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Generalizing the Hilton–Mislin Genus Group

Peter J. Witbooi

University of the Western Cape, Private Bag X17, 7535 Bellville, South Africa

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For any group H, let $\chi(H)$ be the set of all isomorphism classes of groups K such that $K \times \mathbb{Z} \simeq H \times \mathbb{Z}$. For a finitely generated group H having finite commutator subgroup $[H, H]$, we define a group structure on $\chi(H)$ in terms of embeddings of K into H, for groups K of which the isomorphism classes belong to $\chi(H)$. If H is nilpotent, then the group we obtain coincides with the genus group $\mathscr{G}(H)$ defined by Hilton and Mislin. We obtain some new results on Hilton–Mislin genus groups as well as generalizations of known results.

1. INTRODUCTION

We are interested in certain classes of groups, which we now define. Let $\mathscr{X}_0$ be the class of all finitely generated groups that have finite commutator subgroups. Let $\mathscr{N}_0$ (as in [2]) be the subclass of all nilpotent groups in $\mathscr{X}_0$. For any group G, the *non-cancellation set*, $\chi(G)$ (as in the abstract), is the set of all isomorphism classes of groups H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. For certain $\mathscr{X}_0$-groups G, the non-cancellation sets $\chi(G)$ have been studied in [10], for instance. For a finitely generated nilpotent group N, the Mislin genus, $\mathscr{G}(N)$, is defined to be the set of all isomorphism classes of finitely generated nilpotent groups M such that for every prime p, the groups M and N have isomorphic p-localizations; see [7]. For $\mathscr{N}_0$-groups, N, Hilton and Mislin defined an abelian group structure on the set $\mathscr{G}(N)$ in [2]. Various calculations of such Hilton–Mislin genus groups can be found in the literature, for example, in the article by Hilton and Scevenels [3]. In [11] it is shown that for $\mathscr{N}_0$-groups N and M, there is an epimorphism $\mathscr{G}(N) \to \mathscr{G}(N \times M)$ if N is infinite. This is an affirmative answer to a question in [2]. In particular, in [11] we deduce some results on triviality of the genus. Related to such genus studies, we observe some interesting non-cancellation phenomena and non-unique direct sum decompositions

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of groups in $\mathcal{N}_0$. Warfield [8, Theorem 3.5] has shown that for an $\mathcal{N}_0$-group, N, we have $\mathcal{G}(N) = \chi(N)$. The purpose of this article is to generalize the Hilton–Mislin group structure to the non-cancellation sets of $\mathcal{X}_0$-groups. Using the group structure on the non-cancellation set, we prove some results, similar to those in [11], on morphisms between non-cancellation groups. These results imply, inter alia, some theorems on triviality of the non-cancellation set of a $\mathcal{X}_0$-group. Again, as in [11] we can deduce some results on Hilton–Mislin genus groups of $\mathcal{N}_0$-groups.

In Section 2 we prove some basic results on $\mathcal{X}_0$-groups, including a certain pull-back construction for such a group. In Section 3 we have some results on presentations of finite $\mathbb{Z}$-modules. In Section 4 it is shown that for every $\mathcal{X}_0$-group G, the members of $\chi(G)$ can be represented by certain subgroups of G of finite index in G, and we make a detailed study of such subgroups. The group structure is treated in Section 5. In Section 6 we use the methods of [11] to study certain homomorphisms between non-cancellation sets. We generalize some further results on Hilton–Mislin genera of $\mathcal{N}_0$-groups, including some new ones appearing in [11].

2. BASICS OF $\mathcal{X}_0$-GROUPS

We note some of the basic properties of $\mathcal{X}_0$-groups. Throughout this section, G shall denote a $\mathcal{X}_0$-group.

The centre of the group G will be denoted by Z_G. The set of all elements of finite order in G forms a finite normal subgroup, the torsion subgroup, which we shall denote by T_G. The torsionfree quotient, G/T_G, of G is a finite rank free abelian group. A group H belongs to $\mathcal{X}_0$ if and only if H is an extension of a finite group by a finite rank free abelian group.

The class $\mathcal{X}_0$ is closed with respect to taking subgroups and forming finite direct products. In particular, if $G \in \mathcal{X}_0$ then $G \times \mathbb{Z} \in \mathcal{X}_0$, and furthermore if we have a group H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$, then $H \in \mathcal{X}_0$.

