

Wedge cancellation and Mislin genus of certain suspensions

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Abstract

We obtain some results on non-cancellation of spheres as wedge summands of certain $\text{co-}H_0$ -spaces and Mislin genera of such spaces. The spaces concerned are cones on certain maps $X \rightarrow Y$, such that one of the spaces is a sphere, while the other is a wedge of spheres. The main technique is the utilization of certain results on presentations of finite \mathbb{Z} -modules, and we exploit (and compute) the group structure on the genera. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Freyd [2] gave examples showing non-cancellation for wedge sums of spaces in stable homotopy. Studies in the nonstable case appear in the work of Hilton [3], Molnar [9], Bokor [1] and others. The spaces that we consider in this article are 1-connected $\text{co-}H_0$ -spaces, i.e., 1-connected spaces having the rational homotopy type of wedges of spheres. We recall that, dually, a 1-connected space is called an H_0 -space if it has the rational homotopy type of a product of Eilenberg–MacLane spaces. The Mislin genus $\mathcal{G}(X)$ of a nilpotent space X of finite type is the set of all homotopy types of spaces Y such that Y is of finite type and for every prime p the p -localization of X is homotopy equivalent to the p -localization of Y . For $\text{co-}H_0$ -spaces there is a close relationship between Mislin genus and cancellation of wedge summands. This can be observed in the work of Molnar [9] and Bokor [1] for instance. See also the survey article [7] of McGibbon. Zabrodsky [14] introduced a group structure on the Mislin genus for certain H_0 -spaces. This topic is also addressed in [7].

In the article [8], McGibbon enlarges the class of spaces for which the Mislin genera has an Abelian group structure similar to that introduced by Zabrodsky. In particular, in [8,

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Theorem 4], the condition of 1-connectedness on an H_0 -space is replaced by nilpotency. This result was used by Hilton and Scevenels [5] to compare the genera of certain torus bundles with the genera of certain nilpotent groups. More results of this nature are obtained in [10]. Furthermore, for certain co- H_0 -spaces, McGibbon [8, Theorem 5] introduced the structure of an Abelian group on the Mislin genus set of such a space. We shall compute this group structure on the genera of certain mapping cones.

In [10] we study presentations of certain modules over a principal ideal domain. In particular we computed certain sets of equivalence classes of presentations of finite \mathbb{Z} -modules. Some of those results are quoted in this article as Theorems 2.1 and 2.2. The results obtained in [10] shed new light on the computation of genera, in the sense of Hilton and Mislin [4], of certain nilpotent groups, and more generally, it facilitates the study of non-cancellation of groups, as shown in [10]. In this article we use Theorems 2.1 and 2.2 as the main tools for studying non-cancellation phenomena in homotopy, and for the computation of Mislin genus groups in the sense of [8, Theorem 5].

The contents of the different sections are as follows. In Section 2 we quote two results on presentations of finite \mathbb{Z} -modules. In Section 3 we consider the mapping cones of certain types of maps $f: X \rightarrow S^n$, where X is a bouquet of $(m-1)$ -spheres. In particular, for such maps we study (non-)cancellation of S^m in $C_f \vee S^m$. The main results here are Theorems 3.3 and 3.7. In Section 4 we compute, Theorem 4.3, the genus group of C_f for a map f of the type considered in Section 3. Finally, Section 5 is devoted to mapping cones of certain maps $g: S^{m-1} \rightarrow Y$, where Y is a bouquet of n -spheres. This can be regarded as being a continuation of work in [1] of Bokor. We study cancellation of S^n for such mapping cones, and obtain results similar to those of Sections 3 and 4.

Functions between topological spaces will always be assumed to be continuous. We shall often use the same notation for a continuous function and its (pointed) homotopy class—it will be clear from the context what is meant.

2. An equivalence relation on morphism sets

For Abelian groups G and H let $\text{Hom}[G, H]$ be the set of all morphisms $G \rightarrow H$. We define a relation \sim on $\text{Hom}[G, H]$ as follows. For $f_1, f_2 \in \text{Hom}[G, H]$, $f_2 \sim f_1$ if and only if there is an automorphism $\alpha: G \rightarrow G$ such that $f_2 = f_1 \circ \alpha$. This relation can easily be seen to be an equivalence relation. For a finite Abelian group A and a positive integer k , let $E_k(A)$ be the set of all epimorphisms $\mathbb{Z}^k \rightarrow A$. Of course, $E_k(A)$ is nonempty if and only if A can be generated by some finite subset of cardinality at most k . Then \sim is an equivalence relation on $E_k(A)$, called Nielsen equivalence. The set of equivalence classes will be denoted by $E_k^\sim(A)$.

We recall the following results concerning the set $E_k(A)$ from [13,10].

