Non-cancellation for groups with non-abelian torsion

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Abstract

For any group G, we denote by $\chi(G)$ the set of all isomorphism classes of groups H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. If G is finitely generated with finite commutator subgroup, then $\chi(G)$ has a group structure. We develop methods for computing the groups $\chi(G)$ and perform such computations. For a finite normal subgroup F of G, we compare $\chi(G)$ with $\chi(G/F)$.

1. Introduction

In this article[1] we study the non-cancellation set $\chi(G)$ of a group G, where $\chi(G)$ is the set of all isomorphism classes of groups H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. In particular we deal with the class $\mathcal{X}_0$ of finitely generated groups having a finite commutator subgroup. The subclass of all infinite nilpotent groups in $\mathcal{X}_0$ is denoted by $\mathcal{N}_0$.

The class $\mathcal{N}_0$ has been studied with respect to genus and non-cancellation by several authors. The (Mislin) genus $\mathcal{G}(N)$ of a $\mathcal{N}_0$-group N is defined to be the set of all isomorphism classes of finitely generated nilpotent groups M such that for every prime p, the groups M and N have isomorphic p-localizations, $M_p \simeq N_p$. A group structure was imposed by Hilton and Mislin [4] on $\mathcal{G}(N)$, and has proved to be quite useful in computing the genus set. Computations of genera can be found in [8], [4] and in [5] for instance. For an $\mathcal{N}_0$-group N it is known, [11], that $\mathcal{G}(N) = \chi(N)$. This motivated the article [13] in which we impose a group structure on $\chi(H)$ for an $\mathcal{X}_0$-group H, that coincides with the Hilton-Mislin genus group $\mathcal{G}(H)$ if H is a nilpotent group.

In the article [3] there is a study of the relationship between $\mathcal{G}(N)$ and $\mathcal{G}(N/F)$, for a $\mathcal{N}_0$-group N and a finite normal subgroup F of N. In [13] it is shown that for a $\mathcal{X}_0$-group G and a finite characteristic subgroup F of the torsion radical T_G of G, there is an epimorphism $\chi(G) \rightarrow \chi(G/F)$. In this article we consider a more general situation, with F only considered to be a normal subgroup of G. We also develop

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some further methods for computation of non-cancellation sets and genera, and we make some specific computations.

Let $\mathcal{X}_1$ denote the subclass of $\mathcal{X}_0$ of groups G for which the canonical epimorphism, $G \to G/T_G$ onto the torsionfree quotient, admits a section (T_G denoting the torsion radical of G), i.e., G is a semidirect product, $G = T_G \rtimes G/T_G$. The subclass $\mathcal{K}$ of all $\mathcal{X}_1$-groups G for which T_G is abelian is studied in [10]. In particular, in [10] an alternate description of the group $\chi(G)$ is given for a $\mathcal{K}$-group G. We extend the results of [10] to the class $\mathcal{X}_1$.

In Section 2 we study determinants, which are crucial in the computation of $\chi(G)$ for a $\mathcal{X}_1$-group G. We find a simplified way of computing the non-cancellation groups of $\mathcal{X}_1$-groups, generalizing results in [10]. For a $\mathcal{X}_0$-group G and a finite normal subgroup F of G, in Section 3 we define a function $\eta : \chi(G) \to \chi(G/F)$. Section 4 is devoted to a construction based on the regular representation, which yields an isomorphism of non-cancellation sets in certain cases. We calculate $\chi(G)$ for certain $\mathcal{X}_1$-groups which are not metabelian, and these are the first complete calculations of non-cancellation groups for groups with non-abelian torsion, even for G nilpotent. In Section 5 we show that the function η of Section 3 is not always a homomorphism.

For a group G, T_G shall always denote the torsion radical. For a nilpotent group N and a prime p, N_p denotes the p-localization. For $m \in \mathbb{N}$, by $(\mathbb{Z}/m)^*/\pm 1$ we mean the quotient group of the group $(\mathbb{Z}/m\mathbb{Z})^*$ of units mod m, obtained by factoring out the subgroup $\langle -\overline{1} \rangle$.

