# Reduced product objects in model categories

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## 0. Introduction

In his influential 1967 monograph *Homotopical Algebra*, D. Quillen [8] described an abstract approach to homotopy theory enabling analogous theories to be defined in categories other than the category of spaces and continuous maps. Although the starting point in the classical homotopy theory of spaces and maps is the equivalence relation (between maps) of *homotopy*, in Quillen's approach it is that of a *model category*, that is to say, a category **C**, together with three distinguished classes of morphisms, *we*, *cof*, *fib*, called weak equivalences, cofibrations, and fibrations, respectively. These are required to satisfy certain axioms which reflect typical properties of the classes of such maps in topology and they enable the construction of much of the basic machinery of homotopy theory in the category **C**. However, it is not possible in a model category to introduce all possible concepts and prove analogs of all possible theorems that hold in the homotopy theory of spaces: for the simple reason that the axioms of a model category are self-dual whereas the Eckmann-Hilton duality in spaces is known not to be perfect.

Nevertheless additional axioms, if enjoyed by a particular model category, sometimes enable further classical concepts and results to be introduced in **C**. A relatively recent instance of this has been the successful definition by Doerane [2] of a notion of *Lusternik-Schnirelmann category* in a type of model category satisfying the so-called *cube axiom*. In such cases there is a price to be paid: the richer theory is only available in categories **C** for which the additional axioms can be verified.

The purpose of this paper is to show that the cube axiom permits the development of another powerful feature of the homotopy theory of spaces: the existence of James spaces [6]. The classical James space construction associates with a locally countable pointed CW-complex $X$ a space $X_\infty$ and a homotopy equivalence $X_\infty \to \Omega\Sigma X$, where $\Omega$ and $\Sigma$ refer to the loop and (reduced) suspension endofuctors of the category of pointed spaces. It opens the door to the study of the suspension operation via the inclusion $X \to X_\infty$ and to the detection of elements in the cokernel of suspension via the James map $X_\infty \to (X \wedge X)_\infty$. Such considerations have hitherto been out of the reach of abstract homotopy theory, although generalizations, fibrewise and equivariant, of the equivalence $X_\infty \to \Omega\Sigma X$ have been obtained, [3].

In this paper we define the reduced powers $X_n$ of an object $X$ in a suitable model category, or more generally, the objects $(X, A)_n$ and $(X, A)_\infty$ as in the work [4] of Gray, associated with a cofibration $A \to X$. If, in particular, a certain cube axiom is satisfied, we prove the equivalence of the object $X_\infty$ to $\Omega\Sigma X$ generalizing work of I. M. James and others.

## 1. Model categories

Many authors have found it convenient to modify Quillen's axioms as presented in [8]. We use the version given by Hovey [5]:

**1.1 Definition** A *model category* is a category $\mathbf{C}$ with all small limits and colimits together with a model structure on $\mathbf{C}$.

**1.1.1 Definition** A *model structure* on a category $\mathbf{C}$ consists of three classes of morphisms of $\mathbf{C}$ called *weak equivalences, cofibrations* and *fibrations*, and two functorial factorisations $(\alpha,\ \beta)$ and $(\gamma,\ \delta)$ satisfying the following properties:

1. (2-out-of-3) If $f$ and $g$ are morphisms of $\mathbf{C}$ such that $gf$ is defined and two of $f,\ g$ and $gf$ are weak equivalences then so is the third.

2. (Retract) If $f$ and $g$ are morphisms of $\mathbf{C}$ such that $f$ is a retract of $g$ and $g$ is a weak equivalence, cofibration, or fibration, then so is $f$.

3. (Lifting) Define a map (i.e. morphism of $\mathbf{C}$) to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4. (Factorisation) For any map $f : A \to B$,

$$f \;=\; A \overset{\alpha(f)}{\rightarrowtail} B' \underset{\sim}{\overset{\beta(f)}{\twoheadrightarrow}} B \qquad \text{and} \qquad f \;=\; A \underset{\sim}{\overset{\gamma(f)}{\rightarrowtail}} A' \overset{\delta(f)}{\twoheadrightarrow} B \;,$$

indicating that $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

The retract property enables the statement of the lifting property to be strengthened:

**1.1.2 Lemma** ([5, Lemma 1.1.10]). *A map in a model category is a cofibration (trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations).*

In particular, every isomorphism in a model category is a trivial cofibration and a trivial fibration.

