TANNAKA DUALITY FOR COMONOIDS IN COSMOI DRAFT VERSION

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ABSTRACT. A classical result of Tannaka duality is the fact that a coalgebra over some field can be reconstructed from its category of finite dimensional representations, by using the forgetful functor which sends a representation to its underlying vector space. There is also a corresponding recognition result, which characterizes those categories equipped with a functor to finite dimensional vector spaces which are equivalent to the category of finite dimensional representations of a coalgebra.

In this paper we study a generalization of these questions to an arbitrary *cosmos*, that is, a complete and cocomplete symmetric monoidal closed category. Instead of representations on finite dimensional vector spaces we look at representations on objects of the cosmos which have a dual. We give a necessary and sufficient condition which ensures that a comonoid can be reconstructed from its representations, and we succeed in characterising categories of representations of certain comonoids.

1. Introduction

1.1. Tannaka duality concerns the study of the relationship between a group-like object G (ordinary group, compact topological group, algebraic group, group scheme, quantum group, etc.) and its category of representations $\mathrm{Rep}(G)$. This category is naturally equipped with additional structures, the most basic being the forgetful functor which sends a representation of G to its underlying space. That is, for every group-like object G there is a triple $(\mathrm{Rep}(G), V, S)$ where V denotes the forgetful functor and S stands for unspecified additional structures (e.g. a monoidal structure). The question arises whether or not it is possible to go in the other direction: Is there a way to associate a group-like object to a category equipped with suitable additional structures? In other words, is there a way to assign a group-like object $L(\mathscr{A}, \omega, S)$ to a category $\mathscr{A}$ equipped with structures S and a functor ω from $\mathscr{A}$ to the category of spaces? If such a construction exists, there are three natural questions one would like to answer.

(1) The reconstruction problem: If one starts with a group-like object G and then applies $L(-)$ to the associated category of representations, is the resulting group-like object isomorphic to G?

(2) The recognition problem: Is it possible to give a characterization of those triples $(\mathscr{A}, \omega, S)$ which are equivalent to $(\mathrm{Rep}(G), V, S)$ for some group-like object G?

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(3) The description problem: Given a category $\mathscr{A}$ with structures S, is it possible to find conditions for the existence of a functor ω from $\mathscr{A}$ into the category of spaces such that $(\mathscr{A}, \omega, S)$ is equivalent to $(\mathrm{Rep}(G), V, S)$ for some group-like object G?

In the expository book on Quantum Groups [Str07], R. Street proved that in many contexts, 'association in a systematic way' does have a rigorous meaning: Usually $\mathrm{Rep}(-)$ is a contravariant functor from a category of group-like objects to a slice category of **cat**, and this functor often has a left adjoint (see [Str07], Proposition 16.4). Whenever we have an adjunction $L(-) \dashv \mathrm{Rep}(-)$, the reconstruction problem and the recognition problem have a precise mathematical meaning:

(1') Under what conditions is the counit $L(\mathrm{Rep}(G), V, S) \to G$ an isomorphism?

(2') When is the unit $(\mathscr{A}, \omega, S) \to \mathrm{Rep}\big(L(\mathscr{A}, \omega, S)\big)$ an equivalence of categories?

The description problem does not fit so easily into this scheme, and we mention it only for the sake of completeness. We do not attempt to solve it in this paper.

1.2. Group objects in a category of reasonable spaces give rise to *Hopf algebras*: If there is a notion of an algebra of functions on our spaces, and if this is suitably functorial, then the group multiplication induces a comultiplication on the algebra of functions, and the unit induces a counit. For example, if our category of spaces is the category of affine schemes over some field k, then its opposite category is equivalent to the category of algebras over k. In this case, we do in fact have an equivalence between the category of group objects (i.e., affine group schemes over k) and the category of commutative Hopf algebras over k. Representations of an affine group scheme are the same as comodules over the corresponding Hopf algebra H. An *H-comodule* is a vector space M together with a coaction $M \to H \otimes M$, which is compatible with the comultiplication and the counit of H. One has the following correspondence between structures on the vector space H and structures on the category of finite dimensional representations, where the structure on the left is required for the existence of the structure on the right:

comultiplication and counit	$\rightsquigarrow$	necessary for the definition of comodules
multiplication and unit	$\rightsquigarrow$	tensor product of comodules
antipode	$\rightsquigarrow$	duals

In the classical Tannakian duality developed by P. Deligne and N. Saavedra Rivano (see [Saa72], [Del90]; an exposition can be found in [JS90]) one proceeds in two steps. In the first step, one only considers the minimal structure necessary to define the category of comodules: One studies the reconstruction and recognition problem for coalgebras and their categories of comodules. A *coalgebra* is a vector space C, endowed with a coassociative comultiplication $C \to C \otimes C$ and a counit $C \to k$. For each coalgebra C there is a k-linear category $\mathrm{Rep}(C, k)$ of finite dimensional comodules, together with a forgetful functor $V\colon \mathrm{Rep}(C, k) \to \mathrm{Vect}_f$, where Vect_f denotes the category of finite dimensional vector spaces. Conversely, given a k-linear category $\mathscr{A}$ and a k-linear functor $\omega\colon \mathscr{A} \to \mathrm{Vect}_f$,

there is an associated coalgebra $L(\omega)$. In this context, both the reconstruction problem and the recognition problem were solved in [Del90]: Proposition 4.12 shows that coalgebras can be reconstructed from their categories of representations, and Proposition 6.2 gives a characterization of those pairs $(\mathscr{A}, \omega)$ which are equivalent to a category of comodules.
The second step establishes a relationship between additional structures on a coalgebra and additional structures on its category of comodules. If C is a bialgebra, then $\mathrm{Rep}(C, k)$ can be endowed with a symmetric monoidal structure such that the forgetful functor becomes a strong monoidal functor, and in the case of a Hopf algebra, this category is autonomous, i.e., every object has a dual. Conversely, if $\mathscr{A}$ is a symmetric monoidal category and $\omega \colon \mathscr{A} \to \mathrm{Vect}_f$ is strong monoidal functor, then $L(\omega)$ has the structure of a bialgebra (see [Del90], Proposition 6.4). If $\mathscr{A}$ is autonomous, then $L(\omega)$ is a Hopf algebra (see [JS90], Proposition 5).

1.3. It turns out that the above mentioned correspondence is still valid when we replace the category of vector spaces with any complete and cocomplete symmetric monoidal closed category $\mathscr{V}$. Following Bénabou and Kelly, we call such a category a *cosmos.* Instead of Hopf algebras we consider Hopf monoids in $\mathscr{V}$. We also have to replace 'finite dimensional vector space' by 'object with a dual': A comodule of a Hopf monoid has a dual if and only if its underlying object has a dual (see [Str07], Proposition 15.1). We will eventually be interested in the reconstruction of Hopf monoids instead of mere comonoids, and the existence of duals in the category of representations is crucial for the reconstruction of the antipode map of the Hopf monoid (see [Str07], Section 16). In order to apply reconstruction results for Hopf monoids similar to those found in [Str07], we should therefore ask the following question: Is it possible to reconstruct a comonoid C from the category of those C-comodules for which the underlying objects have duals?
For certain classes of cosmoi this question has already been studied: T. Wedhorn studied the reconstruction problem for Hopf algebras over Dedekind rings, and the recognition problem for valuation rings (see [Wed04]). B. Day solved both problems for finitely presentable cosmoi for which the full subcategory of objects with duals is closed under finite limits and colimits (see [Day96]). P. McCrudden used a result of B. Pareigis to solve the reconstruction problem for *Maschkean* categories, which are certain abelian monoidal categories in which all monomorphisms split (see [Par96], [McC02]). All these approaches make the assumption that the category of objects with duals is closed under finite limits, hence they cannot be applied to the case where $\mathscr{V}$ is the category Mod_R of R-modules for a general commutative rings R: An R-module has a dual if and only if it is finitely generated and projective. In the present paper we study the reconstruction and the recognition problem without assuming that the category of objects with duals is closed under finite limits. We succeed in giving a necessary and sufficient condition for solving the reconstruction problem (see Theorem 4.4 and Theorem 6.4), and we provide a partial solution of the recognition problem (see Theorem 5.8 and Theorem 6.5). We now turn to a discussion of these results.

1.4. We fix a cosmos $\mathscr{V}$, and we let $\mathscr{V}^c$ be the full subcategory of objects which have a dual. For a comonoid C, we let $\mathscr{V}^c_C$ be the category of C-comodules whose underlying modules have duals. Such comodules are called *Cauchy comodules* (cf. [Str07], Proposition 10.6). It is well known that in the case where $\mathscr{V}$ is the cosmos of modules over some commutative ring, this category is an R-linear category. For general cosmoi, it is still true that $\mathscr{V}^c_C$ is enriched over $\mathscr{V}$. We will use this additional structure on the category of comodules for our constructions, and we assume that the reader is familiar with the basics of enriched category theory. We use the terminology from [Kel05].
The *comodule functor* $\mathbf{Comon}(\mathscr{V}) \to \mathscr{V}\text{-}\mathbf{cat}/\mathscr{V}^c$ sends a comonoid C to the $\mathscr{V}$-category $\mathscr{V}^c_C$, equipped with the forgetful functor $\mathscr{V}^c_C \to \mathscr{V}^c$. A comonoid C gives rise to a $\mathscr{V}$-comonad on $\mathscr{C}$, given by $C \otimes - \colon \mathscr{V} \to \mathscr{V}$. The category of coalgebras of that monad is precisely the category of comodules of the comonoid $\mathscr{C}$. Moreover, any cocontinuous comonad on $\mathscr{V}$ is of this form: The category of comonoids in $\mathscr{V}$ is equivalent to the category of cocontinuous comonads on $\mathscr{V}$. Using this observation and the semantics-structure duality for $\mathscr{V}$-comonads, we give a new construction of a left adjoint of the comodule functor in Section 3. The resulting adjunction

$$\mathbf{Comon}(\mathscr{V}) \underset{L(-)}{\overset{\mathscr{V}^c_{(-)}}{\rightleftarrows}} \mathscr{V}\text{-}\mathbf{cat}/\mathscr{V}^c \quad (\top)$$

is called the *Tannakian adjunction*.
Using this new construction, we study the counit and the unit of the Tannakian adjunction in Section 4 and Section 5 respectively. The statements of our theorems become less technical if we make some additional assumption about the cosmos $\mathscr{V}$. Recall that a small subcategory $\mathscr{X}$ of an ordinary category $\mathscr{C}$ is called $\mathbf{Set}$-dense if every object C is the colimit of the forgetful functor $(\mathscr{X} \downarrow C) \to \mathscr{C}$, where $(\mathscr{X} \downarrow C)$ denotes the category of morphisms $X \to C$ whose domain lies in $\mathscr{X}$.
The classical reconstruction theorems all rely on the fact that the comonoid in question is the union of subcomonoids whose underlying object has a dual (see e.g. [McC02], [JS90], [Wed04]). A union is of course just a special case of a colimit. In Theorem 6.4 we prove that we get a necessary and sufficient condition for the solution of the reconstruction problem when we consider colimits of diagrams consisting of arbitrary morphisms instead of monomorphisms:

1.5. Theorem. *Assume there is a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is $\mathbf{Set}$-dense in $\mathscr{V}_0$, and let C be a comonoid in $\mathscr{V}$. Then the C-component of the counit of the Tannakian adjunction is an isomorphism if and only if C, considered as a comodule over itself, is the conical colimit of the diagram of Cauchy comodules over C.*

Note that for any commutative ring R, the category of finitely generated free modules is $\mathbf{Set}$-dense in Mod_R, so the above result can be applied to the cosmos Mod_R. Our results concerning the recognition problem require the additional assumption that the category $\mathscr{V}_0$ is locally presentable. In order to state the result we need the *forgetful functor* $V = \mathscr{V}_0(I, -): \mathscr{V}_0 \to \mathbf{Set}$, which sends an object of $\mathscr{V}$ to its 'underlying set'. Moreover, we need the *category of elements* of a functor $F: \mathscr{C} \to \mathbf{Set}$ between ordinary categories. This category is denoted by $\mathrm{el}(F)$, and its objects are pairs (C, x), where $C \in \mathscr{C}$ and $x \in FC$. The morphisms $(C, x) \to (C', x')$ are given by the morphisms $f: C \to C'$ with $Ff(x) = x'$.

1.6. Theorem. *Let $\mathscr{V}$ be a locally finitely presentable cosmos with a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is* **Set***-dense in $\mathscr{V}_0$. Let $\mathscr{A}$ be a $\mathscr{V}$-category which has copowers*[1] *with objects in $\mathscr{X}$ and with their duals, and let $\omega: \mathscr{A} \to \mathscr{V}$ be a $\mathscr{V}$-functor such that $\omega(A) \in \mathscr{V}^c$ for all $A \in \mathscr{A}$. The $(\mathscr{A}, \omega)$-component of the unit of the Tannakian adjunction is an equivalence if*

i) The functor ω reflects isomorphisms,

ii) The category $\mathrm{el}(V\omega_0)$ of elements of $V\omega_0: \mathscr{A}_0 \to \mathbf{Set}$ is cofiltered, and

iii) The functor ω detects and preserves those conical colimits which lie in $\mathscr{V}^c$.

In fact, we prove a more general result in Theorem 6.5. In Section 2 we will see that conditions i) and iii) are necessary conditions: If the $(\mathscr{A}, \omega)$-component of the unit of the Tannakian adjunction is an equivalence of categories, then ω automatically satisfies i) and iii). Condition ii) on the other hand is rather strong. It implies for example that the comonoid C associated to $(\mathscr{A}, \omega)$ is *flat*, i.e., that $C \otimes -: \mathscr{V} \to \mathscr{V}$ preserves finite limits (in the sense of [Kel82]). It would be interesting to know if the conditions i), ii) and iii) are necessary in the case of flat comonoids. More precisely: Is it true that the category of elements $\mathrm{el}(V\omega_0)$ is cofiltered if $\omega: \mathscr{V}^c_C \to \mathscr{V}$ is the forgetful functor for a flat comonoid C? If we do not make the assumption that $\mathscr{V}$ has a **Set**-dense monoidal subcategory $\mathscr{X}$ as in Theorem 1.6, there are obvious counterexamples (see Section 5.9).
In the classical case where $\mathscr{V}$ is the category of vector spaces over some field k it is important to consider comonoids in categories other than $\mathscr{V}$ itself, for example in the category of B-B-bimodules for some k-algebra B (see [Del90], Section 6). If C is such a comonoid, i.e., a B-B-module C with comultiplication $C \to C \otimes_B C$, a C-comodule is a right B-module M, together with a coaction $M \to C \otimes_B M$. The comodules of interest are again the ones for which M is finitely generated and projective (cf. op. cit., Proposition 6.2). We can think of B as a $\mathscr{V}$-category $\overline{\mathscr{B}}$ with one object. The category of right B-modules is then just the category of $\mathscr{V}$-functors $\mathscr{B}^{\mathrm{op}} \to \mathscr{V}$, and a right module is finitely generated and projective if and only if the corresponding $\mathscr{V}$-functor $\mathscr{B}^{\mathrm{op}} \to \mathscr{V}$ lies in the Cauchy completion $\overline{\mathscr{B}}$ of $\mathscr{B}$. All these notions generalize naturally to arbitrary cosmoi $\mathscr{V}$. However, there are lots of examples of cosmoi where Cauchy-completions are rather 'small': If $\mathscr{V}$ is cartesian closed, and if $\mathscr{B} = \mathscr{I}$ is the trivial $\mathscr{V}$-category, then $\mathscr{V}^c \simeq \overline{\mathscr{I}}$ consists of just the terminal object $I = *$. Because of this we work in slightly greater generality: It turns out that all our arguments work for *any* small $\mathscr{V}$-category $\mathscr{B}$.