For every $G \in \mathcal{X}_0$ we define a subgroup F_G as follows. Let n_1 be the exponent of the torsion subgroup T_G, let n_2 be the exponent of the group $\mathrm{Aut}(T_G)$, and let n_3 be the exponent of the torsion subgroup of the centre of G. We define the natural number $n(G) = n_1 n_2 n_3$ and the subgroup

$$F_G = \langle x^n : x \in G \rangle,$$

where $n = n(G)$. Then F_G is a normal subgroup (in fact, a fully invariant subgroup) of G.

PROPOSITION 2.1. *Let $G \in \mathscr{X}_0$ and let $m = n_1 n_2$, for n_1 and n_2 as above, and let*

$$E_G = \langle x^m : x \in G \rangle.$$

Then $E_G < Z_G$ and for the canonical epimorphism $\alpha\colon G \to G/T_G$, we have $\alpha(E_G) = \{g^m T_G : g \in G\}$.

Proof. The second part of the proposition is simple—note that G/T_G is abelian. We now prove that $E_G < Z_G$.

Consider any $g, x \in G$. We shall prove that $g^m x g^{-m} = x$. Let $\zeta\colon G \to G$ be the inner automorphism $a \mapsto gag^{-1}$. Since the commutator subgroup $[G, G]$ is finite, there exists $b \in T_G$ such that $\zeta(x) = bx$. By induction one can prove the following identity ($q \in \mathbb{N}$):

$$\zeta^q(x) = \left[\prod_{i=0}^{q-1} \zeta^i(b)\right] x.$$

The inductive step is as follows:

$$\begin{aligned}\zeta^{r+1}x = \zeta(\zeta^r x) &= \zeta\left[\left\{\prod_{i=0}^{r-1} \zeta^i b\right\} x\right] = \left[\prod_{i=1}^{r} \zeta^i b\right] \zeta(x) \\ &= \left[\prod_{i=1}^{r} \zeta^i b\right] bx = \left[\prod_{i=0}^{r} \zeta^i b\right] x.\end{aligned}$$

Now we note that $\zeta^{n_2}(t) = t$ for each $t \in T_G$. Consequently,

$$\prod_{i=0}^{m-1} \zeta^i(b) = \left[\prod_{i=0}^{n_2-1} \zeta^i(b)\right]^{n_1} = 1,$$

since n_1 is the exponent of T_G. Thus it follows that $g^m x g^{-m} = \zeta^m x = x$. This proves that $E_G < Z_G$. ∎

The following proposition now follows readily. We omit the proof.

PROPOSITION 2.2. *Let $G \in \mathscr{X}_0$ and let $n = n(G)$. Then:*

(a) *the canonical epimorphism $G \to G/T_G$ embeds F_G into G/T_G,*

(b) $[G : F_G] = n^k |T_G|$, *where k is the rank of the free abelian group G/T_G.*

Let G be any $\mathscr{X}_0$-group. Consider the following diagram, in which every homomorphism is a canonical epimorphism onto a quotient group.

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G/T_G \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ G/F_G & \xrightarrow[\alpha']{} & G/(T_G F_G) \end{array} \tag{1}$$

Note that F_G is torsionfree, and α embeds F_G into G/T_G. Also, β embeds T_G into G/F_G. The square is commutative.

THEOREM 2.3. *Let G be any $\mathscr{X}_0$-group, and let $n = n(G)$. Let $H < G$ such that $[G : H]$ is relatively prime to n. Let $\phi = \beta \circ i$ where i: $H \to G$ is the inclusion and β: $G \to G/F_G$ is the canonical epimorphism. Then,*

(a) $F_H \subset \ker \phi$ *and the induced homomorphism* ϕ': $H/F_H \to G/F_G$ *is an isomorphism,*

(b) $T_H = T_G$.

Proof. (a) We note that $\ker \phi = F_G \cap H$. Since $[G : H]$ is relatively prime to n, it follows that if we have $x \in G$ for which $x^n \in H$, then $x \in H$. Thus $F_G \cap H < F_H$, and so $F_H \subset \ker \phi$. On the other hand, clearly $F_H < F_G$. Thus $F_H = \ker \phi$, so that ϕ' is a monomorphism.