Theorem 2.1 (cf. [13, Theorem 3.4]). *Let A be a finite Abelian group of rank m , and let k be any integer bigger than m . Then $E_k^\sim(A)$ is trivial, i.e., for any epimorphisms f and g of \mathbb{Z}^k onto A , $f \sim g$.*

For a finitely generated Abelian group A , the rank, $r(A)$, of A is the minimum of the cardinalities of generating subsets of A . If A is a finite Abelian group, the symbol $\rho(A)$ will be used to denote the smallest of all positive integers n such that $r(nA) < r(A)$. For example, $r(\mathbb{Z}_{10} \times \mathbb{Z}_{25} \times \mathbb{Z}_9) = 2$ and $\rho(\mathbb{Z}_{10} \times \mathbb{Z}_{25} \times \mathbb{Z}_9) = 5$.

The next result is proved in [10].

Theorem 2.2. *Let A be a finite Abelian group. Let k be the rank of A and let $d = \rho(A)$. There is a transitive action of \mathbb{Z}_d^* on $E_k^\sim(A)$ defined as follows. For $f \in E_k(A)$ let us denote its \sim class by $[f]$. If $x \in \mathbb{Z}_d^*$ and x_0 is any element representing x , then*

$$(x, [f]) \mapsto [f \circ \beta],$$

for any endomorphism β of \mathbb{Z}^k such that $\text{Det}(\beta) \equiv \pm x_0 \pmod{d}$.

The isotropy subgroup of the action is the subgroup of \mathbb{Z}_d^* generated by the residue class of -1 .

From this theorem it follows that if $k = r(A)$, we can pick any element of $E_k^\sim(A)$ as neutral element and obtain a group structure on $E_k^\sim(A)$ which is isomorphic to $\mathbb{Z}_d^*/\{1, -1\}$. If k is bigger than the rank of A , then by Theorem 2.1, $E_k^\sim(A)$ is a one-element set, and is trivially a group.

3. Cones on maps into a sphere

Let n and m be integers such that $m - 1 > n > 1$. We are interested in the class $\mathcal{L}(m, n)$ of all suspension maps $g : S^{m-1} \rightarrow S^n$.

We note that since $m - 1 > n$, for every $g \in \mathcal{L}(m, n)$, the order of the homotopy class $[g]$ in the group $\pi_{m-1}(S^n)$ is finite. Furthermore, for a degree r self-map μ of S^n , we have that in $\pi_{m-1}(S^n)$, $[\mu \circ g] = r[g]$. This fact permits us to use simple matrix methods (see also Remark 4.5).

Fix positive integers k, m and n such that $m - 1 > n > 1$. We consider spaces X_1, X_2, \dots, X_k , admitting homotopy equivalences $h_i : S^{m-1} \rightarrow X_i$. Let $X = \bigvee_{i=1}^k X_i$. For a map $f : X \rightarrow S^n$, we let $f_i = f \circ h_i$ for each $i \in \{1, 2, \dots, k\}$. Then the function $f \mapsto (f_1, f_2, \dots, f_k)$ induces a bijection between the set $[X, S^n]$ of homotopy classes of maps $X \rightarrow S^n$, and the k th Cartesian power $[\pi_{m-1}(S^n)]^k$. We now define $\mathcal{L} = \mathcal{L}(m, n, k)$ to be the class of all maps $f : X \rightarrow S^n$ such that $f_i \in \mathcal{L}(m, n)$ for each positive integer $i \leq k$.

Let $\phi : X \rightarrow X$ be a self-map of X . For each $i \in \{1, 2, \dots, k\}$, let $\varepsilon_i : X_i \rightarrow X$ be the inclusion and let $\text{pr}_i : X \rightarrow X_i$ be the projection. For each pair of integers $i, j \in \{1, 2, \dots, k\}$, let l_i be a homotopy inverse of h_i , and define ϕ_{ij} to be the degree of the self-map $l_i \circ \text{pr}_i \circ \phi \circ \varepsilon_j \circ h_j$ of S^n . For each self-map ϕ of X , we obtain a $k \times k$ -matrix, (ϕ_{ij}) , with integer coefficients. The function $\phi \mapsto (\phi_{ij})$ sets up a bijection $[X, X] \rightarrow M_n(\mathbb{Z})$, where $M_n(\mathbb{Z})$ is the set of all $n \times n$ -matrices with integer coefficients. The bijection is such that ϕ is a homotopy equivalence if and only if the matrix (ϕ_{ij}) is unimodular, i.e., the matrix (ϕ_{ij}) is of determinant ± 1 .