[1]Copowers are also called *tensors*, cf. Section 2.12

We give the necessary definitions in Section 2.1.
As already mentioned above, we will use the language of enriched categories throughout. The standard reference is Kelly's book 'Basic concepts of enriched category theory' [Kel05], and for the most part we use the terminology and notation introduced there. The major difference in notation is that we write $\mathcal{P}\mathscr{A}$ for the category of $\mathscr{V}$-functors $\mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, which was denoted by $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ in [Kel05]. In Section 6, we also use some concepts from the theory of locally finitely presentable enriched categories, which was developed in [Kel82]. Another important tool are pasted composites of natural transformations and the theory of mates under adjunction. Both can be found in [KS72].

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2. The comodule functor

2.1. The goal of this section is to give a construction of the right adjoint of the Tannakian adjunction, which sends comonoids to categories of representations. For the entire section, we fix a cosmos $\mathscr{V}$. For $\mathscr{V}$-categories $\mathscr{A}$, $\mathscr{B}$, a *module* M from $\mathscr{A}$ to $\mathscr{B}$ (also called *profunctor* or *distributor*) consists of an object $M(b,a) \in \mathscr{V}$ for each $b \in \mathscr{B}$, $a \in \mathscr{A}$, together with maps

$$M(b,a) \otimes \mathscr{B}(b',b) \to M(b',a) \quad \text{and} \quad \mathscr{A}(a,a') \otimes M(b,a) \to M(b,a')$$

subject to certain natural coherence conditions. There are many equivalent ways to define the category of modules from $\mathscr{A}$ to $\mathscr{B}$: Since $\mathscr{V}$ is closed, a module is the same as a functor $\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}$. By adjunction, this is the same as a functor $\mathscr{A} \to \mathcal{P}\mathscr{B}$. Since $\mathcal{P}\mathscr{B}$ is the free cocompletion of $\mathscr{B}$ (cf. Theorem 3.2), the category of modules is also equivalent to the category $\mathbf{Cocts}[\mathcal{P}\mathscr{A}, \mathcal{P}\mathscr{B}]$ of cocontinuous functors $\mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$. We denote the category of modules from $\mathscr{A}$ to $\mathscr{B}$ by ${}_{\mathscr{A}}\mathcal{M}_{\mathscr{B}}$. Composition of functors in $\mathbf{Cocts}[\mathcal{P}\mathscr{B}, \mathcal{P}\mathscr{B}]$ induces a monoidal structure on ${}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}} \simeq \mathbf{Cocts}[\mathcal{P}\mathscr{B}, \mathcal{P}\mathscr{B}]$. The *comodule functor* (see Definition 2.8) is a functor from the category $\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$ of comonoids in ${}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}$ to the category $\mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}$ of small $\mathscr{V}$-categories over the Cauchy completion $\overline{\mathscr{B}}$ of $\mathscr{B}$. An important special case is the *neutral* case, where $\mathscr{B} = \mathscr{I}$, and the comodule functor sends comonoids in $\mathscr{V} \simeq {}_{\mathscr{I}}\mathcal{M}_{\mathscr{I}}$ to categories over $\mathscr{V}^c$, the full subcategory of $\mathscr{V}$ consisting of objects with duals. In this case the above mentioned equivalence $\mathscr{V} \simeq {}_{\mathscr{I}}\mathcal{M}_{\mathscr{I}} \simeq \mathbf{Cocts}[\mathscr{V}, \mathscr{V}]$ sends an object $M \in \mathscr{V}$ to the cocontinuous functor $M \otimes - \colon \mathscr{V} \to \mathscr{V}$.

From now on, we usually work with **Cocts**$[\mathcal{P}\mathscr{B},\mathcal{P}\mathscr{B}]$ instead of ${}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}$, because the former is a strict monoidal category. Using the equivalence from the previous paragraph one can easily transfer results about one of these categories to the other; we will not do this explicitely. Recognizing that the category of comonoids we are interested in is in fact a full subcategory of the category of comonads also sheds some light on the relationship between the Tannakian adjunction and the semantics-structure duality. We first recall some basic definitions and results from enriched category theory.

2.2. A *comonad* on a $\mathscr{V}$-category $\mathscr{C}$ is a comonoid in the category of endo-$\mathscr{V}$-functors of $\mathscr{C}$ and $\mathscr{V}$-natural transformations. In other words, a comonad consists of a $\mathscr{V}$-functor $T\colon\mathscr{C}\to\mathscr{C}$, a *comultiplication* $\delta\colon T\Rightarrow TT$ and a *counit* $\varepsilon\colon T\Rightarrow \mathrm{id}$ such that the diagrams

$$\begin{array}{ccc} T & \overset{\delta}{\Longrightarrow} & TT \\ {\scriptstyle\delta}\Downarrow & & \Downarrow{\scriptstyle\delta T} \\ TT & \underset{T\delta}{\Longrightarrow} & TTT \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & T & & \\ & /\!/ & \Downarrow{\scriptstyle\delta} & \backslash\!\backslash & \\ T & \underset{\varepsilon T}{\Longleftarrow} & TT & \underset{T\varepsilon}{\Longrightarrow} & T \end{array}$$

are commutative. A morphism of comonads is a natural transformation which is compatible with the counit and the comultiplication. It is a well-known fact that for any adjunction $\eta,\varepsilon\colon F\colon\mathscr{E}\rightleftarrows\mathscr{C}\colon G$, the composite FG is a comonad on $\mathscr{C}$, with comultiplication $F\eta G\colon FG\Rightarrow FGFG$ and counit $\varepsilon\colon FG\Rightarrow\mathrm{id}$. This comonad is called the *density comonad* of F (cf. [Dub70], Chapter II). Thus, for every adjunction, we have an associated comonad. In the next section we will see that the converse is also true.

2.3. For any comonad T on $\mathscr{C}$, there is an associated adjunction $\eta^T,\varepsilon^T\colon V_T\colon\mathscr{C}_T\rightleftarrows\mathscr{C}\colon W_T$, where $\mathscr{C}_T$ is the $\mathscr{V}$-category of *comodules*[2] of T, i.e., of objects M of $\mathscr{C}$ equipped with a morphism $\varrho\colon M\to TM$ such that the diagrams

$$\begin{array}{ccc} M & \overset{\varrho}{\longrightarrow} & TM \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle\delta_M} \\ TM & \underset{T\varrho}{\longrightarrow} & TTM \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \overset{\varrho}{\longrightarrow} & TM \\ & \backslash\!\backslash & \downarrow{\scriptstyle\varepsilon_M} \\ & & M \end{array}$$

are commutative. The hom-object $\mathscr{C}_T(\mathbf{M},\mathbf{M}')$ between the comodules $\mathbf{M}=(M,\varrho)$ and $\mathbf{M}'=(M',\varrho')$ is given by the equalizer

$$\mathscr{C}_T(\mathbf{M},\mathbf{M}') \xrightarrow{\;V_T\;} \mathscr{C}(M,M') \underset{\mathscr{C}(M,\varrho')}{\overset{\mathscr{C}(\varrho,TM')\circ T}{\rightrightarrows}} \mathscr{C}(M,TM')$$

in $\mathscr{V}$, and the structure map of this equalizer gives the functor $V_T\colon\mathscr{C}_T\to\mathscr{C}$. The right adjoint $W_T\colon\mathscr{C}\to\mathscr{C}_T$ sends an object M of $\mathscr{C}$ to (TM,δ_M), called the *cofree comodule* on

[2] Comodules are usually called coalgebras, but in our context this terminology would be misleading: A comonoid gives rise to a comonad, and if $\mathscr{V}$ is the category of modules for some commutative ring, comonoids themselves are called coalgebras.

M, and $(W_T)_{M,M'}\colon \mathscr{C}(M,M') \to \mathscr{C}_T\big((TM,\delta_M),(TM',\delta_{M'})\big)$ is the induced map

$$\begin{array}{ccccc} \mathscr{C}_T\big((TM,\delta_M),(TM',\delta_{M'})\big) & \xrightarrow{V_T} & \mathscr{C}(TM,TM') & \overset{\mathscr{C}(\delta_M,TTM')\circ T}{\underset{\mathscr{C}(TM,\delta_{M'})}{\rightrightarrows}} & \mathscr{C}(TM,TTM') \\ \uparrow{\scriptstyle W_T} & \nearrow{\scriptstyle T} & & & \\ \mathscr{C}(M,M') & & & & \end{array}$$

into the corresponding equalizer. The unit η^T and counit ε^T of this adjunction are given by $(M,\varrho)\xrightarrow{\varrho}(TM,\delta_M) = W_TV_T(M,\varrho)$ and $V_TW_T(M) = TM\xrightarrow{\varepsilon}M$ respectively. It follows that the density comonad of the adjunction $\eta^T,\varepsilon^T\colon V_T\colon\mathscr{C}_T \rightleftarrows \mathscr{C}\colon W_T$ is the comonad T itself.

2.4. If $\mathscr{C}$ is the category $\mathcal{P}\mathscr{B}$ of presheaves on $\mathscr{B}$ and $T\colon\mathcal{P}\mathscr{B}\to\mathcal{P}\mathscr{B}$ is a comonad, then we say that comodule (M,ϱ) is a *Cauchy comodule* if its underlying object M lies in the Cauchy completion of $\mathscr{B}$, and we denote the category of Cauchy comodules of $T\colon\mathcal{P}\mathscr{B}\to\mathcal{P}\mathscr{B}$ by $\mathcal{P}\mathscr{B}^{\mathrm{c}}_T$. This category constitutes the object part of the right adjoint of the Tannakian adjunction. In order to show right adjointness we need a nice description of functors with codomain $\mathcal{P}\mathscr{B}^{\mathrm{c}}_T$ or $\mathscr{C}_T$, which requires the introduction of the following concept.

2.5. A *coaction* of a comonad $T\colon\mathscr{C}\to\mathscr{C}$ on a functor $S\colon\mathscr{A}\to\mathscr{C}$ is a natural transformation $\varrho\colon S\Rightarrow TS$ such that

$$\begin{array}{ccc} S & \overset{\varrho}{\Longrightarrow} & TS \\ {\scriptstyle\varrho}\Downarrow & & \Downarrow{\scriptstyle\delta S} \\ TS & \underset{T\varrho}{\Longrightarrow} & TTS \end{array} \qquad\text{and}\qquad \begin{array}{ccc} S & \overset{\varrho}{\Longrightarrow} & TS \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle\varepsilon S} \\ & & S \end{array}$$

are commutative. Note that if we identify functors $M\colon\mathscr{I}\to\mathscr{C}$ from the trivial $\mathscr{V}$-category to $\mathscr{C}$ with objects in $\mathscr{C}$, then a coaction on M is precisely a morphism $\varrho\colon M\to TM$ which turns M into a comodule.

2.6. Proposition. *Let $\varrho\colon S\Rightarrow TS$ be a coaction. Then the assignment $\overline{S}(A) = (SA,\varrho_A)$ extends uniquely to a $\mathscr{V}$-functor $\overline{S}\colon\mathscr{A}\to\mathscr{C}_T$ such that*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\overline{S}} & \mathscr{C}_T \\ & {\scriptstyle S}\searrow \quad \swarrow{\scriptstyle V_T} & \\ & \mathscr{C} & \end{array}$$

is commutative. This gives a bijection, natural in T, between coactions on S and functors $\overline{S}$ which make the above triangle commutative. If $S'\colon\mathscr{A}\to\mathscr{C}_T$ is another $\mathscr{V}$-functor, together with a T-coaction $\varrho'\colon S'\to TS'$, then whiskering with $V_T\colon\mathscr{C}_T\to\mathscr{C}$ gives a bijection *between $\mathscr{V}$-natural transformations $\overline{\varphi}: \overline{S} \Rightarrow \overline{S'}$ and $\mathscr{V}$-natural transformations $\varphi: S \Rightarrow S'$ which make the diagram*

$$\begin{array}{ccc} S & \overset{\varphi}{\Longrightarrow} & S' \\ {\scriptstyle\varrho}\Downarrow & & \Downarrow{\scriptstyle\varrho'} \\ TS & \underset{T\varphi}{\Longrightarrow} & TS' \end{array}$$

commutative.

Proof. The first statement is dual to [Dub70], Proposition II.1.1, and the statement about $\mathscr{V}$-natural transformations between lifts follows from the fact that for any two T-comodules $\mathbf{M}$ and $\mathbf{M}'$, the component $(V_T)_{\mathbf{M},\mathbf{M}'}: \mathscr{C}_T(\mathbf{M}, \mathbf{M}') \to \mathscr{C}(V_T\mathbf{M}, V_T\mathbf{M}')$ of V_T is by definition an equalizer (cf. [Dub70], p. 62). ∎

2.7. The above proposition enables us to show that $\mathscr{C}_T$ and $\mathcal{P}\mathscr{B}^c_T$ are functorial in the comonad T: A morphism of comonads $\varphi: T \Rightarrow T'$ induces a T'-coaction on V_T, with (M, ϱ)-component given by

$$V_T(M, \varrho) = M \xrightarrow{\varrho} TM \xrightarrow{\varphi_M} T'M = T'V_T(M, \varrho),$$

and by Proposition 2.6 this corresponds to a functor $\mathscr{C}_\varphi: \mathscr{C}_T \to \mathscr{C}_{T'}$ making the diagram

$$\begin{array}{ccc} \mathscr{C}_T & \xrightarrow{\mathscr{C}_\varphi} & \mathscr{C}_{T'} \\ & {\scriptstyle V_T}\searrow \quad \swarrow{\scriptstyle V_{T'}} & \\ & \mathscr{C} & \end{array}$$

commutative. Functoriality is an immediate consequence of the uniqueness condition in Proposition 2.6, and in the special case where T is a comonad on $\mathscr{C} = \mathcal{P}\mathscr{B}$, commutativity of the above diagram implies that $\mathcal{P}\mathscr{B}_\varphi$ sends Cauchy comodules to Cauchy comodules.

2.8. Definition. *Let $\mathscr{B}$ be a $\mathscr{V}$-category whose Cauchy completion $\overline{\mathscr{B}}$ is small. The* comodule functor *is the functor*

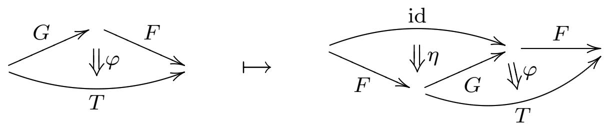
$$\mathcal{P}\mathscr{B}^c_{(-)}: \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}) \longrightarrow \mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}$$

from the category of cocontinuous comonads on $\mathcal{P}\mathscr{B}$ to the category of small $\mathscr{V}$-categories over the Cauchy completion of $\mathscr{B}$, which sends a cocontinuous comonad T to the pair $(\mathcal{P}\mathscr{B}^c_T, V_T)$ consisting of its category of Cauchy comodules together with the forgetful functor $V_T: \mathcal{P}\mathscr{B}^c_T \to \overline{\mathscr{B}}$.

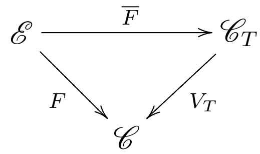
Note that $\mathcal{P}\mathscr{B}^c_T$ really is a small category: For any $M \in \overline{\mathscr{B}}$, there is only a set of morphisms $M \to TM$ which turn M into a T-comodule. This concludes the construction of the right adjoint of the Tannakian adjunction.

2.9. In the remainder of this section we state some important facts about the relationship between comonads and adjunctions which we will need later. We start with the *semantics structure duality* (see Theorem 2.10), which in the classical case $\mathscr{V} = \mathbf{Set}$ is due to Lawvere. Theorem 2.11 is an enriched version of the dual of Beck's famous monadicity theorem. Most of these results are dual versions of results in [Dub70]. Only Proposition 2.13 is a generalization of results in [Dub70] to arbitrary weighted colimits.