We now prove that ϕ is surjective. Consider any $y \in G$. Then $y^n \in F_G$ and for some $m \in \mathbb{N}$ which is relatively prime to n, $y^m \in H$. But then there are $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Thus $y = y^{am} y^{bn}$ with $y^{am} \in H$ and $y^{bn} \in F_G$, and so $yF_G = \phi(y^{am})$. This completes the proof.

(b) Consider any $x \in T_G$. Then for some $m \in \mathbb{N}$ which is relatively prime to the order to x, $x^m \in H$. But this implies that also $x \in H$. Thus $T_G \subset H$, and so $T_G \subset T_H$. Therefore $T_G = T_H$. ▮

The following result is crucial for the proof of the existence of certain embeddings in Section 4. The reader who is not familiar with the categorical notion of pull-back is referred to the book by Hilton and Stammbach [4].

THEOREM 2.4. *Let G be any $\mathscr{X}_0$-group. Then the diagram* (1) *above is a pull-back square in the category of groups.*

Proof. Let $P = \{(x, a) \in G/F_G \times G/T_G : \alpha'(x) = \beta'(a)\}$. Then P is a subgroup of $G/F_G \times G/T_G$. Let π_1: $P \to G/F_G$ and π_2: $P \to G/T_G$ be the restrictions of the relevant projections of $G/F_G \times G/T_G$ onto each of its direct factors. Then the following square is a pull-back square.

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & G/T_G \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \beta'} \\ G/F_G & \xrightarrow[\alpha']{} & G/(T_G F_G) \end{array} \tag{2}$$

Since diagram (2) is a pull-back square and diagram (1) is commutative, there exists a (unique) homomorphism $\phi: G \to P$ such that $\pi_1 \circ \phi = \beta$ and $\pi_2 \circ \phi = \alpha$. In order to complete the proof we must show that ϕ is an isomorphism.

We denote the identity elements of G, G/F_G, and G/T_G by e, e_1, and e_2, respectively. Now suppose that $g \in \ker \phi$. Then,

$$\alpha(g) = \pi_2 \phi(g) = \pi_2(e) = e_2,$$

and it follows that $g \in T_G$. On the other hand,

$$\beta(g) = \pi_1 \phi(g) = \pi_1(e) = e_1,$$

and since β embeds T_G into G/F_G, it follows that $g = e$. Thus ϕ is a monomorphism. We now prove that ϕ is surjective.

Consider any $(x, a) \in P$. Since α is an epimorphism, we can find some $b \in G$ such that $\alpha(b) = a$. Then $x^{-1}\beta(b) \in \ker \alpha'$. Let $t = x^{-1}\beta(b)$, so that $\beta(b) = xt$. In particular we note that $t \in \beta(T_G)$, and we can pick $s \in T_G$ such that $\beta(s) = t$. Now let $g = bs^{-1} \in G$. We prove that $\phi(g) = (x, a)$.

$$\pi_1(\phi g) = \beta(g) = \beta(bs^{-1}) = \beta(b)[\beta(s)]^{-1} = (xt)t^{-1} = x$$

and

$$\pi_2(\phi g) = \alpha(g) = \alpha(bs^{-1}) = \alpha(b)\alpha(s^{-1}) = a,$$

since $\alpha(b) = a$ and $s \in T_G$. This completes the proof. ∎

We note the following result on subgroups of $\mathscr{X}_0$-groups.

PROPOSITION 2.5. *Let G be any infinite $\mathscr{X}_0$-group, and let m be any natural number. Then there is a subgroup H of G such that $[G : H] = m$.*

Proof. The proposition would certainly hold under the additional assumption that the $\mathscr{X}_0$-group G is abelian. Now let M be any subgroup of the free abelian group G/T_G such that $[G/T_G : M] = m$. Let $\pi: G \to G/T_G$ be the projection. Then the subgroup $\pi^{-1}(M)$ of G has index m in G. ∎

A very helpful result that gives a condition under which we are allowed to cancel the infinite cyclic group as a direct factor is the following lemma of Hirshon.

