{T_2^*}F^{T_1} = U^{T_1}T_2^*F^{T_1} = T_2U^{T_1}F^{T_1} = T_2T_1$$

as desired, where we used property (1) of the lift (4.2). Meanwhile applying property (2) simplies the expression for the unit:

$$U^{T_1}\eta_{\oplus^*}F^{T_1} \cdot \eta_{\otimes} = \eta_{\oplus}U^{T_1}F^{T_1} \cdot \eta_{\otimes} = \eta_{\oplus}T_1 \cdot \eta_{\otimes}$$

To see the multiplication for the defined monad, let us first understand the counit of the composite adjunction explicitly. To begin with, an object in $(\mathcal{C}^{T_1})^{T_2^*}$ is a T_2-algebra on a T_1 algebra, and is thus fully described by (A, σ, τ), where σ is a T_1-structure map and τ is a structure map to (A, σ). (A, σ, τ) can be depicted as follows:

$$\begin{tikzcd} T_1T_2A \arrow[r, "\lambda"] \arrow[d, "T_1\tau"] & T_2T_1A \arrow[r, "T_2\sigma"] & T_2A \arrow[d, "\tau"] \\ T_1A \arrow[rr, "\sigma"] & & A \end{tikzcd} \tag{4.10}$$

Note that the top row is T_2^* applied to the T_1 algebra (A, σ) in the bottom row. Here, we have availed of theorem 4.5, first to get the distributive law λ equivalent to the lift, and secondly to identify how this λ is part of the description of the T_2^* action. Note also: τ is both a T_1-algebra morphism and, due to the lifting property of T_2^*, also a T_2-algebra structure map. Thus, referring to the Eilenberg-Moore construction, and once again bearing in mind the lifted nature of T_2^* we derive the following formula for $F^{T_2^*}$:

$$F^{T_2^*}(A, \sigma, \tau) = (T_2 A, T_2\sigma \cdot \lambda_A, \mu_\oplus A)$$

The counit formula from 4.9 reads:

$$F^{T_2^*} F^{T_1} U^{T_1} U^{T_2^*}(A, \sigma, \tau) \xrightarrow{F^{T_2^*}\varepsilon_\otimes U^{T_2^*}} F^{T_2^*} U^{T_2^*}(A, \sigma, \tau) \xrightarrow{\varepsilon_\oplus^*} (A, \sigma, \tau)$$

This can now be read explicitly as:

$$\varepsilon_A : (T_2T_1A, T_2\mu_{\otimes A}{\cdot}\lambda T_1 A, \mu_\oplus T_1 A) \xrightarrow{F^{T_2^*}\varepsilon_\otimes U^{T_2^*}} F^{T_2^*}(T_2A, T_2\sigma{\cdot}\lambda_A, \mu_\oplus A) \xrightarrow{\varepsilon_\oplus^*} (A, \sigma, \tau)$$

The Eilenberg-Moore construction in 3.4 gives an simple description of the counit in an adjunction with algebras: on components it just the structure map at that component. We can apply this interpretation to counits $\varepsilon_\otimes$ and $\varepsilon_\oplus^*$ to easily deduce from the above that $F^{T_2^*}\varepsilon_\otimes U^{T_2^*} = T_2\sigma$ and $\varepsilon_\oplus^* = \tau$. Thus we get the simple explicit definition of the counit for the overall adjunction;

$$\varepsilon_{(A,\sigma,\tau)} = \tau \circ T_2\sigma \tag{4.11}$$

Recalling from the Eilenberg-Moore construction that the multiplication of a monad induced by an adjunction is always just the counit on the free objects, that is, $\mu_A = \varepsilon_{F^{T_2^*} F^{T_1} A}$, we observe that since $F^{T_2^*} F^{T_1} A = (T_2T_1A, T_2\mu_{\otimes A}{\cdot}\lambda_{T_1 A}, (\mu_\oplus T_1)_A)$, by the above formula4.11 for the counit,

$$\mu_A = \mu_\oplus T_1 \cdot T_2T_2T_1A, T_2\mu_\otimes A \cdot \lambda T_1 A$$

which is exactly the multiplication determined by the lift as per 4.5. $\square$

Now, for a lift of T_2, consider again the 'compatible composite' monad $T = T_2T_1$ to which it is equivalent. We observe the following situation

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[dr, bend left, "F^{T_2T_1}"] & (\mathcal{C}^{T_1})^{T_2^*} \arrow[l, bend left, "U"] \arrow[d, dotted, "K"] \\
& \mathcal{C}^{T_2T_1} \arrow[ul, bend left, "U^{T_2T_1}"]
\end{tikzcd}$$

Where the above lemma tells us exactly that both these adjunctions belong to $\Gamma_{T_2T_1}$. Since the Eilenberg-Moore adjunction pictured on the bottom is the terminal object in this category, we have the comparison functor K shown.