2.10. Theorem. [Semantics-structure duality] *Let $\eta, \varepsilon \colon F \colon \mathscr{E} \rightleftarrows \mathscr{C} \colon G$ be an adjunction, and let $T \colon \mathscr{C} \to \mathscr{C}$ be a comonad. The assignment*



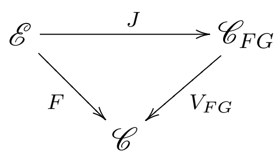
induces a bijection, natural in T, between morphisms of comonads $\varphi \colon FG \Rightarrow T$ and T-coactions $\varrho \colon F \to TF$. Together with Proposition 2.6 this gives a bijection between functors $\overline{F} \colon \mathscr{E} \to \mathscr{C}_T$ which make the diagram



commutative, and morphisms of comonads $\varphi \colon FG \Rightarrow T$. In other words: The assignment which sends an adjunction $F \colon \mathscr{E} \rightleftarrows \mathscr{C} \colon G$ to the comonad FG extends to a left adjoint of the functor which sends a comonad to the adjunction $\eta^T, \varepsilon^T \colon V_T \colon \mathscr{C}_T \rightleftarrows \mathscr{C} \colon W_T$.

Proof. The dual of the first statement follows from [Dub70], Proposition II.1.4 and Proposition II.1.5, and the second assertion follows from the first by Proposition 2.6. ∎

2.11. Theorem. *Let $\eta, \varepsilon \colon F \colon \mathscr{E} \rightleftarrows \mathscr{C} \colon G$ be an adjunction, and let*



be the unit of the semantics-structure adjunction (see Theorem 2.10). Then J is an equivalence if and only if

i) *The functor $F \colon \mathscr{E} \to \mathscr{C}$ reflects isomorphisms.*

ii) *The category $\mathscr{E}$ has equalizers of F-contractible pairs, and F preserves these equalizers.*

If J is an equivalence, then F is called comonadic.

Proof. Note that i) and ii) imply that F reflects equalizers of F-contractible pairs, i.e., a morphism $f\colon A \to B$ in $\mathscr{E}$ is an equalizer of an F-contractible pair $g, h\colon B \to C$ if and only if Ff is an equalizer of Fg and Fh. It follows by the dual of [Dub70], Theorem 2.II.1, that J is an equivalence. Conversely, if J is an equivalence, then i) holds because $V_T\colon \mathscr{C}_T \to \mathscr{C}$ clearly reflects isomorphisms. Condition ii) is again a consequence of [Dub70], Theorem 2.II.1. ∎

2.12. Recall that when we enrich the notion of colimits, we naturally arrive at the concept of a *weighted colimit*[3]: An object K of $\mathscr{V}$-category $\mathscr{C}$ is said to be the colimit of $F\colon \mathscr{D} \to \mathscr{C}$ weighted by $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ if there is a $\mathscr{V}$-natural isomorphism

$$\mathscr{C}(K, M) \xrightarrow{\varphi_M} \mathcal{P}\mathscr{D}\big(J, \mathscr{C}(F-, M)\big)$$

of $\mathscr{V}$-functors. The object K is usually denoted by $J \star F$. We later need the fact that for any small $\mathscr{V}$-category $\mathscr{B}$, the category $\mathcal{P}\mathscr{B}$ of presheaves on $\mathscr{B}$ has all weighted colimits (see [Kel05], Section 3.3). In the special case $\mathscr{D} = \mathscr{I}$, the trivial $\mathscr{V}$-category, giving a weight amounts to giving an object $V \in \mathscr{V}$, and giving a functor $\mathscr{I} \to \mathscr{C}$ is the same as giving an object $C \in \mathscr{C}$. If the colimit of C weighted by V exists, we denote it by $V \odot C$ and we call it the *copower*[4] of V and C. If a $\mathscr{V}$-category has all weighted colimits, then the colimit of $F\colon \mathscr{D} \to \mathscr{C}$ weighted by $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ is given by the coend

$$J \star F = \int^{D \in \mathscr{D}} JD \odot FD$$

(see [Kel05], Section 3.10). The next proposition shows that weighted colimits in the category $\mathscr{C}_T$ of T-comodules are computed the same way as in $\mathscr{C}$.

2.13. Proposition. *Let $F\colon \mathscr{E} \to \mathscr{C}$ be a comonadic functor (see Theorem 2.11). Then F detects and preserves weighted colimits.*

Proof. We use the notation from section 2.2. By definition of comonadicity (see Theorem 2.11) it suffices to check that $V_T\colon \mathscr{C}_T \to \mathscr{C}$ creates colimits for any comonad T on $\mathscr{C}$. Thus let $\mathscr{D}$ be a $\mathscr{V}$-category, $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ a weight, and let $\overline{S}\colon \mathscr{D} \to \mathscr{C}_T$ be a functor such that the colimit $K \in \mathscr{C}$ of $S = V_T\overline{S}\colon \mathscr{D} \to \mathscr{C}$ weighted by J exists. Let

$$\mathscr{C}(K, M) \xrightarrow{\varphi} \mathcal{P}\mathscr{D}\big(J, \mathscr{C}(S-, M)\big)$$

be the $\mathscr{V}$-natural isomorphism which exhibits K as the colimit of S weighted by J (see Section 2.12). By Proposition 2.6, $\overline{S}$ corresponds to some coaction ϱ_0 on S. Let

[3] weighted colimits are called *indexed colimits* in [Kel05]

[4] also known as *tensor* of V and C

$\alpha\colon \mathscr{C}(K,M) \to \mathscr{C}(K,TM)$ be the unique $\mathscr{V}$-natural transformation which makes the diagram

$$\begin{array}{ccc} \mathscr{C}(K,M) & \xrightarrow{\varphi} & \mathcal{P}\mathscr{D}\big(J,\mathscr{C}(S-,M)\big) \\ & & \big\downarrow {\scriptstyle \mathcal{P}\mathscr{D}(J,T)} \\ {\scriptstyle \alpha}\big\downarrow & & \mathcal{P}\mathscr{D}\big(J,\mathscr{C}(TS-,TM)\big) \\ & & \big\downarrow {\scriptstyle \mathcal{P}\mathscr{D}\left(J,\mathscr{C}(\varrho_0,TM)\right)} \\ \mathscr{C}(K,TM) & \xrightarrow[\varphi]{} & \mathcal{P}\mathscr{D}\big(J,\mathscr{C}(S-,TM)\big) \end{array}$$

commutative, and let $\varrho = \alpha(\mathrm{id}_K)\colon K \to TK$. By Yoneda we have $\alpha = \mathscr{C}(\varrho, TM) \circ T$ (both natural transformations send id_K to ϱ), and ϱ is a coaction on K because ϱ_0 is a coaction. Now let $\mathbf{M} = (M, \varrho')$ be any comodule. Then the diagram

$$\begin{array}{ccc} \mathscr{C}(K,M) & \xrightarrow{\varphi} & \mathcal{P}\mathscr{D}\big(J,\mathscr{C}(S-,M)\big) \\ {\scriptstyle \mathscr{C}(K,\varrho')}\big\downarrow\big\downarrow{\scriptstyle \mathscr{C}(\varrho,TM)\circ T} & & {\scriptstyle \mathcal{P}\mathscr{D}\left(J,\mathscr{C}(S-,\varrho')\right)}\big\downarrow\big\downarrow{\scriptstyle \mathcal{P}\mathscr{D}\left(J,\mathscr{C}(\varrho_0,TM)\circ T\right)} \\ \mathscr{C}(K,TM) & \xrightarrow[\varphi]{} & \mathcal{P}\mathscr{D}\big(J,\mathscr{C}(S-,TM)\big) \end{array}$$

is serially commuative, hence φ induces an isomorphism between the equalizers of the pairs in the above diagram. From the definitions in Section 2.2 and from the fact that $\mathcal{P}\mathscr{D}(J,-)$ preserves limits it follows that these equalizers are given by the objects in

$$\mathscr{C}_T\big((K,\varrho),\mathbf{M}\big) \xrightarrow{\varphi} \mathcal{P}\mathscr{D}\big(J,\mathscr{C}_T(\overline{S}-,\mathbf{M})\big)\ ,$$

which shows that φ exhibits (K,ϱ) as colimit of $\overline{S}$ weighted by J. ∎

3. The Tannakian adjunction

3.1. In this section we give a new construction of a left adjoint of the comodule functor from Definition 2.8. This construction is analogous to the construction used in the semantics-structure adjunction (see Theorem 2.10). More precisely, the natural transformation π from Lemma 3.4 plays the same role as the natural transformation $F\eta$ occurring in Theorem 2.10. For the construction of π we need the so called *free cocompletion* of a small $\mathscr{V}$-category $\mathscr{A}$ from [Kel05].

3.2. Theorem. *Let $Y\colon \mathscr{A} \to \mathcal{P}\mathscr{A}$ be the Yoneda embedding, where $\mathscr{A}$ is a small $\mathscr{V}$-category, and let $\mathscr{C}$ be a cocomplete $\mathscr{V}$-category. Then for any cocontinuous $\mathscr{V}$-functor $S\colon \mathcal{P}\mathscr{A} \to \mathscr{C}$ we have*

$$S \cong \mathrm{Lan}_Y K \cong - \star K\colon \mathcal{P}\mathscr{A} \to \mathscr{C}$$

where

$$K = SY\colon \mathscr{A} \to \mathscr{C}\,.$$

Such a cocontinuous S has a right ajoint T given by the $\mathscr{V}$-functor

$$T = \widetilde{K}: \mathscr{C} \to \mathcal{P}\mathscr{A},$$

with $\widetilde{K}(C) = \mathscr{C}(K-, C)$. The full subcategory $\mathbf{Cocts}[\mathcal{P}\mathscr{A}, \mathscr{C}]$ of $[\mathcal{P}\mathscr{A}, \mathscr{C}]$ of cocontinuous functors exists as a $\mathscr{V}$-category, and $S \mapsto SY$ is an equivalence of $\mathscr{V}$-categories

$$[Y, \mathscr{C}]: \mathbf{Cocts}[\mathcal{P}\mathscr{A}, \mathscr{C}] \to [\mathscr{A}, \mathscr{C}].$$

The inverse to this equivalence sends K to (a choice of) $\mathrm{Lan}_Y K$. We let $L_K = \mathrm{Lan}_Y K$, and we write $\alpha_K: K \cong L_K Y$ for the unit of this left Kan extension. We denote the above mentioned adjunction by $\eta^K, \varepsilon^K: L_K: \mathcal{P}\mathscr{A} \rightleftarrows \mathscr{C}: \widetilde{K}$.

Proof. This is (part of) [Kel05], Theorem 4.51. ∎

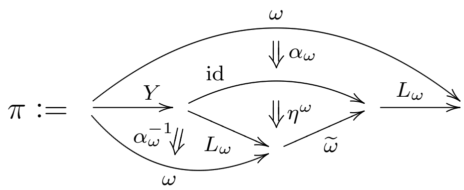
3.3. Proposition. *Let $\mathscr{A}$ be a small $\mathscr{V}$-category, and let $\omega: \mathscr{A} \to \mathcal{P}\mathscr{B}$ be a $\mathscr{V}$-functor whose image is contained in $\overline{\mathscr{B}}$. Then the right adjoint $\widetilde{\omega}: \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{A}$ of L_ω (see Theorem 3.2) is cocontinuous. In particular, the density comonad $L_\omega \widetilde{\omega}$ of $L_\omega: \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ (see section 2.2) is a cocontinuous $\mathscr{V}$-comonad.*

Proof. Since colimits in $\mathcal{P}\mathscr{A} = [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ are computed pointwise (cf. [Kel05], Section 3.3) it suffices to check that for every $A \in \mathscr{A}$,

$$\widetilde{\omega}(-)(A) = [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]\big(\omega(A), -\big): [\mathscr{B}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{V}$$

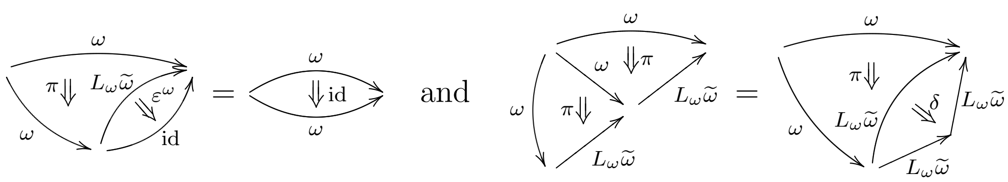
preserves weighted colimits. This follows immediately from the fact that the objects of $\overline{\mathscr{B}}$ are the small projectives in $[\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ (see [Kel05], Section 5.5). ∎

3.4. Lemma. *Let $\mathscr{A}$ be a small $\mathscr{V}$-category, $\omega: \mathscr{A} \to \mathcal{P}\mathscr{B}$ be a $\mathscr{V}$-functor, and let $F: \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$ be a cocontinuous $\mathscr{V}$-functor. Let $\eta^\omega, \varepsilon^\omega: L_\omega: \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}: \widetilde{\omega}$ and $\alpha_\omega: L_\omega Y \cong \omega$ be as in Theorem 3.2. The $\mathscr{V}$-natural transformation*



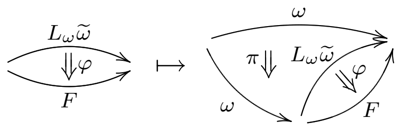
has the following properties:

i) The equalities



hold, where $\delta = L_\omega \eta^\omega \widetilde{\omega}$ denotes the comultiplication of the density comonad of $L_\omega \dashv \widetilde{\omega}$.

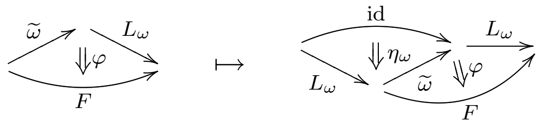
ii) For any cocontinuous $\mathscr{V}$-functor $F\colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$, the assignment



gives a bijection $\mathscr{V}$-Nat$(L_\omega\widetilde{\omega}, F) \to \mathscr{V}$-Nat$(\omega, F\omega)$.

