Pro-objects in $\mathcal{C}$

Let $\mathcal{I}$ be a cofiltered category - it must satisfy
(i) $\forall i, j\ \exists k,\ k \xrightarrow{\alpha} i, k \xrightarrow{\beta} j$
and
(ii) $\forall i \xrightarrow{\alpha,\beta} j \exists k \xrightarrow{\gamma} i, \alpha \circ \gamma = \beta \circ \gamma$.

A pro-object in $\mathcal{C}$ is a functor $\mathcal{I} \xrightarrow{F} \mathcal{C}$.
These pro-objects form the objects of category $pro - \mathcal{C}$.

Morphisms in $pro - \mathcal{C}$, from objects $\mathcal{I} \xrightarrow{F} \mathcal{C}$ to $\mathcal{J} \xrightarrow{G} \mathcal{C}$, are of the following form:
The set $\mathrm{Hom}\left(F\left(i\right), G\left(j\right)\right)$ is contravariantly indexed in i and covariantly indexed in j. Take FIRST the colimit in i and SECOND the limit in j to get the set $\mathrm{Hom}\left(\mathcal{I} \xrightarrow{F} \mathcal{C}, \mathcal{J} \xrightarrow{G} \mathcal{C}\right)$.

So, this is non-empty iff $\forall j\ \exists i,\ \exists F\left(i\right) \xrightarrow{f} G\left(j\right)$.

And an element $\mathbf{f}$ of $\mathrm{Hom}\left(\mathcal{I} \xrightarrow{F} \mathcal{C}, \mathcal{J} \xrightarrow{G} \mathcal{C}\right)$ (a 'pro-morphism') is a set of equivalence classes $\{F\left(i\right) \xrightarrow{\mathbf{f}_j} G\left(j\right)\}$ indexed over all j, where $F\left(i\right) \xrightarrow{F(\alpha)} F\left(i'\right) \xrightarrow{\mathbf{f}_j} G\left(j\right)$ is equivalent to $F\left(i\right) \xrightarrow{\mathbf{f}_j \circ F(\alpha)} G\left(j\right)$, such that for $j \xrightarrow{\beta} j'$, assuming f_j and $f_{j'}$ have the same domain choice i (which can do b/c $\mathcal{I}$ is cofiltered), $G\left(\beta\right) \circ f_j = f_{j'}$.

Composition of pro-morphisms: given pro-objects $\mathcal{I} \xrightarrow{F} \mathcal{C}$, $\mathcal{J} \xrightarrow{G} \mathcal{C}$, $\mathcal{K} \xrightarrow{H} \mathcal{C}$, and pro-morphisms $\mathcal{I} \xrightarrow{F} \mathcal{C} \overset{\mathbf{f}}{\Rightarrow} \mathcal{K} \xrightarrow{H} \mathcal{C}$ and $\mathcal{J} \xrightarrow{G} \mathcal{C} \overset{\mathbf{g}}{\Rightarrow} \mathcal{K} \xrightarrow{H} \mathcal{C}$, define $\mathbf{h} := \mathbf{g} \circ \mathbf{f}$ as follows. For each $k \in \mathcal{K}$ have some $G\left(j\right) \xrightarrow{\mathbf{g}_k} H\left(k\right)$ and some $F\left(i\right) \xrightarrow{\mathbf{f}_j} G\left(j\right)$; define $\mathbf{h}_k := \mathbf{g}_k \circ \mathbf{f}_j$. By the equivalence relation and by definition of lim and colim, this is independent of the choice of j and of i, and also satisfies the necessary composition rules.

Identity pro-morphism: Given pro-object $\mathcal{I} \xrightarrow{F} \mathcal{C}$, its identity map is the pro-morphism $\mathbf{f}$ consisting of identity maps $F\left(i\right) \xrightarrow{\mathbf{f}_i := id_{F(i)}} F\left(i\right)$ for each $i \in \mathcal{I}$.

Pro-homotopy

Let $\mathcal{C}$ be the category of finite topological spaces, with morphisms being continuous maps. We want to define what it means for two pro-morphisms $\mathcal{I} \xrightarrow{F} \mathcal{C} \overset{\mathbf{f},\mathbf{g}}{\Longrightarrow} \mathcal{J} \xrightarrow{G} \mathcal{C}$ to be pro-homotopic.

Let $\mathcal{N}$ be the indexing category of the nonnegative integers, with a unique morphism $n \to m$ iff $n \geq m$. Define $\mathcal{N} \xrightarrow{Z} \mathcal{C}$ as sending n to the finite space $\{a_0, b_1, a_1, b_2, \ldots b_n, a_n\}$ with relations $a_i > b_{i-1}, b_i$ for all i (a contractible 'zigzag') and for $n \geq m$, $Z(n \to m)$ collapses $\{b_i, a_i \mid i > m\}$ to the point a_m.

This gives a pro-object $\mathcal{I} \times \mathcal{N} \xrightarrow{F \times Z} \mathcal{C}$. Define two pro-morphisms $\mathcal{I} \xrightarrow{F} \mathcal{C} \Rightarrow \mathcal{I} \times \mathcal{N} \xrightarrow{F \times Z} \mathcal{C}$, $\mathbf{i}^0$ and $\mathbf{i}^1$. (Note that as $\mathcal{N}$ is cofiltered, so is $\mathcal{I} \times \mathcal{N}$).

For $i \times n \in \mathcal{I} \times \mathcal{N}$ choose $\mathbf{i}^0_{i \times n}$ to be inclusion of $F(i)$ to $F(i) \times a_0 \subset F(i) \times Z(n)$. Choose $\mathbf{i}^1_{i \times n}$ to be inclusion of $F(i)$ to $F(i) \times a_n \subset F(i) \times Z(n)$. (Note that all composition rules hold).

Now define a pro-homotopy between $\mathbf{f}$ and $\mathbf{g}$ to be a pro-morphism $\mathcal{I} \times \mathcal{N} \xrightarrow{F \times Z} \mathcal{C} \overset{\mathbf{h}}{\Rightarrow} \mathcal{J} \xrightarrow{G} \mathcal{C}$ such that $\mathbf{h} \circ \mathbf{i}^0 = \mathbf{f}$ and $\mathbf{h} \circ \mathbf{i}^1 = \mathbf{g}$.

Connection to fundamental group of finite spaces

For a finite topological space X, let $\mathcal{J} = \{*\}$ and let $\mathcal{J} \xrightarrow{G} \mathcal{C}$ map the point to X. This gives a pro-object representing X. Let $\mathcal{I} = \mathcal{N}$ as above and let $\mathcal{I} \xrightarrow{F} \mathcal{C}$ send n to $Y^{(n)}$ and send morphisms to ζ as in my notes on fundamental groups of A-spaces.

Now consider pro-homotopy classes of the pro-morphisms $\mathcal{I} \xrightarrow{F} \mathcal{C} \Rightarrow \mathcal{J} \xrightarrow{G} \mathcal{C}$. The resulting set is isomorphic to $\pi_1(X)$ (by the work in those notes, etc ... need to fill in the details here).