Let $\{e_1, e_2, \dots, e_k\}$ be the canonical \mathbb{Z} -basis of the free Abelian group \mathbb{Z}^k . Then for every $f : X \rightarrow S^n$ there is a homomorphism $\alpha_f : \mathbb{Z}^k \rightarrow \pi_{m-1}(S^n)$ defined by $e_i \mapsto f_i$ for each $i \in \{1, 2, \dots, k\}$. The function $f \mapsto \alpha_f$ induces a one-to-one correspondence between $[X, S^n]$ and the set of all group homomorphisms $\mathbb{Z}^k \rightarrow \pi_{m-1}(S^n)$.

Suppose that $\psi : S^n \rightarrow S^n$ and $\phi : X \rightarrow X$ are homotopy equivalences and that $f, g \in \mathcal{L}$. Note that up to homotopy, there are only two choices for ψ , namely, degree 1 or -1 . Then the square shown below is homotopy commutative,

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ f \downarrow & & \downarrow g \\ S^n & \xrightarrow{\psi} & S^n \end{array}$$

if and only if the associated triangle is commutative. In the triangle, β_ϕ is the homomorphism such that the matrix of β_ϕ with respect to the basis $\{e_1, e_2, \dots, e_k\}$ is (ϕ_{ij}) .

$$\begin{array}{ccc} \mathbb{Z}^k & \xrightarrow{\beta_\phi} & \mathbb{Z}^k \\ \alpha_{\psi \circ f} \searrow & & \swarrow \alpha_g \\ & \pi_{m-1}(S^n) & \end{array}$$

From the observations above (together with an argument similar to that in the paper [9] of Molnar, for instance), we can conclude the following.

Proposition 3.1. For maps f and g in \mathcal{L} , $C_f \simeq C_g$ if and only if there are homotopy equivalences $\psi : S^n \rightarrow S^n$ and $\phi : X \rightarrow X$ such that $\alpha_{\psi \circ f} = \alpha_g \circ \beta_\phi$.

An alternative formulation of Proposition 3.1, in terms of the relation \sim of Section 2, goes as follows.

Proposition 3.2. For maps f and g in \mathcal{L} , $C_f \simeq C_g$ if and only if $\alpha_{\psi \circ f} \sim \alpha_g$, for some homotopy equivalence $\psi : S^n \rightarrow S^n$.

Now for a homotopy equivalence $\psi : S^n \rightarrow S^n$, and a map $f \in \mathcal{L}$, it follows that $\alpha_{\psi \circ f} = \pm \alpha_f$ (i.e., either $\alpha_{\psi \circ f} = \alpha_f$ or $\alpha_{\psi \circ f}$ coincides with the pointwise additive inverse of α_f). Since $\alpha_f \sim -\alpha_f$ anyway, we have yet another formulation of the previous proposition.

Theorem 3.3. For maps f and g in \mathcal{L} , $C_f \simeq C_g$ if and only if $\alpha_f \sim \alpha_g$.

Definition 3.4. For a map $f \in \mathcal{L}$ we define

- (1) $\mathcal{L}_\chi(f)$ to be the set of all homotopy types of spaces Y such that Y is homotopy equivalent to the mapping cone of some map in \mathcal{L} and $Y \vee S^m \simeq C_f \vee S^m$, and
- (2) H_f to be the image of α_f in $\pi_{m-1}(S^n)$.

The group H_f will be seen to play a role similar to the group G_ϕ due to Bokor [1], that will be introduced in Section 5.

Proposition 3.5. *Let f and g be maps in \mathcal{L} , such that the restrictions $f|_{X_k}$ and $g|_{X_k}$ are constant. Then $C_f \simeq C_g$ if and only if $H_f = H_g$.*

Proof. By Theorem 3.3, $C_f \simeq C_g$ if and only if $\alpha_f \sim \alpha_g$. Certainly the condition $\alpha_f \sim \alpha_g$ implies that $H_f = H_g$. By Theorem 2.1, the condition $H_f = H_g$ together with the fact that the rank of H_f is less than k , implies that $\alpha_f \sim \alpha_g$. \square

Immediately from Proposition 3.5 we can deduce the following.