Proof. Both equations in i) can be read off of the respective pasting diagrams. The map in ii) is given by the composite of the assignment

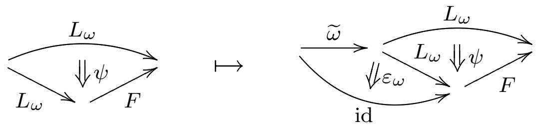
$$\mathscr{V}\text{-Nat}(L_\omega\widetilde{\omega}, F) \longrightarrow \mathscr{V}\text{-Nat}(L_\omega, FL_\omega)\ ,$$



with the map

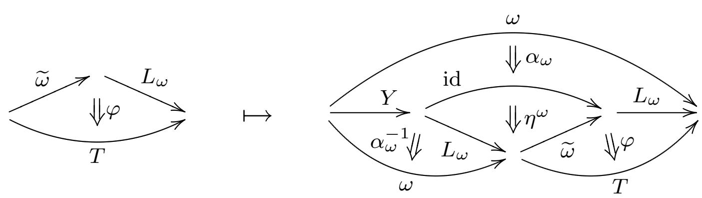
$$\mathscr{V}\text{-Nat}(L_\omega, FL_\omega) \xrightarrow{-Y} \mathscr{V}\text{-Nat}(L_\omega Y, FL_\omega Y) \xrightarrow{\mathscr{V}\text{-Nat}(\alpha_\omega, F\alpha_\omega^{-1})} \mathscr{V}\text{-Nat}(\omega, F\omega)\ .$$

The former has an inverse given by



and whiskering with Y gives a bijection between $\mathscr{V}$-Nat(L_ω, FL_ω) and $\mathscr{V}$-Nat$(L_\omega Y, FL_\omega Y)$ because the Yoneda embedding $Y\colon \mathscr{A} \to \mathcal{P}\mathscr{A}$ is *dense* (see [Kel05], Section 5 for a discussion of dense functors, and op. cit. Proposition 5.16 for the particular result), and because $L_\omega\widetilde{\omega}$ is cocontinuous by Proposition 3.3. Since α_ω is an isomorphism (see Theorem 3.2), so is $\mathscr{V}$-Nat$(\alpha_\omega, F\alpha_\omega^{-1})$. Thus the assignment in ii) really does give a bijection. ∎

3.5. Proposition. [Tannakian adjunction] *Let $\mathscr{A}$ be a small $\mathscr{V}$-category, $\omega\colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ a $\mathscr{V}$-functor with $\omega(\mathscr{A}) \subseteq \overline{\mathscr{B}}$, and let $\eta^\omega, \varepsilon^\omega\colon L_\omega\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}\colon \widetilde{\omega}$ and $\alpha_\omega\colon \omega \cong L_\omega Y$ be as in Theorem 3.2. For $T \in$* **Comon** $({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$*, the assignment*



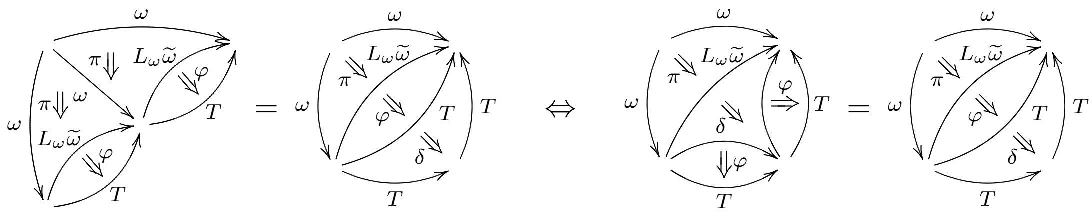
gives a natural bijection between morphisms of $\mathscr{V}$-comonads $\varphi\colon L_\omega\widetilde{\omega} \Rightarrow T$ and T-coactions on ω. Together with the bijection from Proposition 2.6 we get a natural bijection

$$\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})\,(L_\omega\widetilde{\omega}, T) \cong \mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}\big((\mathscr{A}, \omega), (\mathcal{P}\mathscr{B}^{\mathrm{c}}_T, V_T)\big)$$

(here we do not distinguish the category of small $\mathscr{V}$-categories over $\overline{\mathscr{B}}$ from the equivalent category of functors $\omega\colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ whose image is contained in $\overline{\mathscr{B}}$). Thus the assignment $\omega \mapsto L_\omega \widetilde{\omega}$ extends uniquely to a left adjoint of the comodule functor

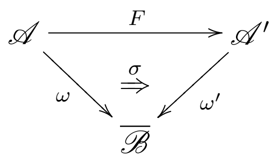
$$\mathcal{P}\mathscr{B}^{c}_{(-)}\colon \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}) \to \mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}.$$

Proof. The last statement follows from the first assertion because $L_\omega \widetilde{\omega}$ is a cocontinuous comonad (see Proposition 3.3). In other words, we only have to show that the bijection from Lemma 3.4 ii) restricts to a bijection between morphisms of comonads and coactions. This is a consequence of the two equations in Lemma 3.4 i): They ensure that the axioms for a morphism of comonads transform into the axioms of a coaction. Since the assignment is bijective, one set of axioms holds if and only if the other does. For example, by part i) we have



and a similar equivalence holds for the axiom involving the counits. Naturality in T can be checked as follows: By Proposition 2.6, we can reduce the question to checking that two coactions are equal, and this follows easily from the definition of $\mathcal{P}\mathscr{B}^{c}_{(-)}$ (see Section 2.8). ■

3.6. Both categories occuring in the Tannakian adjunction are 2-categories in a natural way, and the Tannakian adjunction is in fact a 2-adjunction. Moreover, there is another 2-category which could serve as codomain for the comodule functor: Instead of the slice category $\mathscr{V}$-$\mathbf{cat}/\overline{\mathscr{B}}$ we can consider the category $\mathscr{V}$-$\mathbf{cat}//\overline{\mathscr{B}}$ with the same objects but with morphisms given by natural isomorphisms

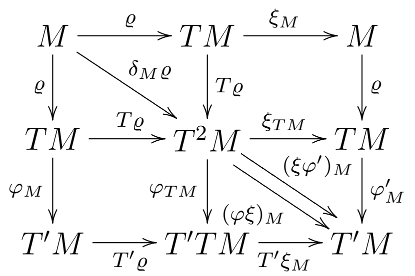


and the comodule functor would then still have a left bi-adjoint. In the remainder of this section we will indicate how the two adjunctions are related.
The category ${}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}$ of modules from $\mathscr{B}$ to $\mathscr{B}$ is a monoidal category, with tensor product given by composition of modules (equivalently, by composition of cocontinuous functors $\mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$). A comonoid in this category is the same as a one-object ${}_{\mathscr{B}}\mathcal{M}^{\mathrm{op}}_{\mathscr{B}}$-category, and a morphism of comonoids $T \to T'$ is the same as a ${}_{\mathscr{B}}\mathcal{M}^{\mathrm{op}}_{\mathscr{B}}$-functor $T' \to T$. We let the 2-cells in the category of comonoids be the ${}_{\mathscr{B}}\mathcal{M}^{\mathrm{op}}_{\mathscr{B}}$-natural transformations. Unraveling these definitions we find that a 2-cell $\xi\colon \varphi \Rightarrow \varphi'\colon T \to T'$ is a $\mathscr{V}$-natural transformation $\xi\colon T \Rightarrow \mathrm{id}$ such that

$$T \xrightarrow{\ \delta\ } T \circ T \underset{\xi\varphi'}{\overset{\varphi\xi}{\rightrightarrows}} T'$$

is commutative.

The 2-cells $F \Rightarrow F' \colon (\mathscr{A}, \omega) \to (\mathscr{A}', \omega')$ in $\mathscr{V}$-$\mathbf{cat}/\overline{\mathscr{B}}$ are simply the $\mathscr{V}$-natural transformations $F \Rightarrow F'$. Similarly, the two-cells $(F, \sigma) \Rightarrow (F', \sigma') \colon (\mathscr{A}, \omega) \to (\mathscr{A}', \omega')$ in $\mathscr{V}$-$\mathbf{cat}//\overline{\mathscr{B}}$ are $\mathscr{V}$-natural transformations $F \Rightarrow F'$; we do not require any compatibility with σ and σ'.
The comodule functor $\mathcal{P}\mathscr{B}^{\mathrm{c}}_{(-)} \colon \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}}) \to \mathscr{V}$-$\mathbf{cat}/\overline{\mathscr{B}}$ is in fact a 2-functor: Given a 2-cell $\xi \colon \varphi \Rightarrow \varphi' \colon T \to T'$ and a Cauchy comodule (M, ϱ), we get a $\mathscr{V}$-natural transformation $\xi V_T \circ V_T \eta^T \colon V_T \Rightarrow V_T$, with (M,ϱ)-component given by $\xi_M \circ \varrho$ (see Section 2.2). Commutativity of



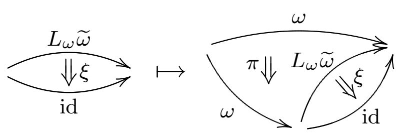
shows that this natural transformation is compatible with the coactions which define $\mathcal{P}\mathscr{B}_\varphi$ (see Section 2.7), hence by Proposition 2.6 it lifts to a 2-cell $\mathcal{P}\mathscr{B}_\xi \colon \mathcal{P}\mathscr{B}_\varphi \Rightarrow \mathcal{P}\mathscr{B}_{\varphi'}$, and we let $\mathcal{P}\mathscr{B}^{\mathrm{c}}_\xi \colon \mathcal{P}\mathscr{B}^{\mathrm{c}}_\varphi \Rightarrow \mathcal{P}\mathscr{B}^{\mathrm{c}}_{\varphi'}$ be its restriction to Cauchy comodules.

3.7. Proposition. *With 2-cells defined as in Section 3.6, the Tannakian adjunction becomes a 2-adjunction. In other words, there is a 2-natural isomorphism of categories*

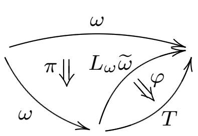
$$\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})\,(L_\omega\widetilde{\omega}, T) \cong \mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}\big((\mathscr{A},\omega), (\mathcal{P}\mathscr{B}^{\mathrm{c}}_T, V_T)\big)$$

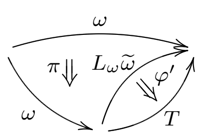
(see Proposition 3.5 for the notation).

Proof. We write $\Gamma \colon \mathscr{V}\text{-}\mathrm{Nat}(L_\omega\widetilde{\omega}, \mathrm{id}) \to \mathscr{V}\text{-}\mathrm{Nat}(\omega,\omega)$ for the bijection

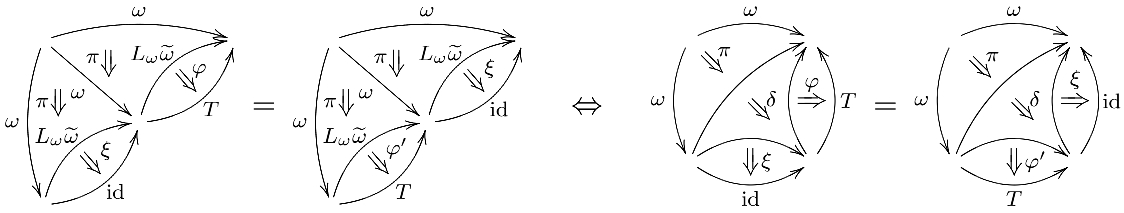


from Lemma 3.4 ii). Let $\varphi, \varphi' \colon L_\omega\widetilde{\omega} \Rightarrow T$ be two morphisms of comonads. We claim that $\xi \colon L_\omega\widetilde{\omega} \Rightarrow \mathrm{id}$ is a 2-cell from φ to φ' if and only if $\Gamma(\xi)$ is compatible (in the sense of Proposition 2.6) with the two coactions

$\Gamma(\varphi) :=$ 

and $\Gamma(\varphi') :=$ 

on ω. This is a consequence of Lemma 3.4: By part i) we have

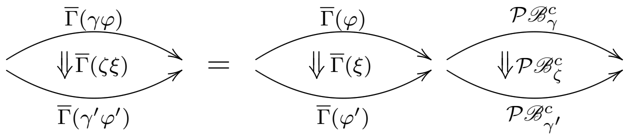


and part ii) implies that the equation on the right holds if and only if $\varphi\xi \circ \delta = \xi\varphi' \circ \delta$, that is, $T\Gamma(\xi) \circ \Gamma(\varphi) = \Gamma(\varphi') \circ \Gamma(\xi)$ if and only if $\xi\colon \varphi \Rightarrow \varphi'\colon L_\omega\widetilde{\omega} \to T$ is a 2-cell (see Section 3.6). By Proposition 2.6 there exist unique lifts $\overline{\Gamma}(\varphi)\colon \mathscr{A} \to \mathcal{P}\mathscr{B}^{c}_{T}$ of $\omega\colon \mathscr{A} \to \overline{\mathscr{B}}$ and $\overline{\Gamma}(\xi)\colon \overline{\Gamma}(\varphi) \Rightarrow \overline{\Gamma}(\varphi')$ of $\Gamma(\xi)\colon \omega \Rightarrow \omega$. We thus get a functor

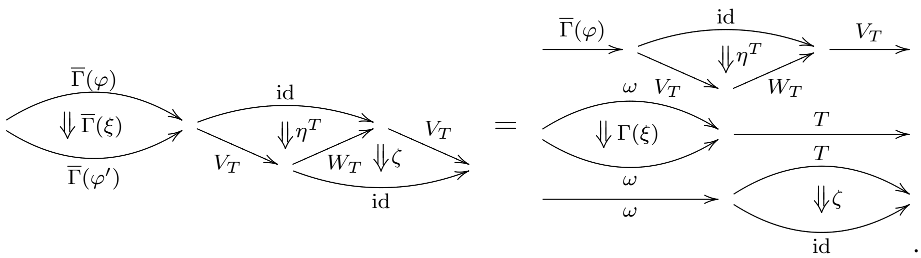
$$\overline{\Gamma}\colon \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})\,(L_\omega\widetilde{\omega}, T) \cong \mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}\big((\mathscr{A},\omega),(\mathcal{P}\mathscr{B}^{c}_{T}, V_T)\big)$$

(functoriality follows again from Lemma 3.4). The object part of $\overline{\Gamma}$ is a bijection by Proposition 3.5, and by the above reasoning $\overline{\Gamma}$ is fully faithful, so it is an isomorphism of categories.

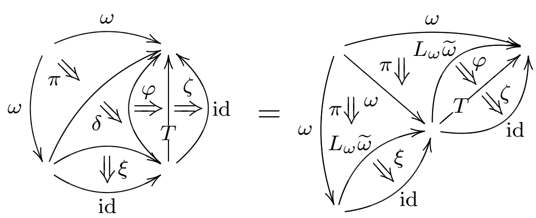
It remains to check that the isomorphism $\overline{\Gamma}$ is 2-natural in T. Let $\zeta\colon \gamma \Rightarrow \gamma'\colon T \to T'$ be a 2-cell in $\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$. We have to check that the equality



holds. By Proposition 2.6 it suffices to check that equality holds if we whisker both sides on the right with $V_{T'}\colon \mathcal{P}\mathscr{B}^{c}_{T'} \to \overline{\mathscr{B}}$, i.e., that $\Gamma(\zeta\xi)$ is equal to



By definition of $\zeta\xi\colon \gamma\varphi \Rightarrow \gamma'\varphi'\colon L_\omega\widetilde{\omega} \to T'$ (see Section 3.6) and of η^T (see Section 2.3), this amounts to showing that



holds. This last equality is a consequence of Lemma 3.4 i). ■

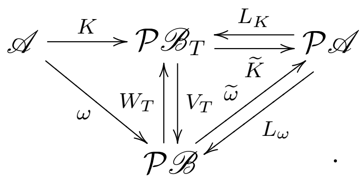
3.8. For $T \in \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$, the functor $V_T\colon \mathcal{P}\mathscr{B}^{\mathrm{c}}_T \to \overline{\mathscr{B}}$ has the following property: Given a $\mathscr{V}$-functor $F\colon \mathscr{A} \to \mathcal{P}\mathscr{B}^{\mathrm{c}}_T$ and a $\mathscr{V}$-natural isomorphism $\beta\colon V_T F \Rightarrow G$, there is a lift $\overline{G}$ of G and a lift $\overline{\beta}\colon F \cong \overline{G}$. Indeed, by Proposition 2.6, F corresponds to a coaction $\varrho\colon V_T F \Rightarrow T V_T F$, and $\varrho' := T\beta \circ \varrho \circ \beta^{-1}\colon G \Rightarrow TG$ defines a coaction on G which satisfies $T\beta \circ \varrho = \varrho' \circ \beta$. Applying Proposition 2.6 again we get the desired lifts.
In other words, the image of the comodule 2-functor is contained in the full sub-2-category $\mathscr{K}$ of $\mathscr{V}\text{-}\mathbf{cat}/\overline{\mathscr{B}}$ whose objects are the *isofibrations*. We write $I\colon \mathscr{K} \to \mathscr{V}\text{-}\mathbf{cat}/\!/\overline{\mathscr{B}}$ for the natural inclusion 2-functor. It turns out that I is a biequivalence: It is clear from the definition of 2-cells that I is locally fully faithful, and the definition of an isofibration implies that I is locally essentially surjective. It remains to show that any object $(\mathscr{A},\omega)$ in $\mathscr{V}\text{-}\mathbf{cat}/\!/\overline{\mathscr{B}}$ is equivalent to an isofibration. This follows from the fact that ω can be written as $\omega = \omega' \circ u$ where ω' is an isofibration and u is an equivalence[5] (see [Lack07], Section 3).
The above considerations imply that the comodule functor with domain $\mathscr{V}\text{-}\mathbf{cat}/\!/\overline{\mathscr{B}}$ (i.e., the composite $I \circ \mathcal{P}\mathscr{B}^{\mathrm{c}}_{(-)}$) does indeed have a left bi-adjoint. Moreover, any property that is stable under bi-equivalence holds for this bi-adjunction if and only if it holds for the Tannakian 2-adjunction. Thus we can use our construction in Proposition 3.7 to deduce facts about the Tannakian bi-adjunction. For example, the T-component of the counit of the bi-adjunction is an equivalence if and only if the T-component of the Tannakian 2-adjunction is, and a 1-cell in $\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$ is an equivalence if and only if it is an isomorphism (recall that $\mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$ is the opposite of the 2-category of one-object ${}_{\mathscr{B}}\mathcal{M}^{\mathrm{op}}_{\mathscr{B}}$-categories). Thus the reconstruction results in Section 4 apply to both the Tannakian 2-adjunction and the Tannakian bi-adjunction. The latter is important because, in the case $\mathscr{B} = \mathscr{I}$, it is an adjunction between monoidal 2-categories. Proposition 6.4 in [McC02] and the computations in [Str07], Section 16 indicate that the left adjoint preserves the tensor product up to isomorphism, and thus lifts to a functor between pseudomonoids on on both sides. By making this precise one could extend our reconstruction results in Section 4 from comonoids to pseudomonoids in the category of comonoids, i.e., one would get reconstruction results for comonoids with additional structure.

