SUBDIVISIONS OF SMALL CATEGORIES

NOTES FOR REU BY J.P. MAY

Let $\mathscr{A}$ be a (small) category. For example, monoids (sets with associative and unital products) can be identified with categories with a single object. Analogously, posets can be identified with those categories $\mathscr{A}$ with at most one arrow between any two objects by defining $x \leq y$ if there is an arrow $x \longrightarrow y$ between the objects x and y of $\mathscr{A}$.

In particular, we write $[n]$ for the poset $\{0 < 1 \cdots < n\}$. If we think of $[n]$ as a finite space, then a continuous map $f\colon [m] \longrightarrow [n]$ is a monotonic, or non-decreasing, function, $f(i) \leq f(j)$ if $i \leq j$. We can equally well regard such a function as a functor $[m] \longrightarrow [n]$. We are perilously close to defining the fundamental notion of a simplicial object in a possibly large category $\mathscr{C}$, so let's do so.

Let Δ denote the category of posets $[n]$ and monotonic maps between them. This category is generated by certain canonical monotonic maps. We have the "face map" $\delta_i\colon [n] \longrightarrow [n+1]$ which is the monomorphism that misses i. That is $\delta_i(j) = j$ if $j < i$ and $\delta_i(j) = j+1$ if $j \geq i$. We also have the surjection $\sigma_i\colon [n+1] \longrightarrow [n]$ that hits i twice. That is, $\sigma_i(j) = j$ if $j \leq i$ and $\sigma_i(j) = j-1$ if $j > i$. Every morphism in Δ is a composite of these morphisms, and they satisfy certain easily determined identities. A simplicial object in $\mathscr{C}$ is a *contravariant* functor $\Delta \longrightarrow \mathscr{C}$. In detail, it is a sequence of objects $C_n \in \mathscr{C}$ together with face maps $d_i\colon C_n \longrightarrow C_{n-1}$ and degeneracy maps $s_i\colon C_n \longrightarrow C_{n+1}$, $0 \leq i \leq n$, such that

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \quad i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if} \quad i < j \\ \mathrm{id} & \text{if} \quad i = j \quad \text{or} \quad i = j+1 \\ s_j \circ d_{i-1} & \text{if} \quad i > j+1. \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \quad \text{if} \quad i \leq j.$$

When $\mathscr{C} = Set$ is the category of sets, we obtain $sSet$, the category of simplicial sets. For example, if S is a topological space, we obtain the simplicial set SX such that S_nX is the set of all continuous maps from the standard topological n-simplex Δ_n to X. Explicitly,

$$\Delta_n = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1, \ \textstyle\sum t_i = 1\} \subset \mathbb{R}^{n+1}.$$

We have the "face maps"

$$\delta_i : \Delta_{n-1} \longrightarrow \Delta_n, \quad 0 \leq i \leq n,$$

specified by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and "degeneracy maps"

$$\sigma_i : \Delta_{n+1} \longrightarrow \Delta_n, \quad 0 \leq i \leq n,$$

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specified by

$$\sigma_i(t_0,\ldots,t_{n+1}) = (t_0,\ldots,t_{i-1},t_i+t_{i+1},\ldots,t_{n+1}).$$

Precomposing with these maps, we obtain the maps d_i and s_i that make SX into a simplicial set. It has long been known that we can use simplicial sets pretty much interchangeably with topological spaces when studying homotopy theory.