**1.1.3 Corollary** ([5, 1.1.11]). *The cofibrations (trivial cofibrations) in a model category are closed under pushout. That is, if*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

*is a pushout square, where $f$ is a cofibration (trivial cofibration), then $g$ is a cofibration (trivial cofibration). Dually, fibrations (trivial fibrations) are closed under pullback.*

If $\mathbf{C}$ is a model category, then, for a particular objects $A$, $B$ of $\mathbf{C}$ one may consider the associated *slash* categories $\mathbf{C}/B$, $A/\mathbf{C}$, $A/\mathbf{C}/B$ etc. Their maps are always maps of $\mathbf{C}$ enjoying additional properties. As pointed out by Hovey, these categories inherit from $\mathbf{C}$ an associated model category structure provided we define a map to be a cofibration, (resp. fibration, weak equivalence) if it is so in the given structure on $\mathbf{C}$.

Every model category $\mathbf{C}$ has an initial object $0$ (the colimit of the empty diagram) and a terminal object $*$. If $0$ and $*$ are isomorphic then $\mathbf{C}$ is *pointed.* Since our goal is a James construction in $\mathbf{C}$ we assume henceforth that $\mathbf{C}$ is pointed and that each object $X$ of $\mathbf{C}$ is *fibrant* and *cofibrant* (i.e. $X \to *$ is a fibration and $* \to X$ is a cofibration).

A commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{1.2}$$

is a *homotopy pullback* if the induced map (shown dotted) in the following diagram is a weak equivalence.

(1.2.1)
$$\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, dotted] \arrow[dd, "k"'] & & C \arrow[dr, "\sim"] \arrow[dd] & \\
& A \times_B C' \arrow[rr] \arrow[dl] & & C' \arrow[dl, "\delta(g)"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

Here it is to be understood that the square with source $A \times_B C'$ is a pullback. The special case $C = *$ of 1.2 is of some significance, for then we call $D$ the *homotopy fibre* of $f$ and denote it by $F_f$. If both $C = A = *$ then we say that $D$ is the loop object of $B$ and denote it by $\Omega B$.

Dually, we define the notions of *homotopy pushout* square and *homotopy cofibre* (i.e. *mapping cone*): specifically the square 1.2 is a homotopy pushout if the induced dotted arrow in the following diagram is a weak equivalence.

(1.2.2)
$$\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, "\alpha(h)"'] \arrow[dd, "k"'] & & C \arrow[dd, "g"] \\
& C' \arrow[ur, "\sim"'] \arrow[dd] & \\
A \arrow[rr] \arrow[dr] & & B \\
& A \vee_D C' \arrow[ur, dotted] &
\end{tikzcd}$$

In the case $C = *$ of 1.2.2, we call $C'$ a *cone* on $D$ and $A \vee_D C'$ a *mapping cone* of $k$. If there is a weak equivalence $X \to *$ then we say that $X$ is *contractible.* In particular each mapping cone of $1 : X \to X$ is a cone on $X$ and is contractible.

The following cube lemma is required in order to prove the equivalence of infinite reduced power with loops-suspension.

**1.3 Cube Axiom**. Suppose that we have a commutative diagram as follows.

$$\begin{tikzcd}
& \bullet \arrow[rr] \arrow[dl] \arrow[dd] & & \bullet \arrow[dl] \arrow[dd] \\
\bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] & \\
& \bullet \arrow[rr] \arrow[dl] & & \bullet \arrow[dl] \\
\bullet \arrow[rr] & & \bullet &
\end{tikzcd}$$
(1.3.1)

If the top and bottom faces are homotopy push-outs and the left and rear faces are homotopy pull-backs, then the remaining two faces are homotopy pull-backs.

This axiom is exactly the same as axiom [2, Cube axiom on p220] in the paper of Doeraene. In the topological case it is very similar to the first cube theorem [7, Theorem 18] of Mather, except that we assume strict commutativity of the diagram. Given this axiom, one can easily prove the following one, and we omit the routine proof.