Proposition 3.6. *Let f and g be maps in \mathcal{L} . Then $C_f \vee S^m \simeq C_g \vee S^m$ if and only if $H_f = H_g$.*

Theorem 3.7. *Fix f in \mathcal{L} . For any $h \in E_k(H_f)$, let $\gamma(h)$ be the (homotopy type of the) cone C_g of a map $g \in \mathcal{L}$ such that $\alpha_g = h$. Then the function $\gamma : E_k(H_f) \rightarrow \mathcal{L}\chi(f)$ induces a bijection,*

$$\Gamma : E_k^\sim(H_f) \rightarrow \mathcal{L}\chi(f).$$

Proof. The function $\gamma : E_k(H_f) \rightarrow \mathcal{L}\chi(f)$ is well-defined, and by Proposition 3.6, γ is surjective. From Proposition 3.5 it follows that the function $\gamma : E_k(H_f) \rightarrow \mathcal{L}\chi(f)$ does factorize through $E_k^\sim(H_f)$, and that the induced map Γ is injective. \square

Corollary 3.8. *Let f be a map in \mathcal{L} .*

- (1) *If H_f is of rank smaller than k , then $\mathcal{L}\chi(f)$ is trivial.*
- (2) *If the rank of H_f is k , then there is a bijection between $\mathcal{L}\chi(f)$ and $\mathbb{Z}_d^*/\{1, -1\}$, where $d = \rho(H_f)$.*

Proof. In view of the bijection of Theorem 3.7, (1) follows by Theorem 2.1 and (2) follows by Theorem 2.2. \square

In fact, similar to the concluding remark in Section 2, the set $\mathcal{L}\chi(f)$ is a group with the homotopy type of C_f as neutral element.

For the following result, we shall use the following notation. Consider any map $f \in \mathcal{L}$. Then the Abelian group H_f can be decomposed as a direct sum of cyclic subgroups, $H_f = F_1 \oplus F_2 \oplus \dots \oplus F_r$, for some positive integer r , $r \leq k$, such that for each $i \in \{2, 3, \dots, r\}$, $|F_{i-1}|$ is an integer multiple of $|F_i|$.

Proposition 3.9. *Let $f \in \mathcal{L}$ be as above. Then there exists $g \in \mathcal{L}$ such that $C_f \simeq C_g$ and such that for each $i \in \{1, 2, \dots, r\}$, the subgroup of $\pi_{m-1}(S^m)$ generated by the homotopy class $[g_i]$ is precisely F_i and for each $i \in \{r + 1, r + 2, \dots, k\}$, $[g_i] = 0$.*

Proof. From [10, Lemma 2.4] it follows that $\alpha_f \sim v$ for some $v: \mathbb{Z}^k \rightarrow H_f$ such that for each $i \in \{1, 2, \dots, r\}$, $v(e_i)$ generates F_i , and for each $i \in \{r + 1, r + 2, \dots, k\}$, $v(e_i) = 0$. We can pick g such that $\alpha_g = v$. \square

4. The genus group

In this section we study spaces of the type introduced in Section 3, and we use the same notation. We shall compute, in detail, the Mislin genus of the mapping cone C_f of a map $f \in \mathcal{L}(m, n, k)$.

Now let us fix a map $f \in \mathcal{L}$. The subgroup of H_f generated by f_i is denoted by F_i . Being only interested in the homotopy type of C_f , by Proposition 3.9 we can assume that f is such that the following condition holds: $H_f = F_1 \oplus F_2 \oplus \dots \oplus F_r$ for some positive integer r , $r \leq k$, and $F_i = 0$ for each $r < i \leq k$, and that for each $i \in \{2, 3, \dots, k\}$, $|F_{i-1}|$ is an integer multiple of $|F_i|$.

Let $Y = (\bigvee_{i=1}^k T_i) \vee S^n$, where for each i , T_i is an m -dimensional sphere. We define a map $g: Y \rightarrow C_f$ as follows. Note that C_f can be regarded to be an m -cellular extension of S^n , and the homotopy classes of the characteristic maps of the m -cells can be labeled $u_1, u_2, \dots, u_k \in \pi_m(C_f, S^n)$, such that for each i , u_i is the homotopy class of the map $(CT_i, T_i) \rightarrow (C_f, S^n)$. Here, CT_i is the cone on T_i .

We choose $g: Y \rightarrow C_f$ as follows:

The restriction $g|_{S^n}$ is the inclusion map onto the subspace S^n of C_f , and under the injection $\pi_m(C_f) \rightarrow \pi_m(C_f, S^n)$, the homotopy class v_i of the restriction $g|_{T_i}$ is mapped onto $|F_i|u_i$. Then g is a maximal map in the sense of [8].

Now let $\phi: Y \rightarrow Y$ be any map. Then with ϕ is associated an integer $r = r_\phi$, the degree of the obvious composition below, and this self-map of S^n will be called ϕ_0 .

$$S^n \xrightarrow{\text{incl}} Y \xrightarrow{\phi} Y \xrightarrow{\text{proj}} S^n.$$

Furthermore, ϕ determines a map

$$\phi_1: \bigvee_{i=1}^k T_i \rightarrow \bigvee_{i=1}^k T_i$$

in a similar manner, and this map determines a matrix M_{ϕ_1} as in Section 3.