2. Determinant

Let B be any (additive) finite abelian group of Prüfer rank $k \geq 1$, i.e., such that the cardinality of a minimal generating subset of B is k. Let t be the greatest common divisor of the orders of the invariant factors of B. Then there exists an epimorphism $\beta : B \to C$, where $C \simeq (\mathbb{Z}/t)^k$. Note that $\ker\beta = \{tx : \ x \in B\}$ is a characteristic subgroup of B and therefore for every $f \in \mathrm{Aut}B$, f induces a unique $f_\beta \in \mathrm{Aut}C$ such that $f_\beta \circ \beta = \beta \circ f$. It is easy to see that for any other epimorphism $\gamma : B \to C$, we have $\det(f_\beta) = \det(f_\gamma)$. This common value will be refered to as the determinant of f and written $\det(f)$.

For B as above, let $E_k(B)$ be the set of all epimorphisms $\mathbb{Z}^k \to B$. Recall that for $f, g \in E_k(B)$, f is said to be Nielsen equivalent to g ($f \sim g$) if there is an automorphism α of $\mathbb{Z}^k$ for which $g \circ \alpha = f$. The necessary properties of this equivalence relation which we use below, are briefly discussed in [10]. The set of equivalence classes is denoted by ${E_k}^{\sim}(B)$, and for $f \in E_k(B)$, the class of f is written $[f]$. We note, in particular, that there is an action of the group $(\mathbb{Z}/t)^*$ on $E_k^{\sim}(B)$ - in this regard we use the notation ${}^r[f]$ for $r \in (\mathbb{Z}/t)^*$ and $f \in E_k(B)$.

2.1 Notation. Let us consider a group $G = T \rtimes_w \mathbb{Z}^k$, for any finite group T and some homomorphism $w : \mathbb{Z}^k \to \mathrm{Aut}T$. Let us assume that the subgroup $\mathrm{Im}w$ of $\mathrm{Aut}T$ is of Prüfer rank k, and let n be the greatest common divisor of the orders of invariant factors of $\mathrm{Im}w$. Throughout this section we fix this notation.

Groups of the type $G = T \rtimes_w \mathbb{Z}^k$ was studied in [10], but with the restriction that T is abelian. Assuming T to be abelian, for a group $H = T \rtimes_v \mathbb{Z}^k$ we found that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$ if and only if the subgroups $\mathrm{Im}v$ and $\mathrm{Im}w$ of $\mathrm{Aut}T$ are conjugate in $\mathrm{Aut}T$. The condition of T being abelian is not necessary as we see in Theorem 2.4 below.

The proof of the following proposition is a routine exercise and we omit it.

Proposition 2.2. *Let G be the group as above. For an action $v : \mathbb{Z}^k \to \mathrm{Aut}T$, let $H = T \rtimes_v \mathbb{Z}^k$. Then H is isomorphic to G if and only if the following condition holds.*
There exist automorphisms $h_0 : T \to T$ and $h_1 : \mathbb{Z}^k \to \mathbb{Z}^k$ such that the following identity holds for every $t \in T$ and every $z \in \mathbb{Z}^k$.

$$[h_0 \circ w(z)](t) = [v(h_1 z)](h_0 t) \tag{2.1}$$

Corollary 2.3. *Let G and H be groups as in Proposition 2.2. If $H \simeq G$ then the subgroups $\mathrm{Im}v$ and $\mathrm{Im}w$ of $\mathrm{Aut}T$ are conjugate in $\mathrm{Aut}T$.*

Proof. We use Proposition 2.2. Since h_1 is surjective, we have $\mathrm{Im}(v \circ h_1) = \mathrm{Im}(v)$. Thus the corollary follows from Proposition 2.2. □

Theorem 2.4. *Let G and H be the groups as in Proposition 2.2 above. Then $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$ if and only if the subgroups $\mathrm{Im}v$ and $\mathrm{Im}w$ of $\mathrm{Aut}T$ are conjugate in $\mathrm{Aut}T$.*