**1.4 Proposition**. *Suppose that* $\mathbf{C}$ *is a model category satisfying the Cube Axiom 1.3. Then in* $\mathbf{C}$ *the following condition holds:*
*Given a commutative diagram as in diagram 1.3.1 in which the bottom face is a homotopy push-out and the vertical faces are homotopy pull-backs, then the top face is a homotopy push-out square.*

The following Lemma is an indication of the power of the cube axiom in our context. We use it in the sequel without specific reference.

**1.5 Lemma**. *Taking the product of a rectangular diagram with a fixed object of* $\mathbf{C}$ *preserves the property of being a homotopy pushout.*

**Proof.** There is a cubical diagram whose base is the original rectangle and whose upper face is the desired homotopy pushout. The vertical arrows are product projections which, under our assumptions, are fibrations. Hence the vertical faces are homotopy pullback rectangles and we may apply Proposition 1.4. □

## 2. The Gray construction

For a cofibration $i : A \to X$ in $\mathbf{C}$, we construct objects $(X, A)_n$ in $\mathbf{C}$, for positive integers $n$, which are analogues of the subspaces of the spaces $(X, A)_\infty$ defined by Gray [4]. Fat wedge maps $w_n : W_n(X, A) \to X \times A^n$ are defined as follows (consistent with the construction of Doeraene [2, Definition 3.1]). We take $w_0$ to be the map $* \to X$ and, for $n > 0$, we define $w_n$ inductively as follows. Let

$$e = 1 \times (* \to A) : V \to V \times A$$

be the natural cofibration and note that the outside of the following diagram is commutative.

$$\begin{tikzcd}
W_{n-1}(X,A) \arrow[r, tail, "e"] \arrow[d, "w_{n-1}"'] & W_{n-1}(X,A)\times A \arrow[d, dotted, "\bar{w}"'] \arrow[ddr, "w_{n-1}\times A"] & \\
X\times A^{n-1} \arrow[r, dotted, "\bar{e}"] \arrow[drr, "e"'] & W_n(X,A) \arrow[dr, dotted] & \\
& & X\times A^n
\end{tikzcd} \tag{2.1}$$

The desired map $w_n$ is then the induced map from the pushout of the top left corner. It may be seen (inductively) to be a cofibration by applying Lemma 1.1.2. (Partial lifts of maps from the corner ends yield a full lift by the universal property of the pushout.)

In the case $X = A$ we denote $(X,A)_n$ simply by $A_n$ and likewise $W_n(X,A)$ by $W_nA$. We shall need to recognise $W_n(X,A)$ also as the pushout

$$\begin{tikzcd}
W_{n-1}A \arrow[r, tail, "f"] \arrow[d, "w_{n-1}"'] & X\times W_{n-1}A \arrow[d, dotted, "w'"'] \arrow[ddr, "X\times w_{n-1}"] & \\
A^n \arrow[r, dotted, "\bar{f}"] \arrow[drr, "f\times A^n"'] & W_n(X,A) \arrow[dr, dotted, "w_n"] & \\
& & X\times A^n
\end{tikzcd} \quad , \tag{2.2}$$

where $f = (* \to X)\times 1 : V \to X\times V$.

We now define the objects $(X,A)_n$ $(n \geq 0)$, *folding maps* $\phi_n$ $(n \geq 0)$ and *identification maps* $\mu_n$ $(n \geq 1)$

$$\phi_n : W_n(X,A) \to (X,A)_n \;\; ; \;\; \mu_n : X\times A^{n-1} \to (X,A)_n$$

inductively as follows. We start of by setting $(X,A)_0 = *$, $\phi_0 = * \to *$ and $\nu_1 = 1$. Given $\phi_{n-1}$, then $\mu_n$ is defined by forming a pushout square

$$\begin{tikzcd}
W_{n-1}(X,A) \arrow[r, tail, "w_{n-1}"] \arrow[d, "\phi_{n-1}"'] & X\times A^{n-1} \arrow[d, "\mu_n"'] \\
(X,A)_{n-1} \arrow[r, tail, "j_n"'] & (X,A)_n
\end{tikzcd} \tag{2.3.1}$$

Given $\mu_n$, we now define $\phi_n$. To do so we require yet another way of constructing the object $W_n(X, A)$, this time as the colimit of a larger diagram. The construction, although somewhat technical, is conceptually simple.