LEMMA 2.6 [6, Lemma 1]. *Suppose that G and H are any groups such that $G \times \mathbb{Z} \simeq H \times \mathbb{Z} \times \mathbb{Z}$. Then $G \simeq H \times \mathbb{Z}$.*

3. PRESENTATIONS OF FINITE $\mathbb{Z}$-MODULES

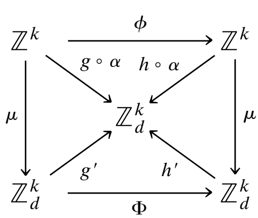
Consider a non-trivial finite abelian group B and let k be an integer which is not less than the Prüfer rank, $\operatorname{rank}(B)$, of B. (For a finite abelian group B, the Prüfer rank is simply the least of the cardinalities of generating subsets of B.) For the free abelian group $\mathbb{Z}^k$, we denote the set of epimorphisms $\mathbb{Z}^k \to B$ by $E_k(B)$. The Nielsen equivalence relation $\sim$ on $E_k(B)$ is defined as follows. For $f_1, f_2 \in E_k(B)$, $f_2 \sim f_1$ if and only if there is an automorphism $\alpha\colon \mathbb{Z}^k \to \mathbb{Z}^k$ such that $f_2 = f_1 \circ \alpha$. This relation can easily be seen to be an equivalence relation. The set of equivalence classes is denoted by $E_k^{\sim}(B)$. In order to describe $E_k^{\sim}(B)$ we introduce the following symbol. For a finite abelian group B, let $\delta(B)$ be the greatest common divisor of the orders of the invariant factors of B. Note that $\delta(B)$ can be defined equivalently to be the integer $\min\{m \in \mathbb{N} : \operatorname{rank}(mB) < \operatorname{rank}(B)\}$—here B is considered to be an *additive* group.

PROPOSITION 3.1. *Let B be any non-trivial finite abelian group of rank k, and let $d = \delta(B)$. Suppose that we have epimorphisms $g, h \in E_k(B)$ and an endomorphism $\phi\colon \mathbb{Z}^k \to \mathbb{Z}^k$ such that $g = h \circ \phi$. Then the cokernel,* $\operatorname{coker}(\phi)$, *of ϕ is a finite group and $|\operatorname{coker}(\phi)|$ is relatively prime to d.*

Proof. There exists an epimorphism $\alpha\colon B \to \mathbb{Z}_d^k$. For elements $(x_1, x_2, \dots, x_k)$ of $\mathbb{Z}^k$, reduction modulo d of coordinates yields a homomorphism $\mu\colon \mathbb{Z}^k \to \mathbb{Z}_d^k$,

$$\mu\colon (x_1, x_2, \dots, x_k) \mapsto (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k).$$

This homomorphism is such that there exist homomorphisms g' and h' making the following diagram commutative.



Then g' and h' are isomorphisms since g' and h' are epimorphisms and $\mathbb{Z}_d^k$ is finite (and consequently hopfian). Therefore Φ is an isomorphism. Thus the determinant $\det \Phi$ is a unit of the ring $\mathbb{Z}_d$. But $\det \Phi$ is the residu class of the integer $\det \phi$. Thus $\det \phi$ is relatively prime to d. Finally, the absolute value $|\det \phi|$ of $\det \phi$ is exactly equal to $|\text{coker } \phi|$. ∎

The following result which we state without proof is essentially the description of the set $E_k^{\sim}(B)$ as, for instance, in the paper by Webb [9].

THEOREM 3.2. *Let B be any non-trivial finite abelian group and let k be an integer which is not less than the rank*, $\text{rank}(B)$, *of B. There is a bijection between the set $E_k^{\sim}(B)$ and the group $\mathbb{Z}_d^*/\{1, -1\}$, where d is the integer defined as follows*:

$$d = \begin{cases} \delta(B) & \text{if } k = \text{rank}(B), \\ 1 & \text{if } k > \text{rank}(B). \end{cases}$$

For monomorphisms ϕ: $\mathbb{Z}^k \to \mathbb{Z}^k$ and a fixed epimorphism $g \in E_k(B)$, the element $[g \circ \phi]$ of $E_k^{\sim}(B)$ is uniquely determined by the image of the integer $\det(\phi)$ *in the group $\mathbb{Z}_d^*/\{1, -1\}$.*

4. SUBGROUPS OF FINITE INDEX IN A $\mathscr{X}_0$-GROUP

THEOREM 4.1. *Let G be any $\mathscr{X}_0$-group, and let $n = n(G)$. Suppose that $H < G$ such that $[G : H]$ is finite and is relatively prime to n. Then $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$.*