Proof. If $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$, then by Corollary 2.3 it follows that $\mathrm{Im}v$ and $\mathrm{Im}w$ are conjugate in $\mathrm{Aut}T$.
Now suppose that there exists $h \in \mathrm{Aut}T$ such that $h(\mathrm{Im}v)h^{-1} = \mathrm{Im}w$. There is an epimorphism $w_1 : \mathbb{Z}^{k+1} \to \mathrm{Im}w$ such that $G \times \mathbb{Z} = T \rtimes_{w_1} \mathbb{Z}^{k+1}$. Similarly, there is an epimorphism $v_1 : \mathbb{Z}^{k+1} \to \mathrm{Im}v$ such that $H \times \mathbb{Z} = T \rtimes_{v_1} \mathbb{Z}^{k+1}$. Let $\tau : \mathrm{Im}w \to \mathrm{Im}v$ be the isomorphism defined by $\tau : \alpha \mapsto h\alpha h^{-1}$. Then the two epimorphisms w_1 and $\tau \circ v_1$ ($\mathbb{Z}^{k+1} \to \mathrm{Im}v$) are Nielsen equivalent since the rank of $\mathrm{Im}v$ and is less than $k+1$. Thus there exists $g \in \mathrm{Aut}\mathbb{Z}^{k+1}$ such that by Proposition 2.2 the pair g, h constitutes an isomorphism $G \times \mathbb{Z} \to H \times \mathbb{Z}$. □

Theorem 2.5. *Let H and K be subgroups of G such that the indices $[G : H]$ and $[G : K]$ are finite and relatively prime to n, and $T_H = T = T_K$. Then:*
(a) $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$
(b) If $[G : H] \equiv \pm[G : K]$ *mod* n*, then* $H \simeq K$*,*

Proof. The (a)-part follows by Theorem 2.4 and the properties of the Nielsen equivalence relation on $E_{k+1}(\mathrm{Im}w)$.
We now prove the (b)-part. By assumption, $H \simeq T \times_\zeta \mathbb{Z}^k$ and $K \simeq T \times_\xi \mathbb{Z}^k$ for some $\zeta, \xi \in E_k(\mathrm{Im}w)$, and moreover, ζ and ξ are Nielsen equivalent. Therefore there exists an automorphism f of $\mathbb{Z}^k$ for which the elements ζ and $\xi \circ f$ of $E_k(\mathrm{Im}w)$ coincide, i.e., $\zeta = \xi \circ f$. Therefore by Proposition 2.2 the function

$$T \times_\zeta \mathbb{Z}^k \to T \times_\xi \mathbb{Z}^k$$

defined by $(t, z) \mapsto (t, f(z))$ is an isomorphism. □

Theorem 2.6. *There is an epimorphism* $\theta_1 : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G)$.

Proof. The description of $\chi(G)$ in [13] is in terms of an integer $n(G)$ which can be seen to be a multiple of n (and we shall write $n(G) = m$). The group structure is decribed by a certain surjective function $\theta : (\mathbb{Z}/m)^*/\pm 1 \to \chi(G)$ which ends up being a group homomorphism. The function θ is defined in terms of subgroups of G, very similar to the way we shall define θ_1 below.
Since $n|m$ there is an epimorphism $(\mathbb{Z}/m)^* \to (\mathbb{Z}/n)^*$, and consequently an epimorphism $\rho : (\mathbb{Z}/m)^*/\pm 1 \to (\mathbb{Z}/n)^*/\pm 1$. Now let z be any integer which is relatively prime to n, and fix any subgroup $G_{(z)}$ of G such that $T < G_{(z)}$ and $[G : G_{(z)}] = z$. Such a subgroup does exist by [13, Theorem 2.5]. By Theorem 2.5 it follows that the rule $z \mapsto [G_{(z)}]$ determines a well-defined function $\theta_1 : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G)$ such that $\theta_1 \circ \rho = \theta$. Since ρ and θ are group homomorphisms and ρ is a surjective function, from the latter identity it follows that θ_1 is a group homomorphism. Since θ is a surjective, also θ_1 is surjective. □

This theorem together with the description of the kernel of θ_1, Theorem 2.7 below, gives us a simplified method for calculating the non-cancellation group, generalizing the work in [10, Section 3].

Theorem 2.7. *Let $m \in \mathbb{Z}$ and suppose that m is relatively prime to n. Then the following conditions are equivalent.*
(a) $\overline{m} \in \ker[(\mathbb{Z}/n)^* \to \chi(G)]$,
(b) There exists an automorphism $\alpha \in \mathrm{Aut}T$ *such that for the inner automorphism* $\tau : v \to \alpha v \alpha^{-1}$ *of G, we have* $\tau(\mathrm{Im}w) = \mathrm{Im}w$ *and for the automorphism* $\sigma : \mathrm{Im}w \to \mathrm{Im}w$ *induced by τ, we have* $\det(\sigma) = \pm\overline{m}^{-1} \in (\mathbb{Z}/n)^*$.