Fix any $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let $S_k$ be the collection of all proper nonempty subsets of $\{1, 2, ..., k\}$. Given any $\sigma \in S_{n+1}$, let $Z_\sigma = Y \times A^{|\sigma|-1}$, where $Y = X$ if $1 \in \sigma$ and otherwise $Y = A$. Let $\sigma_1, \sigma_2, ..., \sigma_k$ be the distinct elements of $\sigma$ in increasing order. For any $\sigma, \tau \in S_{n+1}$ with $\sigma \subset \tau$, let $f_{\sigma,\tau} : Z_\sigma \to Z_\tau$ be a section to the projection map which inserts the base point $*$ in the $i$'th place for every $i$ which is such that $\tau_i \in \tau \backslash \sigma$. Note that if $\sigma, \tau, \rho \in S_{n+1}$ with $\sigma \subset \tau \subset \rho$, then $f_{\tau,\rho} \circ f_{\sigma,\tau} = f_{\sigma,\rho}$. Now let $D$ be the diagram formed by all the maps $f_{\sigma,\tau}$ for $\sigma, \tau \in S_{n+1}$, together with the initial maps into the objects $Z_\sigma$. Let $D_0$ be the "sub-diagram" of $D$ obtained by removing the object $U = Z_{\{1,2,...,n\}}$ and every arrow having $U$ as its target. Then the colimit of diagram $D$ is (isomorphic to) $W_n(X, A)$. An important observation to this end is to note that the colimit of the diagram $D_0$ (exists since **C** has small colimits, and) coincides with $W_{n-1}(X, A) \times A$. Augment the diagram $D$ with maps $g_\sigma : Z_\sigma \to (X, A)_n$ which coincide with $\mu_n$ (in the obvious sense) for all maximal members $\sigma$ of $S_{n+1}$. By induction on $n$ one can prove that the augmented diagram is commutative. Then since $W_n(X, A)$ is a colimit of $D$ there exists a unique map $\phi_n$ such that $\phi_n \circ g_\sigma$ agrees with $f_\sigma$ for all maximal $\sigma$ , where $f_\sigma : Z_\sigma \to W_n(X, A)$ refers to the maps into the colimiting object.

We also define a *multiplication* map

$$\nu_n : X \times A_{n-1} \to (X, A)_n \ ,$$

as follows. Let $\nu_2 = \mu_2$ and then define $\nu_{n+1}$ inductively to be the unique map (dotted arrow) determined by pushout in the following diagram which may be checked to be commutative by considering the appropriate 'diagram $D$'.

$$\begin{tikzcd}
X \times W_{n-1}(A) \arrow[r] \arrow[d, "X\times\phi_{n-1}"'] & X \times A^n \arrow[d, "X\times\mu_n"'] \arrow[ddr, "\mu_{n+1}"] & \\
X \times A_{n-1} \arrow[r] \arrow[drr, "j_{n-1}\circ\nu_n"'] & X \times A_n \arrow[dr, dotted, "\nu_{n+1}"] & \\
 & & (X, A)_{n+1}
\end{tikzcd} \tag{2.3.2}$$

**2.4 Proposition**. *For every object $A$ and for every integer $n$, the object*

*$(CA, A)_n$ is contractible.*

For the proof of this proposition, we require the following.