In order to determine the genus group of C_f in the sense of [8, Theorem 5], we let t (corresponding to C_f) be as in [8]. It is important to note that, in particular, every prime divisor of the exponent $\exp H_f$ of H_f is a divisor of t . We shall see in Theorem 4.3 that these primes are the only ones that matter. We note that $\phi: Y \rightarrow Y$ is a t -equivalence if and only if r_ϕ is relatively prime to t , and the determinant of M_{ϕ_1} is relatively prime to t . If $\phi: Y \rightarrow Y$ is a t -equivalence, then the symbol Z_ϕ will denote the space obtained as the homotopy push-out of the following cotriad

$$S^n \xleftarrow{g} Y \xrightarrow{\phi} Y.$$

Then from [8] we have the following proposition.

Proposition 4.1. *Let $\phi : Y \rightarrow Y$ be a t -equivalence and let Z_ϕ be as above. Then Z_ϕ is of the same genus as C_f .*

Proposition 4.2. *Let $\phi : Y \rightarrow Y$ be a t -equivalence and let Z_ϕ , ϕ_0 and ϕ_1 be as above. Suppose that ϕ is of the form $\phi = \phi_0 \vee \phi_1$. Then $Z_\phi \simeq C_h$ for some map $h : X \rightarrow S^n$ such that $\alpha_h \circ \beta_{\phi_1} = \alpha_{\phi_0 \circ f}$.*

Proof. From the stacked bases theorem for finite rank free Abelian groups (see [6, Theorem 5.1.1] for instance), it follows that the map

$$\phi_1 : \bigvee_{i=1}^k T_i \rightarrow \bigvee_{i=1}^k T_i$$

is homotopy equivalent to some map

$$\zeta : \bigvee_{i=1}^k T_i \rightarrow \bigvee_{i=1}^k T_i$$

which is of the form

$$\bigvee_{i=1}^k \zeta_i : \bigvee_{i=1}^k T_i \rightarrow \bigvee_{i=1}^k T_i.$$

So we assume that ϕ_1 has the shape of such a ζ . For each i , let s_i be the degree of the map ζ_i , and let σ_i be an integer such that $\sigma_i s_i \equiv 1 \pmod{\varepsilon}$, where ε is the exponent of H_f . For each i , let z_i be the image of u_i with respect to the boundary homomorphism $\pi_m(C_f, S^n) \rightarrow \pi_{m-1}(S^n)$. Let $h : X \rightarrow S^n$ be the map corresponding to the homomorphism $\mathbb{Z}^k \rightarrow \pi_{m-1}(S^n)$ determined by,

$$e_i \mapsto r\sigma_i z_i.$$

We now define a particular map $\phi' : C_f \rightarrow C_h$. The restriction, $\phi'|_{S^n}$ is a map of degree r onto the subspace S^n of C_h . Note that there is an obvious way of labeling the (closed) m -cells of C_f as D_1, D_2, \dots, D_k and similarly the m -cells of C_h are E_1, E_2, \dots, E_k . Let $x_i \in \pi_m(C_h, S^n)$, for each $i \in \{1, 2, \dots, k\}$, be the homotopy class of the characteristic map $(CT_i, T_i) \rightarrow (C_h, S^n)$ of the cell E_i . We note that the boundary homomorphism $\pi_m(C_h, S^n) \rightarrow \pi_{m-1}(S^n)$, maps as follows: $x_i \mapsto r\sigma_i z_i$. We continue the construction of the map ϕ' by letting, for each i , $\phi_i|_{D_i}$ be a map into the subspace $S^n \cup E_i$ of C_h , such that under the homomorphism $\pi_m(C_f, S^n) \rightarrow \pi_m(C_h, S^n)$, $u_i \mapsto s_i x_i$.

Next we define a map $g' : Y \rightarrow C_h$. The restriction $g'|_{S^n}$ is the inclusion onto the subspace S^n of C_h , and the restriction $g'|_{(\bigvee_{i=1}^k T_i)}$ is such that the corresponding homomorphism $\mathbb{Z}^k \rightarrow \pi_m(C_h)$, is determined by the function

$$e_i \mapsto \phi'_*(v_i),$$

for the homomorphism $\phi'_* : \pi_m(C_f) \rightarrow \pi_m(C_h)$. Then $\phi' \circ g$ is homotopic to $g' \circ \phi$, i.e., the square below is homotopy commutative.