Proof. Let us assume condition (a). Then G has a subgroup G_0 of index m which is isomorphic to G. Let $h : G_0 \to G$ be an isomorphism and let $i : G_0 \to G$ be the inclusion. Then $G_0 = T \rtimes_{w \circ j} \mathbb{Z}^k$ where $j : \mathbb{Z}^k \to \mathbb{Z}^k$ is an embedding induced by i. Since $[G : \mathrm{Im} i] = m$, in $E_k^{\sim}(\mathrm{Im} w)$ we have $[w \circ j] = {}^{\overline{m}}[w]$. Let $\alpha : T \to T$ and $h_1 : G_0/T \to G/T$ be the isomorphisms induced by h. Then the inner automorphism of $\mathrm{Aut} T$ defined by $\beta \mapsto \alpha\beta\alpha^{-1}$ induces an automorphism τ of $\mathrm{Im} w$. Now we have, for each $z \in \mathbb{Z}^k$,

$$w \circ h_1(z) = \alpha(w \circ j(z))\alpha^{-1} = \tau \circ w \circ j(z),$$

i.e., $[w] = [\tau \circ w \circ j]$. This means that $[w \circ j] = {}^{\overline{m}}[\tau \circ w \circ j]$. Consequently $\det(\tau) = \pm\overline{m}^{-1}$, and the statement (b) follows. Thus we have proved that (a) implies (b). The argument is reversible, and so the theorem follows. □

3. Finite normal subgroup

For any $\mathcal{X}_0$-group G we have a group structure [13] on $\chi(G)$, which we briefly recall. Associated with G we have an integer $n(G)$. We can assume that G is infinite, since for a finite group H, $\chi(H)$ is a one-point set anyway. Let

$$X = \{k \in \mathbb{N} : k \text{ is relatively prime to } n(G)\}.$$

Let us denote the isomorphism class of a group H by $[H]$. Given any $k \in X$, fix any subgroup $G_{(k)}$ of G such that $[G : G_{(k)}] = k$. Such groups do exist if G is infinite.

Then the rule $k \mapsto G_{(k)}$ defines a function $\mu : X \to \chi(G)$. This function μ factorizes through the reduction mod n function, i.e., there exist functions

$$\xi : X \to (\mathbb{Z}/n)^\times \quad\text{and}\quad \sigma : (\mathbb{Z}/n)^\times \to \chi(G)$$

such that $\mu = \sigma \circ \xi$. The function σ is surjective and is such as to induce a group structure on $\chi(G)$ (for which σ becomes a surjective homomorphism of groups). Now consider any $n_1 \in \mathbf{N}$, and let

$$X_1 = \{k \in \mathbb{N} : k \text{ is relatively prime to } nn_1\}.$$

Then the restriction $\mu_1 = \mu | X_1$ of μ factorizes through the reduction mod nn_1 function, and the corresponding function $\sigma_1 : (\mathbb{Z}/nn_1)^\times \to \chi(G)$ induces the same group structure on $\chi(G)$ as σ. In particular we note the following.

Remark 3.1. Given any two $\mathcal{X}_0$-groups G and K, let $m = n(G) \cdot n(K)$, then we have obvious epimorphisms

$$\theta_1 : (\mathbb{Z}/m)^\times \to \chi(G) \quad\text{and}\quad \theta_2 : (\mathbb{Z}/m)^\times \to \chi(K).$$

Now we fix any $\mathcal{X}_0$-group G and a finite normal subgroup F of G. Let $m = n(G)n(G/F)$. We set out to define a function $\eta : \chi(G) \to \chi(G/F)$. Consider any group H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. Let

$Y = \{k \in \mathbb{N} : k \text{ is relatively prime to } m\}$, and

$S_H = \{k \in Y : \text{ for some embedding } \alpha : H \to G,\ [G : \alpha(H)] = k\}$.

Then from [13, Theorem 4.2] it follows that S_H is nonempty. Thus S_H has a least member, k_H. Now suppose that $\alpha : H \to G$ and $\beta : H \to G$ are embeddings such that $[G : \alpha(H)] = k_H = [G : \beta(H)]$. Then $[G/F : \alpha(H)/F] = k_H = [G/F : \beta(H)/F]$, and thus by [13, Theorem 4.3], $\alpha(H)/F \simeq \beta(H)/F$.