Theorem 4.12. *In the diagram above, K is an isomorphism of categories*

Proof. The proof of theorem 3.9 gave an explicit description of the comparison functor. Applying it here, for $f : (A, \sigma, \tau) \to (A', \sigma', \tau')$ in $(\mathcal{C}^{T_1})^{T_2^*}$ we have

$$\begin{aligned} K(A, \sigma, \tau) &= (U(A, \sigma, \tau), U\varepsilon_(A, \sigma, \tau)) = (A, \tau \circ T_2\sigma) \\ Kf &= Uf = f \end{aligned}$$

Where in the simplification we have availed of the explicit formula (4.11) for the counit of the adjunction pictured on the top of the above diagram. Note how the formula for K on objects (A, σ, τ) has an interpretation as a certain composition in the diagram 4.10.

To prove that K is an isomorphism we will exhibit an inverse for it. It is trivial to see that an inverse will also commute serially in the above diagram, and hence will be a valid map of adjunctions in the sense of definition 3.6.

An inverse map to K needs to define for every T_2T_1 algebra (A, θ) a T_2 algebra on a T_1 algebra on A. In the proof for the previous lemma, we commented that this consists of a T_1 structure map σ and a T_2 structure map τ (that is also a T_1 algebra morphism). Here are the candidates:

$$\begin{aligned} \sigma : T_1A &\xrightarrow{\eta_\oplus T_1} T_2T_1A \xrightarrow{\theta} A \\ \tau : T_2A &\xrightarrow{T_2\eta_\otimes} T_2T_1A \xrightarrow{\theta} A \end{aligned}$$

That is, we precompose the T_2T_1-structure map along maps of monads (see 4.4). The reader can check that the fact that these are maps of monads ensures that σ and τ are indeed T_1 and T_2 structure maps respectively. If we recall that the unit of a "free-forgetful" adjunction can be thought of as an "insertion of generators", the definitions of σ and τ have an interpretation: we "locate" the multiplication and additive structure, respectively, in an algebra for both structures. If we again recall diagram 4.10, the condition that τ is a T_1 algebra morphism is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} T_1T_2A \arrow[rr, "\lambda"] \arrow[d, "T_1\tau"'] & & T_2T_1A \arrow[d, "T_2\sigma"] \\ T_1A \arrow[dr, "\sigma"'] & & T_2A \arrow[dl, "\tau"] \\ & A & \end{tikzcd}$$

Beck [4] calls this diagram the "λ distributivity of σ over τ". We verify the commutativity of this diagram (which gives us that the candidate (A, σ, τ) for an inverse map on objects (A, θ) is in fact a member of $(\mathcal{C}_1^T)^{T_2^*})$ and that $\theta = \tau \circ T_2\sigma$ (which gives us that this candidate indeed furnishes an *inverse* to K) in opposite order. Exploiting various aspects of the equivalent data from theorem 4.5, and the fact that θ is a T_2T_1-algebra, we can prove the second fact by the following adjacent attachment of commutative diagrams:

$$\begin{tikzcd}
 & T_2T_1A \arrow[dl, "T_2\eta_\oplus T_1"'] \arrow[d, dotted, "T_2\eta_\otimes\eta_\oplus T_1"] \arrow[dr, dotted, "T_2\eta_\oplus\eta_\otimes T_1"] \arrow[drr, "1"] & & \\
T_2T_2T_1A \arrow[r, dotted, "T_2\eta_\otimes T_2T_1"'] \arrow[d, "T_2\theta"'] & T_2T_1T_2T_1A \arrow[r, dotted, "T_2\lambda T_1"'] & T_2T_2T_1T_1A \arrow[r, dotted, "\mu_\oplus\mu_\otimes"'] & T_2T_1A \arrow[d, "\theta"] \\
T_2A \arrow[r, "T_2\eta_\otimes"] & T_2T_1A \arrow[rr, "\theta"] & & A
\end{tikzcd}$$

The two 'routes' from T_2T_1A to A are equal, so, using the definitions of τ and σ, this gives $\theta = \tau \circ T_2\sigma$. Now, to prove "$\lambda$ distributivity of σ over τ", we again appeal to various to the results contained in the various data of 4.5 and analyze the following adjacent attachment of commutative diagrams:

$$\begin{tikzcd}
 & T_1T_2A \arrow[dl, "T_1T_2\eta_\otimes"'] \arrow[d, dotted, "\eta_\oplus T_1T_2\eta_\otimes"] \arrow[drr, "\lambda"] & & \\
T_1T_2T_1A \arrow[r, dotted, "\eta_\oplus T_1T_2T_1"'] \arrow[d, "T_1\theta"'] & T_2T_1T_2T_1A \arrow[r, dotted, "T_2\lambda T_1"'] \arrow[d, dotted, "T_2T_1\theta"] & T_2T_2T_1T_1 \arrow[r, dotted, "\mu_\oplus\mu_\otimes"'] & T_2T_1A \arrow[d, "\theta"] \\
T_1A \arrow[r, "\eta_\oplus T_1"] & T_2T_1A \arrow[rr, "\theta"] & & A
\end{tikzcd}$$

Like before, we have two equal 'routes' from T_1T_2A from A. Together with the definitons of τ, σ, and the previously proved fact $\theta = \tau \circ T_2\sigma$, we get "$\lambda$-distributivity". This inverse map of K on objects extends trivially to the required inverse functor K^{-1}. $\square$

Recalling our discussion of rings, this theorem now gives an elegant justification and intepretation for the statement that rings are abelian groups *on* monoids.

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