For a simplicial set K, we define a space $|K|$, called the "geometric realization" of K, as follows. As a set

$$|X| = \textstyle\coprod_{n\geq 0}(K_n \times \Delta_n)/(\sim),$$

where the equivalence relation $\sim$ is generated by

$$(k,\delta_i u) \sim (d_i(k),u) \quad \text{for} \quad k\in K_n \quad \text{and} \quad u\in\Delta_{n-1}$$

and

$$(k,\sigma_i v) \sim (s_i(k),v) \quad \text{for} \quad K\in K_n \quad \text{and} \quad v\in\Delta_{n+1}.$$

Topologize $|K|$ by giving

$$|K|^n \equiv \textstyle\coprod_{0\leq n\leq q}(K_n\times\Delta_n)/(\sim)$$

the quotient topology and then giving $|K|$ the topology of the union, so that a subset is closed if it intersects each $|K|^n$ in a closed subset. Write $|k,u|$ for points of $|K|$. Say that (k,u) is nondegenerate if $k\in K_n$ is not of the form $s_i j$ for any i and any $j\in K_{n-1}$ and if $u\in\Delta_n$ is an interior point. Every (k,u) is equivalent to one and only one nondegerate point.

Define $\gamma : |SX| \longrightarrow X$ by

$$\gamma|f,u| = f(u) \quad \text{for} \quad f:\Delta_n\longrightarrow X \quad \text{and} \quad u\in\Delta_n.$$

It is a fact that γ is a weak homotopy equivalence for every space X, although we shall not prove that. There is also a map $\iota\colon K\longrightarrow S|K|$ of simplicial sets specified by $\iota(k)(u) = |k,u|$ for $k\in K_n$ and $u\in\Delta_n$. Again, as we also shall not prove, $|\iota|\colon |K|\longrightarrow |S|K||$ is a homotopy equivalence.

There is a neat relationship between $|-|$ and S. They are left and right adjoint functors, meaning that there is a bijection, natural in both variables between morphism sets:

$$\mathrm{Top}(|K|,X)\cong \mathrm{sSet}(K,SX).$$

It is specified by letting f correspond to g if $f(|k,u|) = g(k)(u)$.

There is also a construction that assigns a simplicial *set* K^s to a simplicial *complex* K. The idea is to allow repeated elements in the sets of simplices. It is easiest to define $\mathscr{K}(X)^s$ for a poset X, and then the construction gives a functor, but one can use any arbitrarily chosen ordering of the set of vertices to apply the definition more generally. One lets the set $\mathscr{K}(X)^s_n$ of n-simplices be the set of sequences $x_0\leq x_1\leq\cdots\leq x_n$. Deleting x_i gives d_i, and repeating x_i gives s_i. The geometric realization gives the geometric realization $|\mathscr{K}(X)^s|$. When X is finite this is homeomorphic to any choice of geometric realization as we defined it earlier, but the definition $|\mathscr{K}(X)| = |\mathscr{K}(X)^s|$ works in general and gives a functor of X.

With this as background, we turn to the homotopy theory of small categories $\mathscr{A}$. We construct a simplicial set $N\mathscr{A}$ called the nerve of $\mathscr{A}$. Regarding $[n]$ as a category, we define the set $N_n\mathscr{A}$ of n-simplices to be the set of functors $[n]\longrightarrow\mathscr{A}$. Regarding a monotonic function $f\colon[m]\longrightarrow[n]$ as a functor, precomposition with f gives us the required contravariant functoriality on Δ. The definition should look very similar to the definition of the total singular functor S from spaces to simplicial sets. It gives us a functor N from $\mathscr{C}at$, the category of small categories and funtors between them, to simplicial sets. We define $B\mathscr{A} = |N\mathscr{A}|$. This is called the *classifying space* of the category $\mathscr{A}$. When G is a group regarded as a category with a single object, BG is called the classsifying space of the group G. These are fundamentally important constructions in topology and its applications.

The nerve functor N is accompanied by a functor $c\colon sSets \longrightarrow \mathscr{C}at$. It is left adjoint to N, meaning that

$$\mathscr{C}at(cK, \mathscr{A}) \cong \mathrm{sSet}(K, N\mathscr{A}).$$

This means that it is conceptually sensible, but it does not have good homotopical properties. For a simplicial set K, the objects of the category cK are the vertices (= 0-simplices) of K. To construct the morphisms, one starts by thinking of the 1-simplices y as maps $d_1y \longrightarrow d_0y$. One forms all words (formal composites) that make sense, that is, whose targets and sources match up. Then one imposes the relations on morphisms determined by

$$s_0x = \mathrm{id}_x \quad \text{for} \quad x \in K_0 \quad \text{and} \quad d_1z = d_0z \circ d_2z \quad \text{for} \quad z \in K_2.$$