Theorem 3.2. *We obtain a well-defined function* $\eta : \chi(G) \to \chi(G/F)$ *by picking for* H *as above, any embedding* $\alpha : H \to G$ *with* $[G : \alpha(H)] = k_H$ *and taking* $\eta([H])$ *to be the isomorphism class* $[\alpha(H)/F]$ *of* $\alpha(H)/F$. □

Remark 3.3. For a $\mathcal{X}_0$-group G and for a finite characteristic subgroup F of the torsion subgroup of G, the function $\eta : \chi(G) \to \chi(G/F)$ defined above, coincides with the epimorphism η of [13, Section 6]. We also wish to note that [13, Theorem 6.4] must be reformulated. Theorem 3.4 (below) is what is actually proved under the statement of [13, Theorem 6.4]. In Section 5 we shall see that η is in general not necessarily a surjective group homomorphism.

Theorem 3.4. *Let* F *be a characteristic subgroup of the torsion radical of an infinite* $\mathcal{X}_0$*-group* G. *Then the function* $\eta : \chi(G) \to \chi(G/F)$ *is a surjective group homomorphism.*

4. A wreath product-like construction

Let T be any finite nilpotent group. Let $p_1, p_2, ..., p_k$ be the distinct prime divisors of T in increasing order. Then T splits as the direct product of its Sylow subgroups,

$$T = T_{p_1} \times T_{p_2} \times ... \times T_{p_k}.$$

For each $i \in \{1, 2, ..., k\}$ let R_{p_i} be the regular wreath product, $R_{p_i} = \mathbb{Z}/p_i \wr T_{p_i}$, and let $R = R_{p_1} \times R_{p_2} \times ... \times R_{p_k}$. Then for each i, R_{p_i} is the Sylow p_i-subgroup of R. To stress the dependence on T, we can write $R = R_T$. Let S be the kernel of the natural epimorphism $R \to T$.

Proposition 4.1. Suppose that the group T is abelian and does not have $\mathbb{Z}/2$ as a Sylow subgroup. Then S is a characteristic subgroup of R.

Proof. Under the given conditions it is easy to prove that S is the centralizer of the commutator subgroup K of R. Since K is a characteristic subgroup of R, it follows that also S is a characteristic subgroup of R. $\square$

4.2 Notation. We find it convenient to use some notation that was introduced in [12]: For a pair of relatively prime natural numbers n and u, the symbol $G(n;u)$ will be used to denote the group $\mathbb{Z}/n \rtimes_h \mathbb{Z}$, where $h : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n)$ is the homomorphism obtained by taking $h(1)$ to be the automorphism $t \mapsto ut$ of $\mathbb{Z}/n$.

4.3 A particular group $N = G(n;u)$. We now fix a relatively prime pair of natural numbers n, u, and we require that the group $N = G(n;u)$ is nilpotent. Let d be the multiplicative order of u modulo n. Let $T = T_N$ be the torsion subgroup of N. Let $\zeta(1)$ be an automorphism of R such that the order of $\zeta(1)$ in $\mathrm{Aut}R$ is d, and for each $t \in T < R$, $\zeta(1) : t \mapsto ut$ (T regarded to be an additive group). We shall prove (Proposition 5.5) that such an automorphism does exist. Then the rule $(j) \mapsto [\zeta(1)]^j$ defines a homomorphism $\zeta : \mathbb{Z} \to \mathrm{Aut}R$. We can thus form a group

$$W_\zeta(N) = R \rtimes_\zeta \mathbb{Z}.$$

For brevity we shall write $W_\zeta(N) = W$. An important connection between W and N becomes clear in Proposition 4.5.