Proof. Certainly the subgroup T_G of H contains all the torsion elements of H, and the quotient group $H_1 = H/T_G$ is a subgroup of the free abelian group $G_1 = G/T_G$. The group operation in G_1 (and H_1) will be denoted by "+" (addition). Since $[G : H]$ is finite, it follows that H_1 is of the same (finite) rank as G_1. Furthermore, $[G_1 : H_1] = [G : H]$. We first define an isomorphism α: $H_1 \times \mathbb{Z}^k \to G_1 \times \mathbb{Z}^k$ having the property that

$$\alpha(\mathbb{Z}^k) < (nG_1) \times \mathbb{Z}^k.$$

By the stacked basis theorem (see [5, Theorem 5.1.1] for instance), there exists a basis $\{v_1, v_2, \ldots, v_k\}$ of G_1 and a basis $\{u_1, u_2, \ldots, u_k\}$ of H_1, together with a sequence $m_1, m_2, \ldots, m_k$ of integers such that for each $i \in \{1, 2, \ldots, k\}$ we have $u_i = m_i v_i$. We note that $\prod_{i=1}^k m_i = [G : H]$, and, in particular, then n is relatively prime to each of the integers m_i. Thus for each i we can find r_i, $s_i \in \mathbb{Z}$ such that $r_i m_i + s_i n = 1$. Now let $\{e_1, e_2, \ldots, e_k\}$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}^k$. We define α by letting for each i

$$\alpha(u_i) = m_i v_i - s_i e_i \quad \text{and} \quad \alpha(e_i) = n v_i + r_i e_i.$$

Since $|r_i m_i + s_i n| = 1$, it follows that α induces an isomorphism between subgroups of $H_1 \times \mathbb{Z}^k$ and $G_1 \times \mathbb{Z}^k$:

$$\langle u_i, e_i \rangle \to \langle v_i, e_i \rangle.$$

Thus α is an isomorphism and has the desired property.

Let us choose, for each $i \in \{1, 2, \dots, k\}$, an element g_i such that $g_i T_G = v_i$, and define a function $\beta\colon \mathbb{Z}^k \to G \times \mathbb{Z}^k$ by the formula

$$\beta\colon \sum_{i=1}^{k} a_i e_i \mapsto \left(\prod_{i=1}^{k} g_i^{n a_i}, \sum_{i=1}^{k} a_i r_i e_i \right).$$

Then β is a homomorphism. We define $\gamma\colon H \to G \times \mathbb{Z}^k$ as the composition

$$H \overset{\Delta}{\to} H \times H \overset{1 \times q}{\to} H \times H_1 \overset{i \times \zeta}{\to} G \times \mathbb{Z}^k,$$

where Δ is the diagonal homomorphism, $q\colon H \to H_1$ is the canonical homomorphism, $i\colon H \to G$ is the inclusion, and $\zeta\colon H_1 \to \mathbb{Z}^k$ is defined by

$$\zeta\colon \sum_{i=1}^{k} c_i u_i \mapsto \sum_{i=1}^{k} s_i c_i e_i.$$

We obtain a well-defined function $\Phi\colon H \times \mathbb{Z}^k \to G \times \mathbb{Z}^k$ through the formula

$$\Phi\colon (h, z) \to \gamma(h)\beta(z).$$

The function Φ is a homomorphism since γ and β are homomorphisms and $\beta(\mathbb{Z}^k)$ belongs to the centre of $G \times \mathbb{Z}^k$. It is not hard to see that Φ induces an isomorphism between the torsion subgroups. Furthermore, Φ induces a homomorphism $H_1 \times \mathbb{Z}^k \to G_1 \times \mathbb{Z}^k$, which coincides with the isomorphism α. Therefore Φ is an isomorphism. By repeated application of Lemma 2.6 it follows that there is an isomorphism $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. ∎

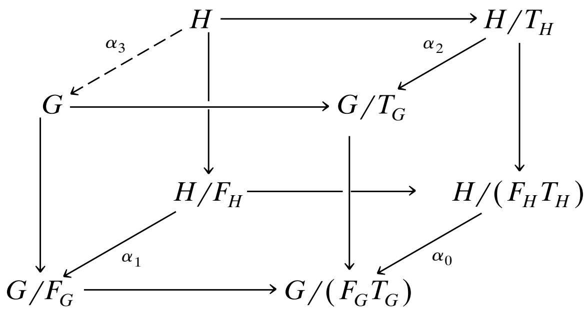
THEOREM 4.2. *Let G be any $\mathscr{X}_0$-group, and let H be any group such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. Then H is isomorphic to a subgroup L of G of finite index in G such that $[G : L]$ is relatively prime to $n = n(G)$.*