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y \\ g \downarrow & & \downarrow g' \\ C_f & \xrightarrow{\phi'} & C_h \end{array}$$

This square is in fact a homotopy pushout square, since the induced map from the mapping cone of ϕ to the mapping cone of ϕ' is (a homology equivalence and hence) a homotopy equivalence. \square

Let U be the closure in Y of the subset $Y \setminus T_k$, i.e., $U = S^n \vee (\bigvee_{i=1}^{k-1} T_i)$. Given any integer s which is relatively prime to t , then we define a map $\psi_s : Y \rightarrow Y$ as follows. The restriction $\psi_s|_{T_k}$ is a map of degree s onto the subspace T_k , and $\psi_s|_U$ is the inclusion onto the subspace U . Let σ be any integer such that $\sigma s \equiv 1 \pmod t$. Let $\eta : X \rightarrow S^n$ be a map (unique up to homotopy) corresponding to the homomorphism $\mathbb{Z}^k \rightarrow \pi_{m-1}(S^n)$ defined as below.

$$\sum_{i=1}^k q_i e_i \mapsto \sum_{i=1}^{k-1} q_i z_i + q_k \sigma z_k.$$

Then from the proof of Proposition 4.2, it follows that $Z_{\psi_s} \simeq C_\eta$. From these observations we can deduce that $\mathcal{G}(C_f) = \mathcal{L}\chi(f)$, and in fact these two objects are isomorphic as groups. Invoking also Theorem 3.7, we obtain the following.

Theorem 4.3. *For $f \in \mathcal{L}$, the genus group $\mathcal{G}(C_f)$ of the co- H_0 -space C_f is trivial if $k > r(H_f)$, and if $k = r(H_f)$, then $\mathcal{G}(C_f) \simeq \mathbb{Z}_d^* / \{1, -1\}$, where $d = \rho(H_f)$.*

We note however that the action of the group \mathbb{Z}_d^* on $\mathcal{L}\chi(f)$ as determined in Section 3, and the action as introduced in [8] of \mathbb{Z}_d^* on $\mathcal{G}(C_f)$, are not the same but differs by the inversion automorphism of \mathbb{Z}_d^* .

Example 4.4. Let us fix a prime $p \geq 5$. Let $q = 2p(p - 1)^2 - 1$, let n be an integer bigger than q and let $m = n + q + 1$. From [11, Theorem 4.15], the p -primary component of the stable homotopy group $\pi_{m-1}(S^n)$ is isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Let $f : S^{m-1} \vee S^{m-1} \rightarrow S^n$ be a map such that the subgroup H_f of the stable group $\pi_{m-1}(S^n)$ is a noncyclic p -group. Then the mapping cone C_f of f has a genus group $\mathcal{G}(C_f) \simeq \mathbb{Z}_p^* / \{1, -1\}$. This group is nontrivial (for $p \geq 5$). If X is a finite wedge sum of more than 2 copies of S^{m-1} and $g : X \rightarrow S^n$ is any map for which H_g is a p -group, then the mapping cone C_g has trivial genus.

Remark 4.5. There are examples of maps $g : S^{m-1} \rightarrow S^n$ which are of finite order in $\pi_{m-1}(S^n)$, for which the genus $\mathcal{G}(C_g)$ of the mapping cone is different from the set of all homotopy classes of spaces X of CW-type for which $C_g \vee S^m \simeq X \vee S^m$ —see [9,

Example 3.5]. This is why in Sections 3 and 4 we restrict to suspension maps. Essentially we only require certain finiteness and linearity conditions as noted when we defined the class $\mathcal{L}(m, n)$.

5. Cones on maps into a wedge sum of spheres

In this section we use the matrix methods as of Bokor [1] and similar to the methods of Section 3, for studying certain mapping cones such as in [1]. In fact the treatment is analogous to Section 3, and we obtain similar results. The pivotal result of this section, Theorem 5.2, together with the algebraic results of Section 2, enables us to sharpen the result [1, Theorem B] of Bokor, and to do genus computations similar to those of Section 4.

The class \mathcal{M} of maps. Let k, m and n be integers such that $m - 1 > n > 1$ and $k > 1$. Let $\bigvee_{i=1}^k S_i^n$ be a bouquet of n -spheres. We define $\mathcal{M} = \mathcal{M}(m, n, k)$ to be the class of all suspension maps $f : S^{m-1} \rightarrow \bigvee_{i=1}^k S_i^n$. We note that every map $f \in \mathcal{M}$ satisfies the conditions (1) and (2) below. Let $[f]$ be the element in $\pi_{m-1}(\bigvee_{i=1}^k S_i^n)$ corresponding to f . By the Hilton–Milnor Theorem (see [12] for instance), we can regard $\prod_{i=1}^k \pi_{m-1}(S_i^n)$ as being a subgroup of $\pi_{m-1}(\bigvee_{i=1}^k S_i^n)$. The first condition satisfied by a member f of \mathcal{M} is the following.

- (1) $[f] \in$ (the torsion subgroup of $\prod_{i=1}^k \pi_{m-1}(S_i^n)$).