4. Reconstruction

4.1. The goal of this section is to give a necessary and sufficient condition for the counit of the Tannakian adjunction (see Proposition 3.5) to be an isomorphism. In order to do this we have to find a suitable description of this counit, hence we fix some notation first. We fix a small $\mathscr{V}$-category $\mathscr{B}$ (with small Cauchy completion) and a cocontinuous comonad $T\colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$. For the sake of brevity we let $\mathscr{A} = \mathcal{P}\mathscr{B}^{\mathrm{c}}_T$ be the category of Cauchy comodules, we write $K\colon \mathscr{A} \to \mathcal{P}\mathscr{B}_T$ for the inclusion functor (see Section 2.4) and we let ω be the composite $\omega = V_T \circ K\colon \mathscr{A} \to \overline{\mathscr{B}}$. By Theorem 3.2, the functors K and ω induce adjunctions $\eta^K, \varepsilon^K\colon L_K\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}_T\colon \widetilde{K}$ and $\eta^\omega, \varepsilon^\omega\colon L_\omega\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}\colon \widetilde{\omega}$. This is

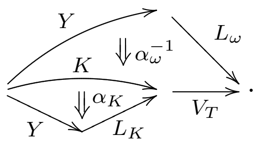
[5]Note that a 1-cell $u\colon (\mathscr{A},\omega) \to (\mathscr{A}',\omega')$ in $\mathscr{V}\text{-}\mathbf{cat}/\!/\overline{\mathscr{B}}$ is an equivalence in $\mathscr{V}\text{-}\mathbf{cat}/\!/\overline{\mathscr{B}}$ if and only if the underlying $\mathscr{V}$-functor $u\colon \mathscr{A} \to \mathscr{A}'$ is an equivalence in $\mathscr{V}\text{-}\mathbf{cat}$.

summarized in the diagram



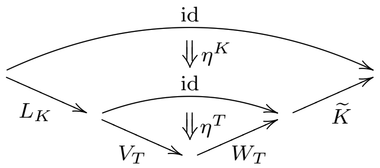
.

Furthermore, we let $\alpha_K\colon K \Rightarrow L_K Y$ and $\alpha_\omega\colon \omega \Rightarrow L_\omega Y$ be the natural isomorphisms from Theorem 3.2. Since the Yoneda embedding is dense (see [Kel05], Proposition 5.16) there is a unique natural isomorphism $\sigma\colon L_\omega \Rightarrow V_T L_K$ such that

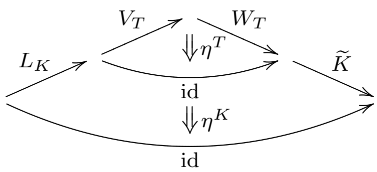
$\sigma Y =$ 

.

Recall that the natural transformations

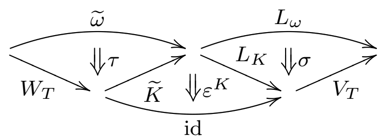


and

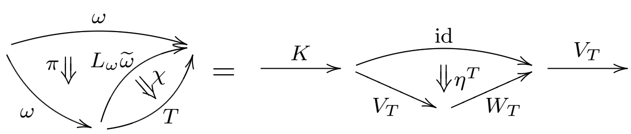


give the unit and counit of the adjunction $V_T L_K\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}\colon \widetilde{K} W_T$. We let $\tau\colon \widetilde{\omega} \Rightarrow \widetilde{K} W_T$ be the mate of $\sigma^{-1}\colon V_T L_K \Rightarrow L_\omega$ under the adjunctions $L_\omega \dashv \widetilde{\omega}$ and $V_T L_K \dashv \widetilde{K} W_T$ (a definition of mates under adjunction can be found in [KS72]).

4.2. Proposition. *With the notation introduced in Section 4.1, the T-component of the counit of the Tannakian adjunction (see Proposition 3.5) is given by*

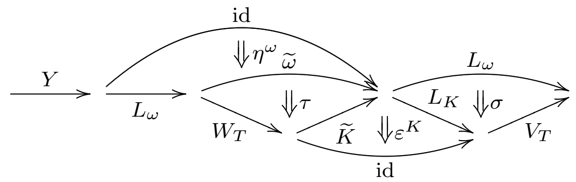
$\chi :=$ 

Proof. We use the notation introduced in Section 4.1. We have to check that the bijection defined in Proposition 3.5 sends χ to the identity functor of $(\mathcal{P}\mathscr{B}^c_, V_T)$ (i.e., to the identity functor of $(\mathscr{A}, \omega)$ with the above notation). By Proposition 2.6 it suffices to check that the two corresponding coactions are the same. With the notation of Lemma 3.4, we have to check that

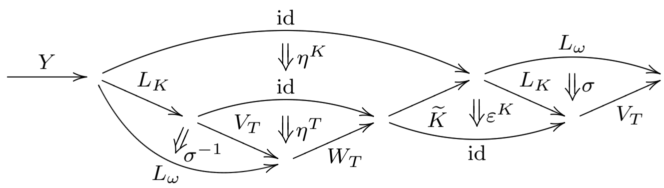


,

because the coaction on the right corresponds to the lift $\mathrm{id}\colon \mathscr{A} \to \mathscr{A}$ of $\omega = V_T K$ (this follows from the definition of η^T in Section 2.3).
Since τ is the mate of σ^{-1} (see Section 4.1), the natural transformation

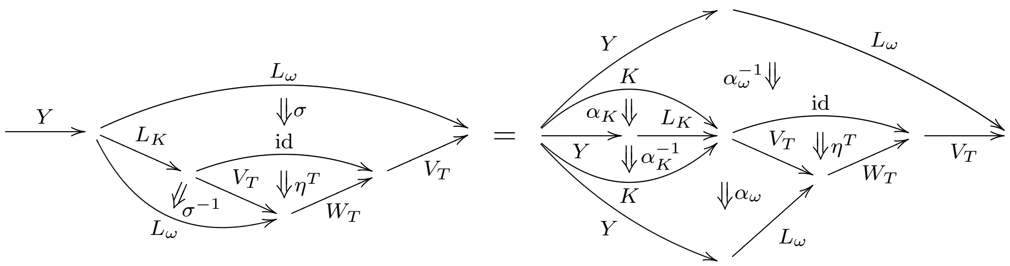
$\zeta :=$ 

is equal to the pasted composite of



.

The triangular identities for η^K and ε^K and the definition of σ (see Section 4.1) imply that the equalities

$\zeta =$ 

hold. The desired equality now follows from the definition of π (see Lemma 3.4). ■

4.3. The above proposition shows that the counit of the Tannakian adjunction is closely related to the counit ε^K of the adjunction $L_K\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B}_T\colon \widetilde{K}$ associated to the inclusion $K\colon \mathscr{A} = \mathcal{P}\mathscr{B}_T^{\mathrm{c}} \to \mathcal{P}\mathscr{B}_T$. For $M \in \mathcal{P}\mathscr{B}_T$, we have

$$L_K \widetilde{K}(M) = \mathcal{P}\mathscr{B}_T(K-, M) \star K = \int^{A \in \mathscr{A}} \mathcal{P}\mathscr{B}_T\big(K(A), M\big) \odot K(A).$$

(see Theorem 3.2), and one can check that the M-component of the counit of $L_K \dashv \widetilde{K}$ is given by the comparison map induced by the canonical maps

$$\mathcal{P}\mathscr{B}_T(KA, M) \odot KA \to M.$$

4.4. Theorem. *We use the notation of Proposition 3.5 and Section 4.1. The T-component of the counit of the Tannakian adjunction is an isomorphism if and only if for each $B \in \mathscr{B}$, the map $\varepsilon^K_{(TYB,\delta_{YB})} \colon L_K \widetilde{K}\big((TYB,\delta_{YB})\big) \to (TYB,\delta_{YB})$ is an isomorphism. This is the case if and only if the canonical maps*

$$\mathcal{P}\mathscr{B}_T\big(K(A),(TYB,\delta_{YB})\big) \odot K(A) \to (TYB,\delta_{YB})$$

exhibit (TYB,δ_{YB}) as coend of

$$\mathcal{P}\mathscr{B}_T\big(K(-),(TYB,\delta_{YB})\big) \odot K(-) \colon \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathcal{P}\mathscr{B}_T.$$

Proof. We use the notation of Proposition 4.2. We have to show that $\chi \colon L_\omega \widetilde{\omega} \to T$ is an isomorphism if and only if $\varepsilon^K_{(TYB,\delta_{YB})} \colon L_K \widetilde{K}(TYB,\delta_{YB}) \to (TYB,\delta_{YB})$ is an isomorphism. Density of the Yoneda embedding and the fact that both $L_\omega \widetilde{\omega}$ and T are cocontinuous (see Proposition 3.3) imply that χ is an isomorphim if and only if χY is. Both σ and τ are isomorphisms, hence χY is an isomorphism if and only if $V_T(\varepsilon^K_{W_T Y})$ is, i.e., if and only if $V_T(\varepsilon^K_{W_T(YB)})$ is an isomorphism for every $B \in \mathscr{B}$. The first claim thus follows from the fact that V_T reflects isomorphisms (see Theorem 2.11). The second assertion follows immediately from the remarks in Section 4.3. ∎

4.5. If we take $\mathscr{V} = \mathbf{Top}$ with monoidal structure given by the cartesian product, and $\mathscr{B} = *$, then every object X of $\mathcal{P}\mathscr{B} \simeq \mathbf{Top}$ has a unique comonoid structure. Moreover, giving an X-coaction on Y is the same as giving a map $Y \to X$. Since **Top** is cartesian, the terminal object is the only object which has a dual. The category of Cauchy comodules of X is therefore the full subcategory of $\mathbf{Top}/X$ generated by the objects $* \to X$. Clearly this category is independent of the topology on X, so it is impossible to reconstruct X from its category of Cauchy comodules. In other words, the counit of the Tannakian adjunction need not be an isomorphism in general.

5. Recognition

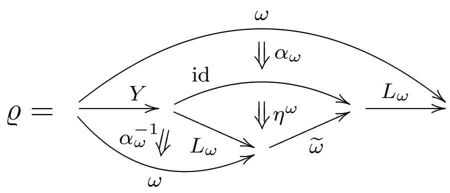
5.1. In this section we study the unit of the Tannakian adjunction. In the entire section we use the notation introduced in Proposition 3.5. As we will see in Proposition 5.2, the unit of the Tannakian is closely related to the unit of the semantics-structure adjunction (see Theorem 2.10). It is well-known when the latter is an equivalence (Beck's monadicity theorem, see Theorem 2.11), and this is the starting point for proving our recognition result.

5.2. Proposition. *Let $\mathscr{A}$, T be as in Proposition 3.5. The unit of the Tannakian adjunction is naturally isomorphic to the composite*

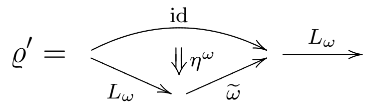
$$\mathscr{A} \xrightarrow{Y} \mathcal{P}\mathscr{A} \xrightarrow{J} \mathcal{P}\mathscr{B}_{L_\omega \widetilde{\omega}}$$

of the Yoneda embedding and the unit $J \colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}_{L_\omega \widetilde{\omega}}$ of the semantics-structure adjunction (see Theorem 2.10).

Proof. By Proposition 2.6 we only have to check that both functors correspond to coactions which are related by a suitable natural isomorphism. By definition of the Tannakian adjunction (see Proposition 3.5), N corresponds to the $L_\omega\widetilde{\omega}$-coaction

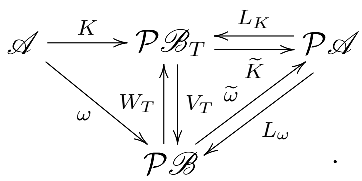


and by Theorem 2.10, J corresponds to the $L_\omega\widetilde{\omega}$-coaction



and the uniqueness part of Proposition 2.6 implies that $JY\colon \mathscr{A} \to \mathcal{P}\mathscr{B}_{L_\omega\widetilde{\omega}}$ corresponds to the coaction $\varrho' Y$. Clearly $L_\omega\widetilde{\omega}\alpha_\omega \circ \varrho = \varrho' Y \circ \alpha_\omega$, so by Proposition 2.6, α_ω lifts to a natural isomorphism between the unit of the Tannakian adjunction and JY. ■

5.3. The above proposition implies that the $(\mathscr{A}, \omega)$-component of the unit of the Tannakian adjunction is fully faithful if J is an equvalence, i.e., if $L_\omega\colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ is comonadic (see Theorem 2.11). However, this is not to be expected in general, because the category $\mathcal{P}\mathscr{B}_T$ is usually not equivalent to a category of presheaves. Hence we first have to analyze the situation where $(\mathscr{A}, \omega)$ is equal to $(\mathcal{P}\mathscr{B}_T^c, V_T)$ more carefully. Writing $K\colon \mathscr{A} \to \mathcal{P}\mathscr{B}_T$ for the inclusion functor, this is summarized by the diagram



Under the additional assumption that K is dense we find that $\mathcal{P}\mathscr{B}_T$ is a reflective subcategory of $\mathcal{P}\mathscr{A}$, and that the comonads $L_\omega\widetilde{\omega}$ and $V_T W_T$ are isomorphic. Moreover, we have $\widetilde{K} \circ K = Y$ (see Theorem 3.2), which suggests that we adopt the following strategy:

a) Find conditions for $\omega\colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ which imply the existence of a reflective subcategory $\mathscr{C}$ of $\mathcal{P}\mathscr{A}$, together with a comonadic adjunction $V\colon \mathscr{C} \rightleftarrows \mathcal{P}\mathscr{B}\colon W$ such that the comonad VW is isomorphic to $L_\omega\widetilde{\omega}$, and

b) find conditions for ω which imply that the Yoneda embedding $Y\colon \mathscr{A} \to \mathcal{P}\mathscr{A}$ factors through the embedding $\mathscr{C} \to \mathcal{P}\mathscr{A}$ from a).

Since comonadic functors reflect isomorphisms, a reasonable candidate for $\mathscr{C}$ is the full subcategory of objects which are orthogonal to the class Σ of morphisms in $\mathcal{P}\mathscr{A}$ which get sent to isomorphisms by $L_\omega \colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ (an object $X \in \mathcal{P}\mathscr{A}$ is *orthogonal* to a class Σ of morphisms if for all $f \in \Sigma$, the morphism $\mathcal{P}\mathscr{A}(f, X)$ is an iomorphism, see [Kel05], Section 6.2). If this subcategory is reflective, then it is the localization of $\mathcal{P}\mathscr{A}$ at Σ, so it can be thought of as the category obtained by formally inverting the morphisms in Σ. We first show that a reflection functor exists if the monoidal category $\mathscr{V}$ is locally presentable.