Proof. First we note that $G/F_G \simeq H/F_H$. In fact there is an isomorphism $\alpha_1\colon H/F_H \to G/F_G$ which induces an isomorphism $\alpha_0\colon H/(F_H T_H) \to G/(F_G T_G)$. Since H/T_H is a free $\mathbb{Z}$-module and G/T_G is a $\mathbb{Z}$-module, there exists a homomorphism $\alpha_2\colon H/T_H \to G/T_G$ such that the following square, in which the two vertical arrows are the canonical epimorphisms, is commutative.

$$\begin{array}{ccc} H/T_H & \xrightarrow{\alpha_2} & G/T_G \\ \downarrow & & \downarrow \\ H/(F_H T_H) & \xrightarrow[\alpha_0]{} & G/(F_G T_G) \end{array}$$

Note that H/T_H and G/T_G are free abelian groups of the same finite rank as the rank of the $\mathbb{Z}_n$-module $G/(F_G T_G)$. Therefore it follows by Proposition 3.1 that the cokernel of α_2 is of finite order relatively prime to n.

Now consider the following diagram (3) in which the unbroken arrows form a commutative diagram. The vertical arrows are the obvious epimorphisms.



(3)

Since the front face is a pull-back square, there exists a (unique) homomorphism α_3: $H \to G$ which is such that diagram (3) is commutative. It readily follows that α_3 is a monomorphism and that $[G : \mathrm{Im}\ \alpha_3] = [G/T_G : \mathrm{Im}\ \alpha_2]$. ∎

THEOREM 4.3. *Let G be any $\mathscr{X}_0$-group and let $n = n(G)$. Let H and L be subgroups of G of finite index. If $[G : H]$ is relatively prime to n and $[G : L] \equiv \pm[G : H] \bmod n$, then $H \simeq L$.*

Proof. The inclusions of H and L into G induce maps fitting into the following commutative diagram.

$$\begin{array}{ccccc} H/T_H & \xrightarrow{\eta'} & G/T_G & \xleftarrow{\lambda'} & L/T_L \\ f\downarrow & & \downarrow & & \downarrow g \\ H/(F_H T_H) & \xrightarrow[\eta]{} & G/(F_G T_G) & \xleftarrow[\lambda]{} & L/(F_L T_L) \end{array}$$

Let k be the rank of the free abelian group H/T_H (which is the same as the rank of L/T_L), and fix any isomorphisms $f_0\colon \mathbb{Z}^k \to H/T_H$ and $g_0\colon \mathbb{Z}^k \to L/T_L$.

By comparing the indices of the subgroups $\operatorname{Im}\eta'$ and $\operatorname{Im}\lambda'$ of G/T_G (these coincide with, respectively, the indices of H and L in G), it follows from Theorem 3.2 that $\eta \circ f \circ f_0 \sim \lambda \circ g \circ g_0$. Thus there is an isomorphism $\alpha\colon \mathbb{Z}^k \to \mathbb{Z}^k$ such that $\eta \circ f \circ f_0 = \lambda \circ g \circ g_0 \circ \alpha$. Now let $\theta = g_0 \circ \alpha \circ f_0^{-1}$.

Note that η and λ are isomorphisms, being induced by (respectively) isomorphisms $\eta_0\colon H/F_H \to G/F_G$ and $\lambda_0\colon L/F_L \to G/F_G$, which are induced by the inclusions of H and L into G. Thus we have a commutative diagram as follows.

$$\begin{array}{ccccc} H/T_H & \xrightarrow{f} & H/(F_H T_H) & \longleftarrow & H/F_H \\ \theta\big\downarrow & & \big\downarrow \lambda^{-1}\circ\eta & & \big\downarrow \lambda_0^{-1}\circ\eta_0 \\ L/T_L & \xrightarrow[g]{} & L/(F_L T_L) & \longleftarrow & L/F_L \end{array}$$

In the diagram above, every vertical arrow is an isomorphism. Taking pull-backs of the relevant triads in the diagram above yields an isomorphism $H \to L$. ∎