This makes good sense since if $K = N\mathscr{A}$, then a 0-simplex is an object x of $\mathscr{A}$, a 1-simplex y is a map $d_1y \longrightarrow d_0y$ and $s_0x = \mathrm{id}_x$, and a 2-simplex z is given by a pair of composable morphisms d_2z and d_0z together with their composite d_1z. Therefore there is a natural map $cN\mathscr{A} \longrightarrow \mathscr{A}$ that is the identity on zero simplices and is induced by the identity on 1-simplices. In fact, it is an isomorphism of categories: it is the identity on objects, and it presents the category in terms of generators given by the morphism sets modulo relations determined by the category axioms. There is also a natural map $K \longrightarrow NcK$. Applied to the boundary of an n-simplex $\Delta[n]$, it is the inclusion $\partial\Delta[n] \longrightarrow \Delta[n]$ when $n > 2$, as we shall see. For the adjunction, a functor $F\colon cK \longrightarrow \mathscr{A}$ is constructed from a map of simplicial sets $g\colon K \longrightarrow N\mathscr{A}$ by letting F be the unique functor that agrees with g on objects (= 0-simplices) and equivalence classes of morphisms (= 1-simplices).

Backing up, we define the standard simplicial n-simplex $\Delta[n]$ to be the simplicial set whose q-simplices are the monotonic functions $\sigma\colon [q] \longrightarrow [n]$; precomposition with monotonic functions $\xi\colon [p] \longrightarrow [q]$ gives the required contravariant functoriality on Δ. The nondegenerate q-simplices in $\Delta[n]$ are the monomorphisms (= strictly monotonic functions) $[q] \longrightarrow [n]$, and there is one for each subset of $[n]$ of cardinality $q + 1$. We may identify the set of all non-degenerate simplices with the poset of non-empty subsets of the set $[n]$ of $n + 1$ elements, ordered by inclusion. In other words, $\Delta[n] = (\mathscr{K}([n])^s$ is the simplicial set determined by the simplicial complex $\mathscr{K}([n])$. A monotonic function $\alpha\colon [m] \longrightarrow [n]$ gives a map $\alpha\colon \Delta[m] \longrightarrow \Delta[n]$ of simplicial sets that sends $\sigma\colon [q] \longrightarrow [m]$ to $\alpha \circ \sigma$. Thus $\Delta[-]$ is a covariant functor from Δ to simplicial sets.

The n-skeleton K^n of a simplicial set K is the subsimplicial set generated by the q-simplices for all $q \leq n$. Visibly, cK depends only on the 2-skeleton K^2. Therefore the inclusion $K^2 \longrightarrow K$ of simplicial sets induces an isomorphism of categories $cK^2 \longrightarrow cK$ for any K. For Δ_n this implies that $\partial\Delta_n \longrightarrow \Delta_n$ induces an isomorphism on application of c when $n > 2$. What is truly amazing is that this extreme loss of information disappears after subdividing twice. This is something I am trying to better understand myself.

The reader will find it easy to believe that there is a subdivision functor on simplicial sets that generalizes the subdivision functor Sd on simplicial complexes in the sense that $(SdK)^s \cong Sd(K^s)$ for a simplicial complex K This allows one to define a subdivision functor on categories by setting $Sd\mathscr{A} = cSdN\mathscr{A}$. One can iterate subdivision, forming functors Sd^2 on both simplicial sets and categories. What is mind blowing at first is that the iterated subdivision $Sd^2\mathscr{A}$ is actually a poset whose classifying space $BSd^2\mathscr{A}$ is homotopy equivalent to $B\mathscr{A}$. I will explain at least the construction in a slow way to try to make the idea transparent.