Proposition 4.4. *The group W is an $\mathcal{N}_0$-group.*

Proof. The homomorphism ζ induces a homomorphism $\zeta' : \mathbb{Z}/d \to \mathrm{Aut}R$. Let d_i be the multiplicative order of u modulo n_i, n_i being the biggest power of p_i dividing n. Then d_i is also the order of the automorphism of R_{p_i} induced by $\zeta(1)$. Since $G(n;u)$ is nilpotent, it follows that for each i, d_i is a power of p_i. Therefore the group

$$Q = R \rtimes_{\zeta'} \mathbb{Z}/d$$

splits as the direct product of its Sylow subgroups, and consequently is nilpotent. Furthermore, W has a torsion-free central subgroup F such that $W/F \simeq Q$. Thus W is nilpotent. The rest is clear. $\square$

Proposition 4.5. $\mathcal{G}(W) \simeq \mathcal{G}(N) \simeq (\mathbb{Z}/d)^*/\pm 1$.

Proof. By [1] we have $\mathcal{G}(N) \simeq (\mathbb{Z}/d)^*/\pm 1$. By Theorem 2.6, there is an epimorphism $(\mathbb{Z}/d)^*/\pm 1 \to \mathcal{G}(W)$.
In order to be able to apply Proposition 4.1 we first prove the proposition in a special case, with the condition that either n is odd or $4|n$.

For the characteristic subgroup S of R we have $W/S \simeq N$, and thus by Theorem 3.4 there is an epimorphism $\mathcal{G}(W) \to \mathcal{G}(N)$. This proves the theorem in the special case.

We now consider the remaining case: suppose that $n = 2m$ and m is odd. Then the Sylow 2-subgroup V of R is such that the action of $\zeta(1)$ on V is trivial. Thus $W \simeq V \times W/V$ and $N = U \times N/U$ where U is the Sylow 2-subgroup of N. Since finite abelian groups are cancellable (see [9]) it follows that $\mathcal{G}(W) \simeq \mathcal{G}(W/V)$ and $\mathcal{G}(N) \simeq \mathcal{G}(N/U)$. Now by the special case we have $\mathcal{G}(W/V) \simeq \mathcal{G}(N/U)$, and this completes the proof. □

Theorem 4.6. *For every* $m \in \mathbb{N}$, $\mathcal{G}(W^m) \simeq \mathcal{G}(N^m)$.

Proof. In order to be able to apply Proposition 4.1, initially we assume that either n is odd or $4|n$, so that by Proposition 4.1 it follows that S^m is a characteristic subgroup of R^m. We note that $W^m/S^m \simeq N^m$. Therefore by Theorem 3.4 there is an epimorphism $\mathcal{G}(W^m) \to \mathcal{G}(N^m)$. Note that for any $m > 1$, we have $\mathcal{G}(N^m) \simeq \mathcal{G}(N^2)$. Furthermore there is an epimorphism $\mathcal{G}(W^2) \to \mathcal{G}(W^m)$. Thus in order to complete the proof (with the special condition on n), it suffices to prove the case $m = 2$.

Now consider any $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 n_2$ and $(n_1, n_2) = 1$. Let d_1 and d_2 be, respectively, the multiplicative orders of u mod n_1 and n_2. Since N is nilpotent it follows that d_1 and d_2 are relatively prime.

The group R splits into a direct product $R = U \times V$ such that $(|U|, n_2) = 1$ and $(|V|, n_1) = 1$. Let α be the automorphism of $(U \times V) \times (U \times V)$ defined by:

$$\alpha : (a, b, x, y) \mapsto (a, y, x, b).$$

There is an obvious choice of a homomorphism $w : \mathbb{Z}^2 \to \mathrm{Aut}(R^2)$ such that $W^2 \simeq R^2 \rtimes_w \mathbb{Z}^2$. Let us denote $w(1,0)$ by w_1 and $w(0,1)$ by w_2. The homomorphism w is such that w_1 fixes the subgroup $(1 \times R)$ and w_2 fixes the subgroup $(R \times 1)$. Let $J = \mathrm{Im} w = \langle w_1, w_2 \rangle < \mathrm{Aut}(R \times R)$. Note that $\zeta(1)$ induces automorphisms ϵ and δ (respectively) of U and V such that the automorphism $\epsilon \times \delta$ of $U \times V$ coincides with $\zeta(1)$ and ϵ and δ are of orders d_1 and d_2 respectively. Thus, α normalizes the subgroup J of $\mathrm{Aut}(R \times R)$. Let τ be the automorphism of J defined by

$$\tau : \lambda \to \alpha\lambda\alpha^{-1},$$

and let $c \in \mathbb{N}$ be such that the determinant in $(\mathbb{Z}/d)^*$ of τ is $\bar{c}$. Then $c \equiv 1 \bmod d_1$ and $c \equiv -1 \bmod d_2$. Thus from [6] it follows that $\mathcal{G}(W^2) \simeq \mathcal{G}(N^2)$. This proves the special case.