THEOREM 4.4. *Let G be any $\mathscr{X}_0$-group and let $n = n(G)$. Let H be a subgroup of G of finite index. If $[G:H]$ is relatively prime to n, then there exists an embedding $\alpha\colon G \to H$ such that $[G:H]\cdot[H:\operatorname{Im}\alpha] \equiv \pm 1 \bmod n$.*

Proof. Let K be any subgroup of H of finite index relatively prime to n, such that $[H:K][G:H] \equiv 1 \bmod n$. Then $[G:K] = [G:H][H:K] \equiv 1 \bmod n$. Therefore by Theorem 4.3, $K \simeq G$, and the conclusion of our theorem follows. ∎

5. GROUP STRUCTURE ON THE NON-CANCELLATION SET

Let X be the set of all integers which are relatively prime to n. From Theorems 4.1, 4.2, and 4.3 it follows that we have a well-defined surjective function $\mu\colon X \to \chi(G)$, defined by the rule $\mu(x) = [H]$ where H is a subgroup of G of index $|x|$, and $[\cdot]$ denotes isomorphism class. In fact this function is shown to factorize through the "reduction mod n"-function

$$\zeta\colon X \to \mathbb{Z}_n^*/\{1, -1\}.$$

Let $\theta\colon \mathbb{Z}_n^*/\{1, -1\} \to \chi(G)$ be the unique function such that $\zeta \circ \theta = \mu$.

THEOREM 5.1. (a) *The fibre* $\theta^{-1}[G]$ *of* θ *over* $[G]$ *is a subgroup of* $\mathbb{Z}_n^*/\{1, -1\}$.

(b) *For any* $[H] \in \chi(G)$, $\theta^{-1}[H]$ *is a coset of* $\theta^{-1}[G]$.

Proof. (a) From Theorem 4.4 it follows that if $u \in \theta^{-1}[G]$ then $u^{-1} \in \theta^{-1}[G]$. Thus $\theta^{-1}[G]$ is closed with respect to inversion of its elements. Given $s, t \in X$ for which we have embeddings $\sigma: G \to G$ and $\tau: G \to G$ such that $[G : \sigma(G)] = s$ and $[G : \tau(G)] = t$ then $[G : \tau \circ \sigma(G)] = st$. This completes the proof that $\theta^{-1}[G] < \mathbb{Z}_n^*/\{1, -1\}$.

(b) Suppose that $r, s, t \in X$ and $\mu(s) = \mu(t)$. Now let L be a subgroup of G such that $[G : L] = s$. Then there is also an embedding $\alpha: L \to G$ such that $[G : \alpha(L)] = t$. If K is a subgroup of L with $[L : K] = r$, then $[K] = \mu(rs)$. But then $[G : \alpha(K)] = rt$, so that also $[K] = \mu(rt)$. Thus we have shown that $\mu(s) = \mu(t)$ implies $\mu(rs) = \mu(rt)$. The (b) part of the theorem follows from the latter fact. ∎

From Theorem 5.1 it follows that there is a bijection

$$\Theta: \left(\mathbb{Z}_n^*/\{1, -1\}\right)/\left(\theta^{-1}[G]\right) \to \chi(G),$$

such that for the canonical epimorphism of semigroups

$$\eta: \mathbb{Z}_n^*/\{1, -1\} \to \left(\mathbb{Z}_n^*/\{1, -1\}\right)/\left(\theta^{-1}[G]\right),$$

we have $\theta = \Theta \circ \eta$. We use Θ to equip $\chi(G)$ with a group structure.

THEOREM 5.2. *Let* G *be any* $\mathscr{X}_0$*-group and let* $n = n(G)$. *Then the function* $\theta: \mathbb{Z}_n^*/\{1, -1\} \to \chi(G)$ *induces a group structure on* $\chi(G)$, *which coincides with the Hilton–Mislin genus group if G is nilpotent.*

Computations of such non-cancellation groups and homotopical applications will appear elsewhere.

6. EPIMORPHISMS BETWEEN NON-CANCELLATION GROUPS

As in [11], the following proposition is quite useful. The elementary proof is omitted.