**2.4.1 Proposition**. *The map $w_n : W_n(CA, A) \to CA \times A^n$ is a weak equivalence.*

*Proof.* The maps $w_0 : * \to CA$ and $w_1 : CA \vee A \to CA \times A$ can be seen to be weak equivalences. Given $n > 0$, suppose that $w_{n-1}$ is a weak equivalence. Then in the diagram

$$\begin{tikzcd}
CA \times A^{n-1} \arrow[d] & W_{n-1}(CA, A) \arrow[l, "w_{n-1}"'] \arrow[r] \arrow[d, "w_{n-1}"'] & W_{n-1}(CA, A) \times A \arrow[d, "w_{n-1} \times A"] \\
CA \times A^{n-1} & CA \times A^{n-1} \arrow[l] \arrow[r] & CA \times A^n
\end{tikzcd} ,$$

the vertical arrows are weak equivalences and the push-outs of the cotriads in the top and bottom rows are $W_n(CA, A)$ and $CA \times A^n$ respectively. The unique map $W_n(CA, A) \to CA \times A^n$ determined by push-out coincides with $w_n$. Hence, by the cube lemma [5, Lemma 5.2.6] (c.f. [1, II Lemma (1.2)]), $w_n$ is a weak equivalence. □

**Proof of Proposition 2.4**. For each $n \in \mathbb{N} \cup \{0\}$ there is a cofibration $j_n : (CA, A)_n \to (CA, A)_{n+1}$. The map $j_0 : (CA, A)_0 \to (CA, A)_1$ coincides with $* \to CA$, and is therefore a weak equivalence. Since $(CA, A)_0$ is contractible, it suffices to prove that each $j_n$ is a weak equivalence. By Proposition 2.4.1, the map $w_{n-1}$ in diagram 2.3.1 is a weak equivalence if $X = CA$. Since the square is a push-out, it follows that $j_n$ is a weak equivalence, completing the proof of Proposition 2.4. □

## 3. The Main Theorem

For a cofibration $i : A \to X$, the object obtained as the push-out of the cotriad $* \leftarrow A \to X$ is denoted by $X/A$ and is weakly equivalent to each mapping cone of $i$.

**Definition 3.1**. For a cofibration $A \to X$, we define $V_1(X, A) = W_1(X, A)$ and, for $n \geq 2$, the object $V_n(X, A)$ so that the following square is a push-out.

(3.1.1)
$$\begin{tikzcd}
X \times W_{n-1}(A) \arrow[r, "X\times\phi_{n-1}"] \arrow[d, "w'"'] & X \times A_{n-1} \arrow[d] \\
W_n(X,A) \arrow[r] & V_n(X,A)
\end{tikzcd}$$

For the trivial cofibration $A = A$ we write $V_n(A)$ instead of $V_n(A,A)$.

**Proposition 3.2**. *The object $P$ obtained in the following push-out square, is (isomorphic to) $V_n(X,A)$.*

$$\begin{tikzcd}
* \times A_{n-1} \arrow[r] \arrow[d] & * \times A_n \arrow[d] \\
X \times A_{n-1} \arrow[r] & P
\end{tikzcd}$$

**Proof**. Consider the commutative diagram below.

$$\begin{tikzcd}
* \times W_{n-1}(A) \arrow[r] \arrow[d] & * \times A^n \arrow[d] \\
X \times W_{n-1}(A) \arrow[r] \arrow[d] & W_n(X,A) \arrow[d] \\
X \times A_{n-1} \arrow[r] & V_n(X,A)
\end{tikzcd}$$

We note (comparing with 2.2) that the upper square is a push-out. The lower square is a push-out, by definition of $V_n(X,A)$. Thus the outer square is a push-out. Now we turn to the following commutative diagram.

$$\begin{tikzcd}
* \times W_{n-1}(A) \arrow[r] \arrow[d] & * \times A^n \arrow[d] \\
* \times A_{n-1} \arrow[r] \arrow[d] & * \times A_n \arrow[d] \\
X \times A_{n-1} \arrow[r] & V_n(X,A)
\end{tikzcd}$$

We have shown above that the outer square is a push-out. By the definition of the objects $A_n$ it follows that the upper square is a push-out and hence the lower square is also. This completes the proof of Proposition 3.2. □

**Theorem 3.3**. *The following square in which the vertical arrows are versions of $\nu$ and the horizontal arrows are the obvious cofibrations, is a homotopy* *push-out (hpo) square .*

$$\begin{tikzcd}
A \times A_n \arrow[r] \arrow[d] & X \times A_n \arrow[d] \\
A_{n+1} \arrow[r] & (X,A)_{n+1}
\end{tikzcd}$$