For each $i \in \{1, 2, \dots, k\}$, let $f_i = \text{pr}_i \circ f$ where $\text{pr}_i : \bigvee_{j=1}^k S_j^n \rightarrow S_i^n$ is the map that collapses the subspace $\bigvee_{j \neq i} S_j^n$ to the basepoint. If f satisfies condition (1), then the function f and the sequence (f_1, f_2, \dots, f_k) uniquely determines each other. The second condition satisfied by maps $f \in \mathcal{M}$ is:

- (2) For each $i \in \{1, 2, \dots, k\}$, and for each integer r , if ρ is a self-map of degree r on S_i^n , then $[\rho \circ f_i] = r[f_i]$.

Definition 5.1 (from [1]). Given a map $f : S^{m-1} \rightarrow \bigvee_{i=1}^k S_i^n$ in \mathcal{M} , we define G_f to be the subgroup of $\pi_{m-1}(S^n)$ generated by the subset $\{f_i \mid i = 1, \dots, k\}$.

The set of homotopy classes of maps in \mathcal{M} is a subgroup of $\pi_{m-1}(\bigvee_{i=1}^k S_i^n)$. We briefly revisit the detail of the algebraic approach to studying the relevant mapping cones C_f as for instance in [1].

For a self-map h of $\bigvee_{i=1}^k S_i^n$, let λ_{ij} be the degree of the self-map $s_i \circ h \circ t_j$ of S^k , where $t_j : S^k \rightarrow \bigvee_{i=1}^k S_i^n$ is the obvious inclusion map into the j th wedge summand and $s_j : \bigvee_{i=1}^k S_i^n \rightarrow S^k$ is a map which is constant on the subspace $\bigvee_{i \neq j} S_i^n$ and its restriction to S_j^n is of degree 1. Associating with each such h the matrix $M_h = (\lambda_{ij})$, we obtain a one-to-one correspondence between the set of homotopy classes of self-maps of $\bigvee_{i=1}^k S_i^n$ and the set of all integral $n \times n$ -matrices. Given \mathcal{M} , let A be the subgroup of $\pi_{m-1}(S^n)$ of all the homotopy classes of maps g of the form $g = s_1 \circ f$ for f in \mathcal{M} , where s_1 is as above. Let $\{e_1, e_2, \dots, e_k\}$ be the canonical \mathbb{Z} -basis of the free Abelian group \mathbb{Z}^k . Then for every f in \mathcal{M} there is a homomorphism $\alpha_f : \mathbb{Z}^k \rightarrow A$ defined by $e_i \mapsto f_i$ (and we

acknowledge some minor abuse of notation, regarding f_i as having S^n as its codomain) for each $i \in \{1, 2, \dots, k\}$. We note that the notation α_f is different from the same symbol used in Sections 3 and 4, but also the context is different, so there cannot be any confusion. The function $f \mapsto \alpha_f$ sets up a one-to-one correspondence between the set of homotopy classes of maps in \mathcal{M} and the set of all group homomorphisms $\mathbb{Z}^k \rightarrow A$. The functions $h \mapsto M_h$ and $f \mapsto \alpha_f$ are such that a square as below is homotopy commutative,

$$\begin{array}{ccc} S^{m-1} & \xrightarrow{\varepsilon} & S^{m-1} \\ f \downarrow & & \downarrow g \\ \bigvee_{i=1}^k S_i^n & \xrightarrow{h} & \bigvee_{i=1}^k S_i^n \end{array}$$

if and only if the associated triangle is commutative. In the triangle, L_h is the homomorphism such that the matrix of L_h with respect to the basis $\{e_1, e_2, \dots, e_k\}$ is the transpose $(M_h)^t$ of M_h .

$$\begin{array}{ccc} \mathbb{Z}^k & \xleftarrow{L_h} & \mathbb{Z}^k \\ \alpha_f \searrow & & \swarrow \alpha_{g \circ \varepsilon} \\ & A & \end{array}$$

Furthermore, if the diagram is commutative, then the induced map of mapping cones $C_f \rightarrow C_g$ is a homotopy equivalence if and only if h and ε are homotopy equivalences and the latter statement is equivalent to saying that M_h is unimodular and ε is of degree ± 1 . It is easy to see that if the square above is homotopy commutative and ε is of degree -1 , then there exists a map $h_1 : \bigvee_{i=1}^k S_i^n \rightarrow \bigvee_{i=1}^k S_i^n$ such that $h_1 \circ f$ is homotopic to g and such that M_h and M_{h_1} has the same determinant up to a factor ± 1 . In particular, we have the following theorem.