The remaining case of $n = 2m$ for some odd m follows similarly as in the proof of Proposition 4.5. □

5. The genus of $W \times N$

Let N and W be as in paragraph 4.3, and let us recall that the torsion subgroups of N and W are T and R respectively, and we recall that R is a semidirect product $R = S \rtimes T$ for some action of T on S, and $W = R \rtimes_\zeta \mathbb{Z}$. In what follows we calculate the group $\mathcal{G}(W \times N)$.

The torsion subgroup of $W \times N$ is the group $R \times T$, and $W \times N = (R \times T) \rtimes_\xi \mathbb{Z}^2$, where $\xi : \mathbb{Z}^2 \to \mathrm{Aut}(R \times T)$ is such that $\xi(1,0)$ is the automorphism $(h,t) \mapsto (\zeta(h), t)$, and $\xi(0,1)$ is the automorphism $(h,t) \mapsto (h, ut)$. We write $\xi(1,0) = v$ and $\xi(0,1) = w$. We must compute the kernel of the epimorphism $\theta : (\mathbb{Z}/d)^*/\pm 1 \to \mathcal{G}(W \times N)$. Let J be the image of v in $\mathrm{Aut}(R \times T)$, i.e., J is the subgroup generated by the subset $\{v, w\}$. For any $\alpha \in \mathrm{Aut}(R \times T)$, let τ_α be the inner automorphism of $\mathrm{Aut}(R \times T)$, $\tau_\alpha : \beta \to \alpha^{-1}\beta\alpha$. We are particularly interested in the subgroup J_* of $\mathrm{Aut}(R \times T)$ of all automorphisms α for which $\tau_\alpha(J) = J$, i.e., J_* is the normalizer of J in $\mathrm{Aut}(R \times T)$.

In our main calculations of this section we shall use the following notation and impose the following condition on the group N.

Condition 5.1. We shall assume that for the group N of paragraph 4.3, the torsion is such that if p is a prime factor of n then also $p^2 | n$.

Notation: The elements of $R \times T$ can be written as triples (b, x, y), with $b \in S$ and $x, y \in T$, and then (b, x) denotes a typical element of R.

Proposition 5.2. *Let us assume Condition 5.1. Given any $\alpha \in J_*$, then the automorphism f of J induced by τ_α is exactly the identity automorphism of J.*

Proof. There exist $r, s \in \mathbb{N}$ such that $f(v) = v^r w^s$. We prove that $s \equiv 0 \bmod d$ and $r \equiv 1 \bmod d$. In this calculation it is important to note that the subgroup $C = \{(a, 0, y) : \ a \in S, \ y \in T\}$ is a characteristic subgroup of $R \times T$. This is so because C is the centralizer of the commutator subgroup of $R \times T$.

Now pick any generator y of T. Then $\alpha(0,0,y) = (b, 0, y_1)$ for some $b \in S$ and some $y_1 \in T$. Thus $v\alpha(0,0,y) = (b', 0, y_1)$ for some $b' \in S$. But then,

$$\begin{aligned}(b', 0, y_1) &= v\alpha(0,0,y)\\ &= \alpha v^r w^s (0,0,y)\\ &= \alpha v^r (0,0,u^s y)\\ &= \alpha(0,0,u^s y)\\ &= u^s \alpha(0,0,y)\\ &= u^s (b, 0, y_1)\end{aligned}$$

$= (u^s b, 0, u^s y_1)$.

In particular then, $u^s y_1 = y_1$. Now y_1 must of the same order as y (which is n) due to Condition 5.1 and since α is an isomorphism. So it follows that

$u^s \equiv 1 \bmod n$, and so $s \equiv 0 \bmod d$.

Now pick any generator x of T and let $\alpha(0, x, 0) = (a, x_1, z)$. If $tx_1 = 0$ for some $t \in \mathbb{N}$, then $t\alpha(0, x, 0)$ belongs to the centralizer C of the commutator subgroup of $R \times T$. Since α is an automorphism and C is a characteristic subgroup of $R \times T$, it follows that $(0, tx, 0) \in C$. But this implies that $tx = 0$. Thus x and x_1 have the same order.