For a set C and a simplicial set L, one can form a new simplicial set $C \times L$ by letting $(C \times L)_q = C \times L_q$, and similarly letting the faces and degeneracies be induced by those of L. A simplicial set K can be reconstructed from the disjoint union over n of the simplicial sets $K_n \times \Delta[n]$ for $n \geq 0$ by taking equivalence classes under the equivalence relation generated by

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \tag{1}$$

for $k \in K_n$, $\sigma \in \Delta[m]_q$, and $\alpha\colon [m] \longrightarrow [n]$. Here $\alpha^*(k) \in K_m$ is given by the fact that K is a contravariant functor from Δ to sets and $\alpha_*(\sigma) \in \Delta[n]_q$ is given by the fact that $\Delta[-]$ is a covariant functor from Δ to simplicial sets. The simplicial structure is induced from the simplicial structure on the $\Delta[n]$. The point is that an arbitrary pair (k, τ) in $K_n \times \Delta[n]_q$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]_q$, where $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical q-simplex in $\Delta[q]$, and $\tau\colon [q] \longrightarrow [n]$ is viewed as a morphism of Δ, so that $\tau = \tau_*(\iota_q)$. Identifying equivalence classes of q-simplices with elements of K_q in this faction, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p).$$

We define $Sd\Delta[n] = (\mathscr{K}'[n])^s$. That is, we take the simplicial set associated to the barycentric subdivision of the simplicial complex $\mathscr{K}[n]$, where we again regard $[n]$ as a poset. Just like $\Delta[-]$, this gives a covariant functor $Sd\Delta[-]$ from Δ to simplicial sets. We use it in exactly the same way as above to construct SdK as a quotient of the disjoint union of the $\mathscr{K}[n] \times Sd\Delta[n]$ by the equivalence relation of the same form as (1). This is obviously sensible! We then define $Sd\mathscr{A} = cSdN\mathscr{A}$.

————————- (Notes below)

It is classical that we have a canonical equivalence $SdK \longrightarrow K$. Applying this with $K = N\mathscr{A}$ gives a canonical (realization) equivalence $SdN\mathscr{A} \longrightarrow N\mathscr{A}$. Checking that the functor c, preserves such equivalences (if it does) and using that $c \circ N \cong \mathrm{Id}$, this gives a canonical equivalence $Sd\mathscr{A} \longrightarrow \mathscr{A}$. We claim that a little more explicitness shows that $SdSd\mathscr{A}$ is a poset. That is, want to reinterpret directly in terms of categories. (I haven't had time yet).

Historical fact: Don W. Anderson claimed in passing that the last claim is easily verified, in his 1978 paper referred to in del Hoya's miserable attempt at a paper.

More accurately and conceptually, $SdK = K \otimes_\Delta Sd\Delta[-]$, as we have defined it above, can be rewritten as $K \otimes_\Delta \Delta(-,-)$. Here the category Δ is giving a cosimplicial simplicial set, namely the represented bifunctor $\Delta(-,-)$. We have the simplicial sets $\Delta[n]([q]) = \Delta([q],[n])$, one for each fixed n. Fixing q and letting n vary, we have covariant functors. In defining the tensor product we use the covariant functoriality, but then the simplicial structure on the result is induced by the contravariant functoriality. In categorical situations like this there is an evident associativity isomorphism of simplicial sets

$$(K \otimes_\Delta L) \otimes_\Delta M \cong K \otimes_\Delta (L \otimes_\Delta M)$$

where K is simplicial and L and M are cosimplicial simplicial. Inductively, this implies that

$$Sd^n K \cong K \otimes_\Delta Sd^n \Delta[-].$$

This gives a good hold on these functors, since $Sd^n\Delta[-] = (\mathscr{K}^{(n)}\Delta[-])^s$ is just the classical iterated barycentric subdivision, regarded as a simplicial set.

Can we say anything useful about $c(N\mathscr{A} \otimes_\Delta L)$ in general?

Can we simplify the proof (almost impossible to dig out of the literature) that $Sd^2 K$ comes from a classical simplicial complex?