Theorem 5.2. *For maps f and g in \mathcal{M} , there is a homotopy equivalence $C_f \simeq C_g$ if and only if there is an automorphism η of the Abelian group \mathbb{Z}^k such that $\alpha_f \circ \eta = \alpha_g$.*

In what follows, we find it convenient to write $\bigvee^k S^n$ instead of the more cumbersome $\bigvee_{i=1}^k S_i^n$.

Corollary 5.3. *Suppose that $f : S^m \rightarrow \bigvee^k S^n$ and $g : S^m \rightarrow \bigvee^k S^n$ are maps in \mathcal{M} such that $G_f = G_g$ and the group G_f is of rank strictly smaller than k . Then $C_f \simeq C_g$.*

Proof. Since $\text{Im}(\alpha_f) = G_f = G_g = \text{Im}(\alpha_g)$ and the rank of $\text{Im}(\alpha_f)$ is smaller than k , from Theorem 2.1 it follows that there is an automorphism μ of \mathbb{Z}^k , for which $\alpha_g = \alpha_f \circ \mu$. Thus by Theorem 5.2 we obtain a homotopy equivalence $C_f \simeq C_g$. \square

The next corollary is a sharpening of the equivalence of the statements (2) and (3) in the theorem [1, Theorem 0.3]. A similar result is found in a paper by Molnar [9, Theorem 3.8(ii)]. But first we note the following:

If $f : S^{m-1} \rightarrow \bigvee^k S^n$ is any map, and σ is the inclusion map, $\bigvee^k S^n \subset (\bigvee^k S^n) \vee S^n$, then $C_{\sigma \circ f} \simeq C_f \vee S^n$.

Corollary 5.4. Suppose that $f : S^{m-1} \rightarrow \bigvee^k S^n$ and $g : S^{m-1} \rightarrow \bigvee^k S^n$ are maps in \mathcal{M} such that $G_f = G_g$. Then $C_f \vee S^n \simeq C_g \vee S^n$.

Proof. Let ϕ and γ be maps $S^{m-1} \rightarrow (\bigvee^k S^n) \vee S^n$ defined by $\phi = \sigma \circ f$ and $\gamma = \sigma \circ g$, where σ is the inclusion map, $\bigvee^k S^n \subset (\bigvee^k S^n) \vee S^n$. Then $G_\phi = G_f = G_g = G_\gamma$ and, $q \circ \phi$ is null-homotopic, $q : (\bigvee^k S^n) \vee S^n \rightarrow S^n$ being the map that collapses the subspace $\bigvee^k S^n$ to the base point. Thus by Theorem 5.3, $C_\phi \simeq C_\gamma$. But $C_\phi \simeq C_f \vee S^n$ and $C_\gamma \simeq C_g \vee S^n$, and so the corollary follows. \square

We thus obtain a result similar to Corollary 3.8. To this end we define, for $f \in \mathcal{M}$, the set $\mathcal{M}_\chi(f)$ to be the set of all homotopy types of spaces Y such that $Y \simeq C_g$ for some $g \in \mathcal{M}$, and $Y \vee S^n \simeq C_f \vee S^n$.

Theorem 5.5. For $f \in \mathcal{M}$, let $d = \rho(G_f)$. Then $\mathcal{M}_\chi(f)$ is in one-to-one correspondence with $\mathbb{Z}_d^*/\{1, -1\}$ if $k = r(G_f)$, and if $k > r(G_f)$ then $\mathcal{M}_\chi(f)$ is trivial.

Using Corollary 5.4 and [1, Theorem A] we obtain the following:

Theorem 5.6 (cf. [1, Theorem B]). Let $f, g : S^{m-1} \rightarrow \bigvee^k S^n$ be maps in \mathcal{M} . Suppose that the torsion subgroup of $\pi_{m-1}(S^n)$ can be generated by a subset consisting of N elements. If $k > N$, then C_f and C_g are of the same genus if and only if they are homotopy equivalent.

Through methods similar to those in Section 4, we can compute the Mislin genus group, in the sense of [8, Theorem 5], of these mapping cones. We state the result, which is very similar to Theorem 4.3, without proof.

Theorem 5.7. For $f \in \mathcal{M}$, the genus group $\mathcal{G}(C_f)$ of the co- H_0 -space C_f is trivial if $k > r(G_f)$, and if $k = r(G_f)$, then $\mathcal{G}(C_f) \simeq \mathbb{Z}_d^*/\{1, -1\}$, where $d = \rho(G_f)$.

For reasons similar to those mentioned in Remark 4.5, (see also the conditions (1) and (2) at the beginning of Section 5) we restricted the discussion to suspension maps. The example [9, Example 3.4] shows that we cannot use the methods of this section to compute the genus of the mapping cone of an arbitrary element of finite order in $\pi_{m-1}(S^n)$.

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