Now $v\alpha(0, x, 0) = (a', ux_1, z)$ for some $a' \in S$. Since $v\alpha = \alpha v^r w^s$, we have:

$$\begin{aligned}(a', ux_1, z) &= \alpha v^r w^s (0, x, 0)\\ &= \alpha v^r (0, x, 0)\\ &= \alpha(0, u^r x, 0)\\ &= u^r \alpha(0, x, 0)\\ &= u^r (a', x_1, z).\end{aligned}$$

This implies that $(u^r - u)x = 0$, i.e., $r \equiv 1 \bmod d$. We have shown that $f(v) = v$.

Next we show that $f(w) = w$. Now let $r, s \in N$ be such that $f(w) = v^r w^s$. Starting with an element $(0, x, 0)$, where x is some generator of T, one can show that $r \equiv 0 \bmod d$. It can also be shown, calculating images of $(0, 0, z)$ for some generator z of T, that $s \equiv 1 \bmod d$. Thus, in fact, f is the identity map of J. □

Theorem 5.3. $\mathcal{G}(W \times N) \simeq (\mathbb{Z}/d)^*/\pm 1$.

Proof. This follows by Theorem 2.7 and Proposition 5.2. □

The following example shows that the function η of Section 3 is not always a homomorphism.

Example 5.4. We use the notation of paragraph 4.3, and choose a particular group N. Let $N = G(1225; 36)$. Note that $n = 5^2 7^2$ and $u = 5.7 + 1$, and therefore by [2, Lemma 1.1] it follows that N is nilpotent. Also, N satisfies Condition 5.1. Again let $W = W(N)$. Then, factoring out the finite normal subgroup $(0, S)$ of $W \times W$, we consider the function $\eta : \mathcal{G}(W^2) \to \mathcal{G}(W \times N)$ defined in Section 3.
By [1] we have $\mathcal{G}(N) \simeq (\mathbb{Z}/35)^*/\pm 1$. We can identify the group $\mathcal{G}(N)$ with the subgroup $\langle \overline{2} \rangle < (\mathbb{Z}/35)^*$, and so $\mathcal{G}(N)$ is cyclic and of order 12. By [6] the group $\mathcal{G}(N^2)$ is isomorphic to the quotient group $\langle \overline{2} \rangle / \langle \overline{2}^6 \rangle$, and by Theorem 4.6, $\mathcal{G}(N^2) \simeq \mathcal{G}(W^2)$. By Theorem 5.3 we have $\mathcal{G}(W \times N) \simeq (\mathbb{Z}/35)^*/\pm 1$. Let H denote a group that embeds in W^2 with index 2 and such that H contains all the torsion

elements of W^2. Let $[H]$ denote the element of $\mathcal{G}(W^2)$ represented by H. Then in the group $\mathcal{G}(W^2)$ we have $6[H] = 0$ whereas in the group $\mathcal{G}(W \times N)$, $6\eta[H]$ is non-zero. Therefore η is not a homomorphism. This matter is pursued in [7]. $\square$

We have one outstanding matter to clarify. Let F be a finite group acting regularly on a set X. Let $\rho : F \to \Sigma_X$ be the permutation representation. For any group H, let $H \wr F$ be the regular wreath product. Then for any automorphism α_0 of F, the action $\rho \circ \alpha_0$ of F on X is regular and so there exists a permutation π of X, giving rise to an automorphism α_1 of H^X such that for each $z \in F$, in $\mathrm{Aut}(H^X)$ we have

$$\alpha_1[\rho(z)] = [\rho \circ \alpha_0(z)]\alpha_1,$$

and then the pair of automorphisms α_0, α_1 determines an automorphism α of $H \wr F$. Since the centralizer of $\mathrm{Im}\rho$ in Σ_X is trivial, π is uniquely determined by α_0, and there is a standard way of converting π into an automorphism of H^X. In this way α_0 determines α in such a way that α and α_0 have the same order. From this, and using the fact that a finite nilpotent group splits as the direct product of its Sylow subgroups, we deduce the following proposition.

Proposition 5.5. *For any finite nilpotent group T and any automorphism α_0 of T, there exists an automorphism α of R (R is as in the first paragraph of Section 4) such that α and α_0 have the same order.* $\square$

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