PROPOSITION 6.1. *Suppose that we have groups* A, B, *and* C *together with a homomorphism* $\beta: A \to C$ *and a surjective group homomorphism* $\gamma: A \to B$. *If* $\alpha: B \to C$ *is a function (between sets) such that* $\alpha \circ \gamma = \beta$, *then* α *is a homomorphism. If, moreover,* β *is surjective, then* α *is also surjective.*

For $\mathscr{X}_0$-groups G and H and for groups K belonging to $\chi(G)$, the rule $K \mapsto K \times H$ induces a well-defined function

$$\phi\colon \chi(G) \to \chi(G \times H).$$

Similar to [11], we have the next result.

THEOREM 6.2 (cf. [11, Theorem 3]). *Let G and H be any $\mathscr{X}_0$-groups, and suppose that G is infinite. Then the function $\phi\colon \chi(G) \to \chi(G \times H)$ is a surjective homomorphism of groups.*

Proof. Let $m = n(G \times H)$. Then m is a multiple of $q = n(G)$. Thus there are epimorphisms $\theta_2\colon \mathbb{Z}_m^* \to \chi(G \times H)$ and $\theta_1\colon \mathbb{Z}_m^* \to \chi(G)$—the latter homomorphism factorizes through the obvious epimorphism $\mathbb{Z}_m^* \to \mathbb{Z}_q^*$. Let x be any positive integer which is relatively prime to m, and let $\bar{x}$ be its residue class modulo m. Choose a subgroup K_x of G such that $[G : K_x] = x$. Such a K_x does exist by Theorem 2.5 since G is infinite. Then $\theta_1(\bar{x}) = [K_x]$. Since $[G \times H : K_x \times H] = [G : K_x] = x$, it follows that $\theta_2(\bar{x}) = [K_x \times H]$. Thus $\phi \circ \theta_1 = \theta_2$, and the theorem follows by Proposition 6.1. ∎

COROLLARY 6.3 (cf. [11, Corollary 4]). *Let N and M be any $\mathscr{X}_0$-groups. If N is infinite and $\chi(N)$ is trivial, then $\chi(N \times M)$ is trivial.*

Now let us consider a $\mathscr{X}_0$-group G together with a finite characteristic subgroup F of G. We now construct a certain function $\eta\colon \chi(G) \to \chi(G/F)$.

It is not hard to see that if K is any group such that $K \times \mathbb{Z} \simeq G \times \mathbb{Z}$, then F is a characteristic subgroup of $K \times \mathbb{Z}$. More precisely, such a group K has a (unique) subgroup F' such that for every isomorphism $h\colon G \times \mathbb{Z} \to K \times \mathbb{Z}$, we have $\alpha(F) = F'$. Thus (by a five-lemma argument) it follows that $K/(F') \times \mathbb{Z} \simeq G/F \times \mathbb{Z}$. We show that if for a group K in $\chi(G)$ we let $K \mapsto K/F'$, then we obtain a function $\eta\colon \chi(G) \to \chi(G/F)$. We show now that η is well defined.

Suppose that we have groups $K_1, K_2 \in \chi(G)$ and an isomorphism $h\colon K_1 \simeq K_2$. Then h induces an isomorphism $F_1 \to F_2$, where F_1 and F_2 are the subgroups of K_1 and K_2, respectively, corresponding to F. Therefore, h induces an isomorphism $K_1/F_1 \to K_2/F_2$. This proves that η is a well-defined function. So indeed we have a function $\eta\colon \chi(G) \to \chi(G/F)$.

The next theorem is modelled on a result of Hilton [1, Theorem 2.1].

THEOREM 6.4. *Let F be a finite characteristic subgroup of the infinite $\mathscr{X}_0$-group G. Then the function $\eta\colon \chi(G) \to \chi(G/F)$ is a surjective group homomorphism.*

Proof. Let m be the lowest common multiple of $n(G)$ and $n(G/F)$. Then there are epimorphisms $\theta_1\colon \mathbb{Z}_m^* \to \chi(G)$ and $\theta_2\colon \mathbb{Z}_m^* \to \chi(G/F)$. Let x be any positive integer which is relatively prime to m, and let $\bar{x}$ be its residue class modulo m. Let K_x be any subgroup of G such that $[G : K_x] = x$. Such a K_x does exist by Theorem 2.5. Then $\theta_1(\bar{x}) = [K_x]$. Since $[G/F : K_x/F] = [G : K_x] = x$, we have $\theta_2(\bar{x}) = [K_x/F]$. Thus $\phi \circ \theta_1 = \theta_2$, and the theorem follows by Proposition 6.1. ∎

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