**Proof**. The morphism $X \times w_{n-1} : X \times W_{n-1}(A) \to X \times A^n$ can be factorized as the composition of two cofibrations (see 2.2):

$$X \times W_{n-1}(A) \xrightarrow{w'} W_n(X,A) \xrightarrow{w_n} X \times A^n \, .$$

We have two commutative diagrams inducing arrows via 3.1.1 :

$$\begin{tikzcd}
 & X{\times}W_{n-1}A \arrow[r, "X\times\phi_{n-1}"] \arrow[d, "w'"'] & X{\times}A_{n-1} \arrow[d] \arrow[dd, bend left=60, "X\times j_n"] \\
\mathbf{A} & W_n(X{,}A) \arrow[r] \arrow[d, "w_n"'] & V_n(X{,}A) \arrow[d, dotted, "v_n"'] \\
 & X{\times}A^n \arrow[r, "X\times\mu_n"'] & X{\times}A_n
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X{\times}W_{n-1}A \arrow[r, "X\times\phi_{n-1}"] \arrow[d, "w'"'] & X{\times}A_{n-1} \arrow[d] \arrow[dr, "\nu_n"] & \\
\mathbf{B} & W_n(X{,}A) \arrow[r] \arrow[rr, bend right=15, "\phi_n"'] & V_n(X{,}A) \arrow[r, dotted, "\alpha_n"] & (X{,}A)_n
\end{tikzcd}$$

By the cube axiom, the vertical composite of the two squares in **A** is a hpo. Since the upper square is a pushout, the lower square is a hpo. Next, considering the diagram

(3.3.1)

$$\begin{tikzcd}
W_n(X,A) \arrow[rr] \arrow[dr, "\phi_n"] \arrow[dd, "w_n"'] & & V_n(X,A) \arrow[dl, "\alpha_n"'] \arrow[dd, "v_n"] \\
 & (X,A)_n & \\
X \times A^n \arrow[rr, crossing over, "X\times\mu_n"] \arrow[dr, "\mu_{n+1}"'] & & X \times A_n \arrow[dl, "\nu_{n+1}"] \\
 & (X,A)_{n+1} \arrow[from=2-2, "j_{n+1}"'] &
\end{tikzcd}$$

we recognize the push-out square defining the object $(X,A)_{n+1}$. In view of **B** the top triangle is commutative and we may check that the remainder of the diagram is commutative. We have shown that the square at the back is an hpo. It now follows that the right front square is an hpo, since the back and left front squares are. Note that the right front square is also the right hand face of the following commutative cube.

3.3.2

$$\begin{tikzcd}
& V_nA \arrow[rr] \arrow[dl] \arrow[dd] & & V_n(X,A) \arrow[dl, "\alpha_n"'] \arrow[dd, "v_n"] \\
A_n \arrow[rr] \arrow[dd] & & (X,A)_n \arrow[dd] & \\
& A \times A_n \arrow[rr] \arrow[dl] & & X \times A_n \arrow[dl, "\nu_{n+1}"] \\
A_{n+1} \arrow[rr] & & (X,A)_{n+1} &
\end{tikzcd}$$

In the left face we have a similar square (for the special case $X = A$). Thus the proof will be complete if we can show that the upper face is a hpo (since the left and right faces are hpo). This we now prove by induction.
In the case $n = 1$, the relevant square is as follows and is obviously a hpo.

$$\begin{tikzcd}
W_1(A) \arrow[r] \arrow[d] & W_1(X,A) \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

So the case $n = 1$ of the theorem follows. Now assume that $n \geq 2$ and consider the following commutative diagram. By Proposition 3.2, the top left is an hpo, so that the composed square on the top row is an hpo.

$$\begin{tikzcd}
* \times A_{n-1} \arrow[r] \arrow[d] & A \times A_{n-1} \arrow[r] \arrow[d] & X \times A_{n-1} \arrow[d] \\
* \times A_n \arrow[r] & V_n(A) \arrow[r] \arrow[d] & V_n(X,A) \arrow[d] \\
& A_n \arrow[r] & (X,A)_n
\end{tikzcd}$$