5.4. LEMMA. *Let $\mathscr{A}$ be a small $\mathscr{V}$-category. If $\mathscr{V}_0$ is locally presentable, then $\mathcal{P}\mathscr{A}_0$ is locally presentable.*

PROOF. Since $\mathcal{P}\mathscr{A}$ is cocomplete as a $\mathscr{V}$-category, its underlying ordinary category $\mathcal{P}\mathscr{A}_0$ is cocomplete, so it suffices to show that there is a regular cardinal λ and a set of λ-small objects which forms a strong **Set**-generator of $\mathcal{P}\mathscr{A}_0$. By assumption we can choose a regular cardinal λ such that $\mathscr{V}_0$ is locally λ-presentable. Thus there exists a set $\mathscr{G}$ of λ-small objects of $\mathscr{V}_0$ which forms a strong **Set**-generator of $\mathscr{V}_0$. For $A \in \mathscr{A}$ and $G \in \mathscr{G}$, the object $\mathscr{A}(-, A) \odot G$ is λ-small: By definition of copowers (see Section 2.12) and by Yoneda we have for each $F \in \mathcal{P}\mathscr{A}_0$

$$\mathcal{P}\mathscr{A}_0\big(\mathscr{A}(-, A) \odot G, F\big) \cong [G, \mathcal{P}\mathscr{A}\big(\mathscr{A}(-, A), F\big)]_0 \cong [G, FA]_0 \cong \mathscr{V}_0(G, FA),$$

and both $\mathscr{V}(G, -)$ and $F \mapsto FA$ preserve λ-filtered colimits. The latter preserves in fact all colimits, because colimits in $\mathcal{P}\mathscr{A}$ are computed pointwise.
Let $\alpha \colon F \Rightarrow F'$ be a $\mathscr{V}$-natural transformation, i.e., a morphism in $\mathcal{P}\mathscr{A}_0$. Since the above isomorphism is natural, we find that $\mathcal{P}\mathscr{A}_0\big(\mathscr{A}(-, A) \odot G, \alpha\big)$ is an isomorphism if and only if $\mathscr{V}(G, \alpha_A)$ is. This implies that the set $\{\mathscr{A}(-, A) \odot G | G \in \mathscr{G}\}$ forms a strong **Set**-generator of $\mathcal{P}\mathscr{A}_0$. Thus $\mathcal{P}\mathscr{A}_0$ is locally presentable. ∎

5.5. LEMMA. *Let $\omega \colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ and $L_\omega \colon \mathcal{P}\mathscr{A} \rightleftarrows \mathcal{P}\mathscr{B} \colon \widetilde{\omega}$ be as in Proposition 3.5, and let Σ be the class of morphisms f in $\mathcal{P}\mathscr{A}_0$ for which $L_\omega f$ is an isomorphism. If $\mathscr{V}_0$ is locally presentable, then the full subcategory $\mathscr{C} = \Sigma^\perp$ of $\mathcal{P}\mathscr{A}$ generated by the objects which are orthogonal to Σ is reflective, i.e., the inclusion functor $I \colon \mathscr{C} \to \mathcal{P}\mathscr{A}$ has a left adjoint $R \colon \mathcal{P}\mathscr{A} \to \mathscr{C}$.*

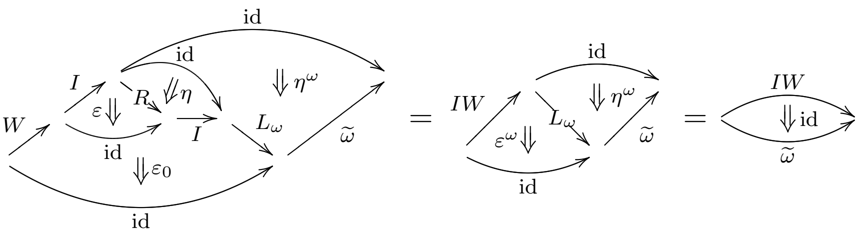
PROOF. The functor L_ω preserves copowers, so for any morphism $f \colon X \to X'$ in Σ and any object $V \in \mathscr{V}$, the morphism $V \odot f \colon V \odot X \to V \odot X'$ is in Σ. Hence an object $F \in \mathcal{P}\mathscr{A}_0$ is orthogonal to Σ in the enriched sense if and only if it is orthogonal in the classical sense (cf. [Kel05], Section 6.2).
By Lemma 5.4, the categories $\mathcal{P}\mathscr{A}_0$ and $\mathcal{P}\mathscr{B}_0$ are locally presentable, and the functor $(L_\omega)_0 \colon \mathcal{P}\mathscr{A}_0 \to \mathcal{P}\mathscr{B}_0$ preserves all colimits because it is a left adjoint, so it is accessible. It follows from [AR94], Remark 2.50 and op. cit., Section 2.60, that the full subcategory of $\mathrm{Mor}(\mathcal{P}\mathscr{A}_0)$ generated by Σ is an accessible, accessibly embedded subcategory. Thus there is a regular cardinal λ and a subset $\Sigma_0 \subseteq \Sigma$ such that every element of Σ is a λ-filtered colimit of elements of Σ_0, and this colimit is computed as in $\mathrm{Mor}(\mathcal{P}\mathscr{A}_0)$. It follows immediately that $\Sigma^\perp = \Sigma_0^\perp$, and [AR94], Theorem 1.39 implies that the inclusion $I_0 \colon \mathscr{C}_0 \to \mathcal{P}\mathscr{A}_0$ has a left adjoint $R_0 \colon \mathcal{P}\mathscr{A}_0 \to \mathscr{C}_0$. The claim now follows from [Kel05], Theorem 4.85. ∎

5.6. Lemma. *Let $\mathscr{V}_0$ be locally presentable. With the notation from Lemma 5.5, the functor $\widetilde{\omega}\colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{A}$ factors through the inclusion $I\colon \mathscr{C} \to \mathcal{P}\mathscr{A}$, i.e., there is a functor $W\colon \mathcal{P}\mathscr{B} \to \mathscr{C}$ such that $\widetilde{\omega} = IW$. The restriction $V = L_\omega I$ of L_ω to $\mathscr{C}$ is left adjoint to W, and V reflects isomorphisms.*

Proof. We have to check that for any $Y \in \mathcal{P}\mathscr{B}$, the object $\widetilde{\omega}(Y)$ lies in $\mathscr{C}$, i.e., that for any morphism $f\colon X \to X'$ in Σ, the map

$$\mathcal{P}\mathscr{A}\big(f, \widetilde{\omega}(Y)\big)\colon \mathcal{P}\mathscr{A}\big(X', \widetilde{\omega}(Y)\big) \to \mathcal{P}\mathscr{A}\big(X, \widetilde{\omega}(Y)\big)$$

is an isomorphism. By adjunction, this is is equivalent to $\mathcal{P}\mathscr{A}(L_\omega f, Y)$ being an isomorphism, which is evident: The morphism $L_\omega f$ is an isomorphism by definition of Σ (see Proposition 5.5). It is clear that the restriction of a left adjoint still gives a left adjoint, so it remains to check that V reflects isomorphisms.
Let η and ε be the unit and counit of the adjunction $R\colon \mathcal{P}\mathscr{A} \rightleftarrows \mathscr{C}\colon I$ from Proposition 5.5, and let η_0 and ε_0 be the unit and counit of $V\colon \mathscr{C} \rightleftarrows \mathcal{P}\mathscr{B}\colon W$. We clearly have $I\eta_0 = \eta^\omega$ and $\varepsilon_0 = \varepsilon^\omega$, so it might seem pedantic to use different names, but doing so clarifies the pasting diagram



which shows that $L_\omega \eta$ is the mate of the identity natural transformation $IW = \widetilde{\omega}$ under the adjunctions $L_\omega \dashv \widetilde{\omega}$ and $VR \dashv IW$. This implies that $L_\omega \eta$ is an isomorphism, i.e., that the components of η lie in Σ. Commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & IR(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle IR(f)} \\ X' & \xrightarrow[\eta_{X'}]{} & IR(X') \end{array}$$

shows that a morphism f of $\mathcal{P}\mathscr{A}_0$ lies in Σ if and only if $R(f)$ lies in Σ. Since $R(f)$ is a morphism of $\mathscr{C} = \Sigma^\perp$, this is the case if and only if $R(f)$ is an isomorphism. Now let f be a morphism in $\mathscr{C}$ such that $Vf = L_\omega I(f)$ is an isomorphism. Then $I(f)$ lies in Σ, and the above reasoning shows that $RI(f)$ is an isomorphism. But $\varepsilon\colon RI \Rightarrow \mathrm{id}$ is a natural isomorphism, which implies that f is an isomorphism. Thus V reflects isomorphisms, as claimed. ∎

5.7. Since the functor V constructed in the above lemma reflects isomorphisms, it has a chance of being comonadic (cf. Theorem 2.11). In other words, Lemma 5.6 is the first step towards a solution of a) in Section 5.3. In order to solve b) we need some more background on weighted colimits. Let $J \star G$ be the colimit of $G\colon \mathscr{D} \to \mathscr{E}$ weighted by $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$, and let

$$\mathcal{P}\mathscr{D}\big(J, \mathscr{E}(G-, E)\big) \xrightarrow{\varphi} \mathscr{E}(J \star G, E)$$

be the corresponding natural isomorphism. The identity of $J \star G$ corresponds under φ to the *unit*

$$J \xrightarrow{\lambda} \mathscr{E}(G-, J \star G) \ ,$$

of $J \star G$, which has the property[6] that for any $\mathscr{V}$-natural transformation $\alpha\colon J \to \mathscr{E}(G-, E)$, there is a unique morphism $a\colon J \star G \to G$ such that $\alpha = \mathscr{E}(G-, a)$. In particular, if $L\colon \mathscr{E} \to \mathscr{E}'$ is a $\mathscr{V}$-functor such that the colimit $J \star LG$ exists, there is a unique morphism $\widehat{L}\colon J \star LG \to L(J \star G)$ for which the diagram

$$\begin{array}{ccc} J & \xrightarrow{\quad\lambda\quad} & \mathscr{E}(G-, J \star G) \\ {\scriptstyle \lambda'}\big\downarrow & & \big\downarrow{\scriptstyle L} \\ \mathscr{E}'(LG-, J \star LG) & \xrightarrow[\mathscr{E}(LG-,\widehat{L})]{} & \mathscr{E}'\big(LG-, L(J \star G)\big) \end{array}$$

where λ' denotes the unit of $J \star LG$, is commutative. The morphism $\widehat{L}\colon J \star LG \to L(J \star G)$ is called the *comparison morphism*, and we say that L *preserves* the colimit $J \star G$ if $\widehat{L}$ is an isomorphism. As one would expect, if $L\colon \mathscr{E} \to \mathscr{E}'$ is a left adjoint, then it preserves all colimits which exist (see [Kel05], Section 3.2).

5.8. THEOREM. *Let $\mathscr{V}$ be a complete and cocomplete symmetric monoidal closed category with $\mathscr{V}_0$ locally presentable. With the notation of Proposition 3.5, let*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad N \quad} & \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega \widetilde{\omega}} \\ & {\scriptstyle \omega}\searrow \quad \swarrow{\scriptstyle V_{L_\omega \widetilde{\omega}}} & \\ & \mathcal{P}\mathscr{A} & \end{array}$$

be the unit of the Tannakian adjunction, and write Σ for the class of morphisms f of $\mathcal{P}\mathscr{A}_0$ for which $L_\omega(f)$ is an isomorphism. The functor $N\colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega \widetilde{\omega}}$ is an equivalence of $\mathscr{V}$-categories if the following hold:

i) The functor $\omega\colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ reflects isomorphisms,

ii) The left adjoint $L_\omega\colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ preserves equalizers of L_ω-contractible pairs in $\Sigma^\perp$, and

[6] This is a consequence of the $\mathscr{V}$-Yoneda Lemma. In general, having this property is weaker than being the weighted colimit.

iii) Whenever $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ is a weight and $G\colon \mathscr{D} \to \mathscr{A}$ is a $\mathscr{V}$-functor such that the weighted colimit $J \star \omega G$ lies in the subcategory $\overline{\mathscr{B}} \subseteq \mathcal{P}\mathscr{B}$, then $J \star G$ exists and is preserved by $\omega\colon \mathscr{A} \to \mathcal{P}\mathscr{B}$.

Moreover, if $\mathbf{\Phi}$ is a class of weights with the property that for each $X \in \mathcal{P}\mathscr{A}$ there is a weight $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ in $\mathbf{\Phi}$ and a functor $G\colon \mathscr{D} \to \mathscr{A}$ such that $X \cong J \star YG$, then $N\colon \mathscr{A} \to \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega\widetilde{\omega}}$ is an equivalence if i) and ii) hold, and iii) holds for all weights in $\mathbf{\Phi}$.

Proof. For any $X \in \mathcal{P}\mathscr{A}$ we have $X \cong X \star Y$ (see [Kel05], Formula (3.17)). It follows that for $\mathbf{\Phi}$ the class of all weights $\mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, the conditions in the second statement are satisfied. In other words, the first statement follows from the second.
By Proposition 5.2, the composite of $N\colon \mathscr{A} \to \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega\widetilde{\omega}}$ with the inclusion $\mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega\widetilde{\omega}} \to \mathcal{P}\mathscr{B}_{L_\omega\widetilde{\omega}}$ is isomorphic to the composite

$$\mathscr{A} \xrightarrow{Y} \mathcal{P}\mathscr{A} \xrightarrow{J} \mathcal{P}\mathscr{B}_{L_\omega\widetilde{\omega}}$$

of the Yoneda embedding and the unit of the semantics structure adjunction. Let $\mathscr{C}$ be the full subcategory $\Sigma^{\perp}$ of $\mathcal{P}\mathscr{A}$ consisting of objects which are orthogonal to Σ. Let $R\colon \mathscr{C} \rightleftarrows \mathcal{P}\mathscr{A} \colon I$ be as in Lemma 5.5, and let $V\colon \mathscr{C} \rightleftarrows \mathcal{P}\mathscr{B}\colon W$ be as in Lemma 5.6. We write $J'\colon \mathscr{C} \to \mathcal{P}\mathscr{A}_{VW} = \mathcal{P}\mathscr{A}_{L_\omega\widetilde{\omega}}$ for the $V \dashv W$ component of the unit of the semantics structure adjunction (see Theorem 2.10). Since the unit of $V \dashv W$ is equal to the unit of $L_\omega \dashv \widetilde{\omega}$ by construction, the diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{I} & \mathcal{P}\mathscr{A} \\ & {\scriptstyle J'}\searrow \quad \swarrow{\scriptstyle J} & \\ & \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega\widetilde{\omega}} & \end{array}$$

is commutative. By assumption ii), $V = L_\omega I\colon \mathscr{C} \to \mathcal{P}\mathscr{B}$ preserves equalizers of V-contractible pairs, and by Lemma 5.6 it reflects isomorphisms. Thus J' is an equivalence (see Theorem 2.11). We will now show that the Yoneda embedding factors through the full subcategory $\mathscr{C} = \Sigma^{\perp}$, and therefore that $JY = J'Y$ is fully faithful. If this holds, then condition iii) implies that all objects X of $\mathscr{C}$ with $V(X) \in \overline{\mathscr{B}}$ lie in the essential image of the Yoneda embedding, and since the equivalence $J'\colon \mathscr{C} \to \mathcal{P}\mathscr{B}_{L_\omega\widetilde{\omega}}$ satisfies $V_{L_\omega\widetilde{\omega}}J' = V$, it follows that $J'Y\colon \mathscr{A} \to \mathcal{P}\mathscr{B}^{\mathrm{c}}_{L_\omega\widetilde{\omega}}$ is an equivalence.
In other words, it remains to show that for each $A \in \mathscr{A}$, the presheaf $\mathscr{A}(-, A)$ lies in $\Sigma^{\perp}$. Since $L_\omega\colon \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ is left adjoint, Σ is closed under copowers, so it suffices to check that for any $f_0\colon X \to X'$ in Σ, and any morphism $g\colon X \to \mathscr{A}(-, A)$, the unique dotted arrow

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathscr{A}(-,A) \\ {\scriptstyle f_0}\downarrow & \nearrow & \\ X' & & \end{array}$$

exists. Since L_ω preserves colimits, the pushout $f\colon \mathscr{A}(-,A) \to X''$ of f_0 along g lies in Σ, too. It follows that the dotted arrow in the above diagram exists and is unique if and only if there is a unique dotted arrow making

$$\begin{array}{ccc} \mathscr{A}(-,A) & = & \mathscr{A}(-,A) \\ {\scriptstyle f}\downarrow & \nearrow & \\ X'' & & \end{array}$$

commutative. By assumption, there is a weight $J\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ in the class $\mathbf{\Phi}$ and a functor $G\colon \mathscr{D} \to \mathscr{A}$ such that $X'' \cong J \star YG$, so we might as well assume that f is a morphism from $\mathscr{A}(-,A)$ to $J \star YG$.

Step 1: Existence. By the remarks in Section 5.7 we have a chain of isomorphisms

$$\omega(A) \xrightarrow{\alpha_\omega} L_\omega YA \xrightarrow{L_\omega f} L_\omega(J \star YG) \xrightarrow{\widehat{L_\omega}^{-1}} J \star L_\omega YG \xrightarrow{J\star\alpha_\omega G} J \star \omega G\ ,$$

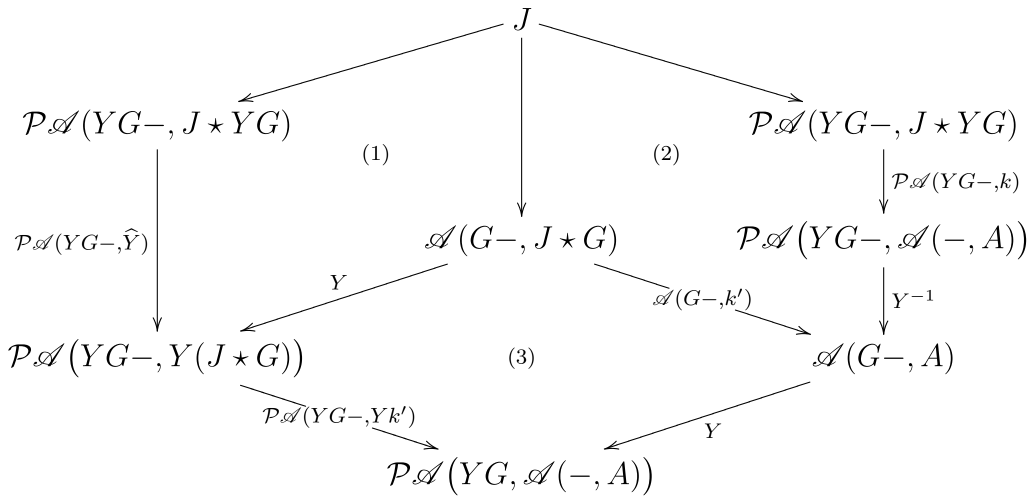
which shows that $J \star \omega G$ lies in $\overline{\mathscr{B}}$. By iii) it follows that the colimit $J \star G \in \mathscr{A}$ exists, and that $\widehat{\omega}\colon J \star \omega G \to \omega(J \star G)$ is an isomorphism. The fact that $\alpha_\omega\colon \omega \Rightarrow L_\omega Y$ is an isomorphism implies that $\widehat{L_\omega Y}\colon J \star L_\omega G \to L_\omega Y(J \star G)$ is an isomorphism. An easy application of the Yoneda Lemma and the definition of the comparison morphism (see Section 5.7) show that $\widehat{L_\omega Y} = L_\omega(\widehat{Y}) \circ \widehat{L_\omega}$, and it follows that $L_\omega(\widehat{Y})$ is an isomorphism. Since the Yoneda embedding is full, the composite $\widehat{Y} \circ f\colon \mathscr{A}(-,A) \to \mathscr{A}(-, J \star G)$ is of the form $Yh = \mathscr{A}(-,h)$ for a unique morphism $h\colon A \to J \star G$. Note that $L_\omega Yh = L_\omega(\widehat{Y}) \circ L_\omega(f)$ is an isomorphism, by the above argument and by assumption on f. Since $\omega \cong L_\omega Y$ it follows that $\omega(h)$ is an isomorphism, and condition i) implies that h itself is an isomorphism. Thus

$$\begin{array}{ccc} \mathscr{A}(-,A) & = & \mathscr{A}(-,A) \\ {\scriptstyle f}\downarrow & & \\ J * YG & \nearrow{\scriptstyle \mathscr{A}(-,h^{-1})} & \\ {\scriptstyle \widehat{Y}}\downarrow & & \\ \mathscr{A}(-, J \star G) & & \end{array}$$

gives the desired lift.

Step 2: Uniqueness. We use the notation introduced above. We have to show that for any $k\colon J \star YG \to \mathscr{A}(-,A)$ with $kf = \mathrm{id}$, the equality $k = Yh^{-1} \circ \widehat{Y}$ holds. This follows from the fact that the target $\mathscr{A}(-,A)$ of k lies in the image of the Yoneda embedding, and within that image, $Y(J \star G)$ *is* the colimit of YG weighted by J, which means that $\mathscr{A}(-,A)$ cannot detect whether or not $\widehat{Y}\colon J \star YG \to Y(J \star G)$ is an isomorphism. We explain this more carefully. From Section 5.7 we know that there is a unique morphism

$k' \colon J \star G \to A$ such that part (2) of the diagram



is commutative, where the unlabeled arrows denote the respective units. Part (1) is commutative by definition of $\widehat{Y}$ (see Section 5.7), and part (3) is commutative by functoriality of the Yoneda embedding. It follows that $Yk' \circ \widehat{Y} = k$, and therefore that $Yk' \circ Yh = Yk' \circ \widehat{Y} \circ f = kf = \mathrm{id}$. But h is an isomorphism, hence we must have $k' = h^{-1}$ and $k = Yk' \circ \widehat{Y} = Y(h^{-1}) \circ \widehat{Y}$, which shows that the lift is unique. ∎

5.9. We now take a closer look at the different conditions in Theorem 5.8. From Proposition 2.13 it follows that iii) is a necessary condition, and from Theorem 2.11 we know that i) is a necessary condition, too. On the other hand, in the proof of Theorem 5.8 we have seen that under the given conditions, the inclusion functor $K \colon \mathscr{A} \to \mathcal{P}\mathscr{B}_{L_\omega \widetilde{\omega}}$ of the category of Cauchy comodules of $L_\omega \widetilde{\omega}$ in the category of all comodules is a dense functor (cf. [Kel05], Proposition 5.16). But this implies that the right adjoint $\widetilde{K} \colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{A}$ is fully faithful. Therefore the counit ε^K is an isomorphism, so by Theorem 4.4 it follows that the $L_\omega \widetilde{\omega}$-component of the counit of the Tannakian adjunction is an isomorphism. Hence condition ii) is not a necessary condition. Also, this condition is probably the most difficult to check. In some cases it might be easier to check whether L_ω preserves *all* finite limits. If this is the case, ω is called a *flat* functor (cf. [Kel82], Section 6).

6. Applications to a special class of cosmoi

6.1. When working with additive categories enriched over the category of modules for some commutative ring, the notion of weighted colimits is generally not needed. For example, any presheaf is a conical colimit of representable functors. In this section we show that this is related to the fact that the category of modules over a commutative ring has a small **Set**-dense subcategory. Let $\mathscr{V}$ be a monoidal category for which the full subcategory $\mathscr{V}^c$ of objects with duals is **Set**-dense. More generally, we assume that there is a full monoidal subcategory $\mathscr{X}$ of $\mathscr{V}^c$ which is **Set**-dense. Of course, this implies that

$\mathscr{V}^c$ is **Set**-dense, too, but it is sometimes easier to work with a smaller category. If $\mathscr{V}$ is the category of modules over some commutative ring, we could for example take $\mathscr{X}$ to be the full subcategory of finitely generated free modules.
To each $\mathscr{V}$-category $\mathscr{A}$ which has copowers with objects in $\mathscr{X}$ we can associate an ordinary category endowed with an action of $\mathscr{X}_0$. Such a category is called a $\mathscr{X}_0$-*actegory* (in [McC00]) or $\mathscr{X}_0$-*representation* (in [GP97]). Since we assume that $\mathscr{X}_0$ is **Set**-dense, the assignment which sends a $\mathscr{V}$-category $\mathscr{A}$ which has copowers with objects in $\mathscr{X}$ to the $\mathscr{X}_0$-actegory $\mathscr{A}_0$, with action given by $- \odot -: \mathscr{X}_0 \times \mathscr{A}_0 \to \mathscr{A}_0$ is in fact a fully faithful 2-functor (see [GP97], Theorem 3.4). This means that giving a $\mathscr{V}$-functor $F: \mathscr{A} \to \mathscr{A}'$ between $\mathscr{V}$-categories which have copowers with objects in $\mathscr{X}$ is the same as giving an ordinary functor $F_0: \mathscr{A}_0 \to \mathscr{A}'_0$, together with morphisms $\widehat{F}: X \odot F_0 A \to F_0(X \odot A)$ making the diagrams

$$\begin{array}{ccc} F_0A & \xrightarrow{l'} & I \odot F_0A \\ & \searrow^{F_0 l} & \downarrow{\scriptstyle \widehat{F}} \\ & & F_0(I \odot A) \end{array} \quad \text{and} \quad \begin{array}{ccc} X \odot (X' \odot F_0A) & \xrightarrow{a'} & (X \otimes X') \odot F_0A \\ \downarrow{\scriptstyle X \odot \widehat{F}} & & \\ X \odot F_0(X' \odot A) & & \downarrow{\scriptstyle \widehat{F}} \\ \downarrow{\scriptstyle \widehat{F}} & & \\ F_0\big(X \odot (X' \odot A)\big) & \xrightarrow[F_0 a]{} & F_0\big((X \otimes X') \odot A\big) \end{array}$$

commutative. The arrows l and a correspond to the canonical isomorphisms $\mathrm{id} \cong [I, -]$ and $[X, [X', -]] \cong [X \otimes X', -]$ under the $\mathscr{V}$-natural isomorphisms which define the respective copowers, and $\widehat{F}$ is given by the map of the same name introduced in Section 5.7. Moreover, we know that copowers with objects in $\mathscr{X}$ are absolute colimits (see [Str83]), so the morphisms $\widehat{F}: X \odot FA \to F(X \odot A)$ are isomorphisms. Still under the assumption that $\mathscr{X}_0$ is **Set**-dense and that $\mathscr{A}$, $\mathscr{A}'$ have copowers with objects in $\mathscr{X}$, giving a $\mathscr{V}$-natural transformation $\alpha: F \Rightarrow F': \mathscr{A} \to \mathscr{A}'$ is the same as giving an ordinary natural transformation $\alpha: F_0 \Rightarrow F'_0$ such that

$$\begin{array}{ccc} X \odot F_0A & \xrightarrow{X \odot \alpha_A} & X \odot F'_0A \\ \downarrow{\scriptstyle \widehat{F}} & & \downarrow{\scriptstyle \widehat{F'}} \\ F_0(X \odot A) & \xrightarrow[\alpha_{X \odot A}]{} & F'_0(X \odot A) \end{array}$$

is commutative.

6.2. Proposition. *Assume there is a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is* **Set***-dense in $\mathscr{V}_0$. Let $\mathscr{A}$ be a $\mathscr{V}$-category which has copowers with objects in $\mathscr{X}$, and let $\mathscr{C}$ be a $\mathscr{V}$-category which has powers*[7] *with arbitrary objects of $\mathscr{V}$. Under these assumptions, a $\mathscr{V}$-functor $K: \mathscr{A} \to \mathscr{C}$ is $\mathscr{V}$-dense if and only if the underlying ordinary functor $K_0: \mathscr{A}_0 \to \mathscr{C}_0$ is* **Set***-dense.*

[7] Powers are the dual notion of copowers.

PROOF. The assumption that powers with arbitrary arbitrary objects exist in $\mathscr{C}$ implies that K is $\mathscr{V}$-dense if and only if the map $\mathscr{C}_0(C,D) \to \mathscr{V}\text{-Nat}\big(\mathscr{C}(K-,C),\mathscr{C}(K-,D)\big)$ which sends $g\colon C \to D$ to the $\mathscr{V}$-natural transformation $\mathscr{C}(K-,g)$ is a bijection of sets (see [Kel05], Section 5.1). For $C \in \mathscr{C}$, let $(K \downarrow C)$ be the category with objects the morphisms $\varphi\colon KA \to C$, $A \in \mathscr{A}$, and morphisms $\varphi \to \varphi'$ the morphisms in $\mathscr{A}_0$ which make the evident triangle commutative. From [Kel05], Formula 5.4, we know that K_0 is **Set**-dense if and only if each object C is the colimit of the canonical cocone on the functor $V_C\colon (K \downarrow C) \to \mathscr{C}$ which sends φ to its domain. We write S_D for the set of cocones on V with vertex D, and we let $V = \mathscr{V}_0(I,-)\colon \mathscr{V}_0 \to \mathbf{Set}$ be the canonical forgetful functor. Let $\chi\colon \mathscr{V}\text{-Nat}\big(\mathscr{C}(K-,C),\mathscr{C}(K-,D)\big) \to S_D$ be the map which sends α to the cocone $\big(V\alpha_A(\varphi)\big)_{\varphi\in(K\downarrow C)}$. The composite

$$\mathscr{C}_0(C,D) \longrightarrow \mathscr{V}\text{-Nat}\big(\mathscr{C}(K-,C),\mathscr{C}(K-,D)\big) \xrightarrow{\chi} S_D$$

sends g to the cocone $(g\varphi)_{\varphi\in(K\downarrow C)}$. This composite is a bijection if and only if C is the colimit of the canonical cocone, i.e., if and only if K_0 is **Set** dense. If we can show that χ is a bijection, then K_0 is **Set**-dense if and only if K is $\mathscr{V}$-dense, as claimed.
We now construct an inverse for χ, as follows. Given a cocone $\gamma = (\gamma_\varphi)_{\varphi\in(K\downarrow C)}$, we let

$$\beta_A\colon \mathscr{C}_0(KA,C) \to \mathscr{C}_0(KA,D)$$

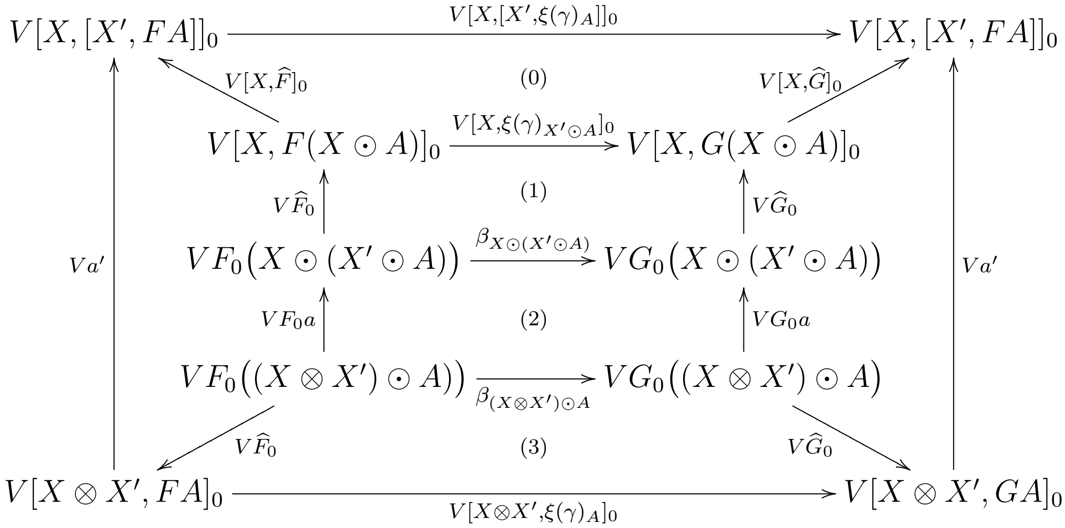
be the map with $\beta_A(\varphi) = \gamma_\varphi$. We write $F,G\colon \mathscr{A} \to \mathscr{V}^{\mathrm{op}}$ for the functors $\mathscr{C}(K-,C)$ and $\mathscr{C}(K-,D)$ respectively. Note that we have $VF_0 = \mathscr{C}_0(KA,C)$, and β is a natural transformation between the **Set**-valued functors VF_0 and VG_0. We first use the density assumption to lift this to a natural transformation $\xi(\gamma)\colon F_0 \to G_0$ between the underlying ordinary $\mathscr{V}_0$-valued functors F and G, and we then show that $\xi(\gamma)$ is in fact $\mathscr{V}$-natural. Since the copower of B by X in $\mathscr{V}^{\mathrm{op}}$ is given by $[X,B]$, and because all $\mathscr{V}$-functors preserve copowers with objects in $\mathscr{V}^c$ (see [Str83]), we get isomorphisms