Thus the top right square (2) is an hpo. By the inductive hypothesis the vertical composite of the right hand squares is an hpo. Thus the lower square is an hpo and the induction is complete. □

In any category, the direct limit of any sequence of maps,

$$A_1 \to A_2 \to A_3 \to ...,$$

is defined and it may or may not exist for a given sequence. For convenience (and a little abusively) we suppress the role of the morphisms in the sequence above, and denote the direct limit by $\lim(A_n)$ if it exists. In a model category the direct limits exist for all sequences of maps, and direct limit is functorial in the obvious way. Nevertheless, the following proposition is much more generally applicable. Its proof is simple and we omit it.

**Proposition 3.4**. *Suppose that (in any category) we have a sequence of pushout squares*

$$\begin{tikzcd} A_n^{(0)} \arrow[r] \arrow[d] & A_n^{(1)} \arrow[d] \\ A_n^{(2)} \arrow[r] & A_n^{(3)} \end{tikzcd},$$

*and for each* $i \in \{0, 1, 2, 3\}$ *and* $n \in \mathbb{N}$*, a given map*

$$f_n^{(i)} : A_n^{(i)} \to A_{n+1}^{(i)},$$

*which is such that we actually have a sequence of maps of squares. If each of the four direct limits lim*$(A_n^{(i)})$ *exist, then they form a pushout square.*

For our final result we need the relevant model category to satisfy the following condition on direct limits.

**Condition 3.5**: *Given any sequence of maps with* $A_n$ *contractible for each* $n \in \mathbb{N}$*,*

$$A_1 \to A_2 \to A_3 \to ...,$$

*then* $\lim(A_n)$ *is contractible.*

The limit of the sequence, $(X, A)_1 \to (X, A)_2 \to (X, A)_3 \to ...$, we denote by $(X, A)_\infty$, and the object $(X, X)_\infty$ is denoted by $X_\infty$.

**Theorem 3.6**. *If* **C** *satisfies the cube axiom and Condition 3.5, then for a cofibration* $A \to X$*, there is a map* $f_\infty : (X, A)_\infty \to X/A$ *having* $A_\infty$ *as its homotopy fibre.*

**Proof**. For the following commutative diagram we form the pushouts of the cotriads in the top row and in the bottom row.

$$\begin{tikzcd} A^{n+1} \arrow[d] & A \times A^n \arrow[l, "\mu_{n+1}"'] \arrow[r, "\text{incl}"] \arrow[d, "\text{proj}"'] & X \times A^n \arrow[d, "\text{proj}"] \\ * & A \arrow[l] \arrow[r] & X \end{tikzcd}$$

By Theorem 3.3, the object obtained as the pushout of the upper cotriad is $(X, A)_{n+1}$. The object obtained as the pushout of the cotriad in the bottom cotriad is $X/A$. Then there exists a unique map $f_{n+1} : (X, A)_{n+1} \to X/A$, completing a commutative cube diagram. We have a sequence (indexed by $n$) of such cubes, and the limit of this sequence is a cube as below:

$$\begin{tikzcd}
& A \times A_\infty \arrow[rr, "\text{incl}"] \arrow[dl, "\mu_\infty"'] \arrow[dd] & & X \times A_\infty \arrow[dl] \arrow[dd] \\
A_\infty \arrow[rr] \arrow[dd] & & (X,A)_\infty \arrow[dd, "f_\infty"] & \\
& A \arrow[rr] \arrow[dl] & & X \arrow[dl] \\
* \arrow[rr] & & X/A &
\end{tikzcd}$$

The proof is completed through application of Proposition 3.4, Condition 3.5, and the Cube Axiom 1.3. □

**Corollary 3.7**. *Suppose that* $\mathbf{C}$ *satisfies the cube axiom and Condition 3.5. For any object* $A$ *of* $\mathbf{C}$*, the homotopy fibre of the initial morphism* $* \to \Sigma A$ *is* $A_\infty$*, i.e.,* $A_\infty$ *is weakly equivalent to* $\Omega\Sigma A$.

**Proof**. This is deduced from Theorem 3.6, using Proposition 2.4 and Condition 3.5. □

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