$$F(X\odot A) \xrightarrow{\widehat{F}} [X,FA] \quad \text{and} \quad G(X\odot A) \xrightarrow{\widehat{G}} [X,GA]\,,$$

and the composite $V\widehat{G}_0 \circ \beta_{X\odot A} \circ V\widehat{F}_0^{-1}\colon V[X,\mathscr{C}(KA,C)]_0 \to V[X,\mathscr{C}(KA,D)]_0$ is natural in X. Since $\mathscr{V}_0(X,-)$ is naturally isomorphic to $V[X,-]_0$, it follows by **Set**-density of $\mathscr{V}^c$ in $\mathscr{V}$ that there is a unique morphism $\xi(\gamma)_A\colon \mathscr{C}(KA,C) \to \mathscr{C}(KA,D)$ in $\mathscr{V}$ such that

$$\begin{array}{ccc}
\mathscr{V}_0\big(X,\mathscr{C}(KA,C)\big) & \xrightarrow{\mathscr{V}_0(X,\xi(\gamma)_A)} & \mathscr{V}_0\big(X,\mathscr{C}(KA,D)\big) \\
\downarrow{\cong} & & \downarrow{\cong} \\
V[X,\mathscr{C}(KA,C)]_0 & \xrightarrow{V[X,\xi(\gamma)_A]_0} & V[X,\mathscr{C}(KA,D)]_0 \\
\downarrow{V\widehat{F}_0^{-1}} & & \downarrow{V\widehat{G}_0^{-1}} \\
\mathscr{C}_0\big(K(X\odot A),C\big) & \xrightarrow[\beta_{X\odot A}]{} & \mathscr{C}_0\big(K(X\odot A),D\big)
\end{array}$$

is commutative for every $X \in \mathscr{V}^c$. Hence part (1) and (3) of the diagram



are commutative. Part (2) is commutative since β is natural, and the two pentagons are instances of the coherence diagrams in Section 6.1. The outer diagram is commutative because a' is natural, hence it follows that part (0) is commutative. Since $\mathscr{V}^c$ is dense we find that $[X',\xi(\gamma)_A]\circ\widehat{F} = \widehat{G}\circ\xi(\gamma)_{X'\odot A}$. The considerations in Section 6.1 therefore imply that $\xi(\gamma)$ is a $\mathscr{V}$-natural transformation. The second coherence diagram of Section 6.1 implies that $V\xi(\gamma)_A = \beta_A$ for all objects $A \in \mathscr{A}$, and it follows that $\chi\big(\xi(\gamma)\big) = \gamma$. Moreover, if we start with a $\mathscr{V}$-natural transformation $\alpha\colon F \Rightarrow G$ and construct the β_A associated to the cocone $\chi(\alpha)$, we clearly get $\beta_{X\odot A} = V\alpha_{X\odot A}$, i.e., $V\xi\big(\chi(\alpha)\big)_{X\odot A} = V\alpha_{X\odot A}$. Both α and $\xi(\gamma)$ are $\mathscr{V}$-natural, hence we must have $V[X,\xi\big(\chi(\alpha)\big)_A]_0 = V[X,\alpha_A]_0$, and by density of $\mathscr{X}$ it follows that $\xi\big(\chi(\alpha)\big) = \alpha$. In other words, the assignment which sends a cocone γ to the $\mathscr{V}$-natural transformation $\xi(\gamma)$ constructed above gives the desired inverse to χ. ∎

6.3. Corollary. *Assume there is a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is* **Set***-dense in $\mathscr{V}_0$. Let $\mathscr{A}$ be a $\mathscr{V}$-category which has copowers with objects in $\mathscr{X}$. Fix a presheaf $F \in \mathcal{P}\mathscr{A}$, and let $(Y \downarrow F)$ be the category of representable functors over F. Then F is the conical colimit of the forgetful functor $(Y \downarrow F) \to \mathcal{P}\mathscr{A}$.*

Proof. Since the Yoneda embedding is always $\mathscr{V}$-dense and because $\mathcal{P}\mathscr{A}$ is complete, it follows by Proposition 6.2 that F is the ordinary colimit of the canonical cocone on the forgetful functor $(Y \downarrow F) \to \mathcal{P}\mathscr{A}_0$. But $\mathcal{P}\mathscr{A}$ has arbitrary powers, hence the notion of conical colimit and ordinary colimit coincide. ∎

6.4. Theorem. *Assume there is a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is* **Set***-dense in $\mathscr{V}_0$. Let $\mathscr{B}$ be a $\mathscr{V}$-category with small Cauchy completion. Then the T-component of the counit of the Tannakian adjunction is an isomorphism if and only if for each $B \in \mathscr{B}$, (TB,δ_B) is the conical colimit of the diagram of Cauchy comodules over (TB,δ_B).*

Proof. We use the notation from Theorem 4.4. By Proposition 2.13 it follows that the category $\mathscr{A} = \mathcal{P}\mathscr{B}_T^c$ has copowers with objects in $\mathscr{X}$ (note that for $B \in \overline{\mathscr{B}}$, $X \in \mathscr{X}$, we have $X \odot B \in \overline{\mathscr{B}}$ because $\mathscr{X}$ consists of objects with duals). By Corollary 6.3, the presheaf $F = \mathcal{P}\mathscr{B}_T\big(K-, (TB, \delta_B)\big)$ on $\mathscr{A}$ is the colimit of the diagram of representable functors over F. Since $L_K(\mathscr{A}(-, A)) \cong KA$ and because L_K preserves colimits, we find that $L_K\widetilde{K}(TB, \delta_B)$ is the colimit of the diagram $V_{(TB,\delta_B)} \colon \big(K \downarrow (TB, \delta_B)\big) \to \mathcal{P}\mathscr{B}_T$ of Cauchy comodules over (TB, δ_B). It is not hard to check that the comparison morphism induced by the canonical cocone is precisely the (TB, δ_B)-component of ε^K, so the conclusion follows from Theorem 4.4. ∎

6.5. Theorem. *Assume that there is a full monoidal subcategory $\mathscr{X} \subseteq \mathscr{V}$ consisting of objects with duals such that $\mathscr{X}_0$ is* **Set***-dense in $\mathscr{V}_0$, and assume that $\mathscr{V}_0(I, -) \colon \mathscr{V} \to$* **Set** *preserves filtered colimits. Let $\mathscr{A}$ be a $\mathscr{V}$-category which has copowers with objects in $\mathscr{X}$ and with their duals, let $\mathscr{B}$ be a $\mathscr{V}$-category, and let $\omega \colon \mathscr{A} \to \mathcal{P}\mathscr{B}$ be a $\mathscr{V}$-functor. Then the unit of the Tannakian adjunction is an equivalence if*

i) The functor ω reflects isomorphisms,

ii) For each object $B \in \mathscr{B}$, the category $\mathrm{el}\big(V(\mathrm{ev}_B\omega)_0\big)$ of elements of $V(\mathrm{ev}_B\omega)_0 \colon \mathscr{A}_0 \to$ **Set** *is cofiltered, where $\mathrm{ev}_B \colon \mathcal{P}\mathscr{B} \to \mathscr{V}$ denotes the evaluation functor, and*

iii) The functor ω detects and preserves those conical colimits which lie in $\overline{\mathscr{B}}$.

If these conditions are satisfied, then the comonad $L_\omega\widetilde{\omega} \colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$ preserves finite limits.

Proof. We use the notation from Proposition 3.5 and Theorem 5.8. First note that $\mathscr{V}_0$ is locally finitely presentable: For $X \in \mathscr{X}$ we have

$$\mathscr{V}_0(X, -) \cong \mathscr{V}_0(I, [X, -]) \cong \mathscr{V}_0(I, X^\vee \otimes -),$$

which preserves filtered colimits by our assumption on the unit object I. But $\mathscr{X}_0$ is **Set**-dense, so it is in particular a strong generator, which shows that $\mathscr{V}_0$ is indeed locally finitely presentable. It follows that the Cauchy completion $\overline{\mathscr{B}}$ of $\mathscr{B}$ is small (see [Joh89]), hence it makes sense to speak about the Tannakian adjunction in this context (cf. Proposition 3.5). Moreover, Theorem 5.8 can be applied. Condition i) coincides with condition i) in Theorem 5.8, and if we let $\mathbf{\Phi}$ be the class of conical weights, then Corollary 6.3 and iii) imply that condition iii) of Theorem 5.8 is satisfied. Therefore it suffices to check that ii) implies that $L_\omega \mathcal{P}\mathscr{A} \to \mathcal{P}\mathscr{B}$ preserves the necessary coequalizers. We will in fact show that L_ω preserves all *finite* limits (see [Kel82], Section 4 for a definition).
To see this, we first note that the functor

$$(\mathrm{ev}_B)_{B\in\mathscr{B}} \colon \mathcal{P}\mathscr{B} \to \prod_{B\in\mathscr{B}} \mathscr{V}$$

preserves and reflects all limits and all colimits, because both limits and colimits in $\mathcal{P}\mathscr{B}$ are computed pointwise. Hence it suffices to check that for a fixed $B \in \mathscr{B}$, the functor

$\mathrm{ev}_B L_\omega \colon \mathcal{P}\mathscr{A} \to \mathscr{V}$ preserves finite limits, i.e., that $\mathrm{ev}_B L_\omega$ is *left exact* in the terminology of [Kel82]. Since the functor $(\mathrm{ev}_B)_{B\in\mathscr{B}}$ preserves all colimits, it also preserves left Kan extensions (see [Kel05], Proposition 4.14). Thus $\mathrm{ev}_B L_\omega$ is naturally isomorphic to the left Kan extension of $\mathrm{ev}_B\omega$ along the Yoneda embedding $Y \colon \mathscr{A} \to \mathcal{P}\mathscr{A}$. This reduces the problem to showing that $\mathrm{ev}_B\omega$ is *flat*, and by [Kel82], Section (6.3) it suffices to check that $\mathrm{ev}_B\omega \in [\mathscr{A}, \mathscr{V}]$ is a filtered (conical) colimit of representable functors.
Note that $\mathscr{A}^{\mathrm{op}}$ has copowers with objects in $\mathscr{X}$: Since any $X \in \mathscr{X}$ has a dual $X^\vee$, and because copowers with objects in $\mathscr{X}$ are preserved by any functor (see [Str83]), we have natural isomorphisms

$$\mathscr{A}(B, X^\vee \odot A) \cong X^\vee \otimes \mathscr{A}(B, A) \cong [X, \mathscr{A}(B, A)],$$

which shows that $\mathscr{A}$ has powers with objects in $\mathscr{X}$ (equivalently, that $\mathscr{A}^{\mathrm{op}}$ has copowers with objects in $\mathscr{X}$). Proposition 6.2, applied to the $\mathscr{V}$-category $\mathscr{A}^{\mathrm{op}}$ and the contravariant Yoneda embedding $Y' \colon \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathscr{V}]$, shows that Y' is **Set**-dense. It follows that $\mathrm{ev}_B\omega$ is isomorphic to the conical colimit of the forgetful functor $(Y' \downarrow \mathrm{ev}_B\omega) \to [\mathscr{A}, \mathscr{V}]$. By the weak Yoneda lemma, the category $(Y' \downarrow \mathrm{ev}_B\omega)$ is isomorphic to the opposite of the category of elements of $V(\mathrm{ev}_B\omega)_0$. The latter is filtered by assumption ii), hence the remarks in [Kel82], Section (6.3) show that $\mathrm{ev}_B\omega$ is indeed flat. The second statement follows immediately: The functor $\widetilde{\omega}$ is right adjoint, so it preserves all limits, and we have just shown that L_ω preserves finite limits. ∎

6.6. Corollary. *Let R be a commutative ring, B an R-algebra, and let $\mathscr{V}$ be the category* Mod_R *of R-modules. Let $\mathscr{A}$ be an additive R-linear category, equipped with an R-linear functor $\omega \colon \mathscr{A} \to \mathrm{Mod}_B$ such that $\omega(A)$ is finitely generated and projective for all $A \in \mathscr{A}$. If ω reflects isomorphisms, detects and preserves those (ordinary) colimits which are finitely generated projective, and if the category $\mathrm{el}(\omega)$ of elements of ω is cofiltered, then the $(\mathscr{A}, \omega)$-component of the unit of the Tannakian adjunction is an equivalence of categories. Moreover, the comonad $L_\omega\widetilde{\omega} \colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$ preserves finite limits.*

Proof. We let $\mathscr{X}$ be the full monoidal subcategory of Mod_R consisting of finitely generated free modules. Since $\mathscr{A}$ is additive, it has copowers with objects in $\mathscr{X}$: The copower of $A \in \mathscr{A}$ with R^n is simply the n-fold direct sum $\oplus_{i=1}^n A$. Moreover, the dual of R^n is isomorphic to R^n, which shows that $\mathscr{A}$ has copowers with objects in $\mathscr{X}$ and with their duals. The Cauchy completion of B, considered as a one-object $\mathscr{V}$-category, is the full subcategory of Mod_B consisting of finitely generated projective modules. Thus the image of $\omega(A)$ is an object of the Cauchy completion of B, and it makes sense to speak of the $(\mathscr{A}, \omega)$-component of the unit of the Tannakian adjunction (cf. Proposition 3.5). Since the forgetful functor $V \colon \mathrm{Mod}_R \to \mathbf{Set}$ is conservative (i.e., it reflects isomorphisms), the conical colimit of a functor exists if and only if the corresponding ordinary colimit exists (see [Kel05], Formula (3.4)). Hence all the conditions in Theorem 6.5 are satisfied. ∎

We conclude with a few remarks about the necessity of the conditions in Theorem 6.5. More precisely, we are interested in the following question: Under what assumptions on the cosmos $\mathscr{V}$, the small category $\mathscr{B}$ and the comonad $T \colon \mathcal{P}\mathscr{B} \to \mathcal{P}\mathscr{B}$ does the functor

$V_T\colon \mathcal{P}\mathscr{B}_T^c \to \mathcal{P}\mathscr{B}$ satisfy the conditions in Theorem 6.4? Since this theorem is a consequence of Theorem 5.8, the general remarks from Section 5.9 apply: Conditions i) and iii) hold without any further assumptions. If $\overline{\mathscr{B}} \subseteq \mathcal{P}\mathscr{B}$ is closed under finite limits and if $T \in \mathbf{Comon}\,({}_{\mathscr{B}}\mathcal{M}_{\mathscr{B}})$ preserves finite limits, then the functor $V_T\colon \mathcal{P}\mathscr{B}_T^c \to \mathcal{P}\mathscr{B}$ satisfies condition ii) of Theorem 6.4. Indeed, it is well known that $\mathcal{P}\mathscr{B}_T$ has finite limits under these conditions, and that V_T preserves them. Since $\overline{\mathscr{B}}$ is closed under finite limits, so is $\mathcal{P}\mathscr{B}_T^c$, and it follows that the category of elements of $V(\mathrm{ev}_B V_T)_0$ is cofiltered for every $B \in \mathscr{B}$. In the special case of Corollary 6.6, where $\mathscr{V} = \mathrm{Mod}_R$ for some commutative ring R and $\mathscr{B}$ is an R-algebra B, seen as a one-object $\mathscr{V}$-category, the category $\overline{\mathscr{B}}$ is simply the category of finitely generated projective B-modules. It is closed under finite limits if B is a *hereditary* Noetherian algebra, i.e., if submodules of finitely generated projective B-modules are again finitely generated projective.

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