1 A sketch of the idea

Start with a quasi-category K that is not a Kan complex, and a non-degenerate simplex $y \in K_2$.

L will be the simplicial set generated by having two copies of y in K, y_1 and y_2. This means that L_n will consist of copies of simplices in K_n. If x does not have y as a face, there will be only one copy of x; if x has y as one face of itself there will be two copies of x; etc.

Specifically, for $x \in K_n$, look for injective maps $[2] \overset{\alpha}{\hookrightarrow} [n]$ such that $\alpha^* x = y$. For a given copy of x in L_n, x', we can choose either $\alpha^* x' = y_1$ or $\alpha^* x' = y_2$, for each such α. Therefore it makes sense to have a distinct copy of x for each such set of choices; that is, the number of copies of x in L is set to be $2^{|\{[2] \overset{\alpha}{\hookrightarrow} [n] \ | \ \alpha^* x = y\}|}$, and for each copy, we keep track of these choices.

Faces and degeneracies are defined in such a way to make these choices consistent.

2 Define L

Fix a quasi-category K that is not a Kan complex, and a non-degenerate simplex $y \in K_2$.

For $n \geq 0$, define $\mathcal{M}_n := \{[2] \overset{\alpha}{\hookrightarrow} [n]\}$, a subset of $\mathbf{\Delta}([2], [n])$.

Define a new simplicial set L as follows. Define $L_n = \{x \times S \in K_n \times \mathcal{P}(\mathcal{M}_n) \ | \ \forall \alpha \in S, \ \alpha^* x = y\}$.

And for any map $[m] \overset{\phi}{\to} [n]$, define $\phi^*(x \times S) := (\phi^* x) \times (\phi^* S)$, where $\phi^* S := \{\alpha \in \mathcal{M}_m \ | \ \phi \circ \alpha \in S\}$.

(Trivially, this agrees with identities and compositions, and so this gives a contravariant functor $\mathbf{\Delta} \to \mathcal{S}et$ defining L).

3 Maps between K and L

I will construct two maps of simplicial sets, $K \xrightarrow{i} L$ and $L \xrightarrow{r} K$, such that $r \circ i = id_K$.

For $x \in K_n$, define $i(x) := x \times \emptyset \in L_n$. For any $[m] \xrightarrow{\phi} [n]$ we have $\phi^* i(x) = \phi^*(x \times \emptyset) = (\phi^* x) \times (phi^* \emptyset) = (\phi^* x) \times \emptyset = i(\phi^* x)$. Thus i is a natural transformation of functors $\Delta \to \mathcal{S}et$ and is therefore a map of simplicial sets.

For $x \times S \in L_n$, define $r(x \times S) := x$. For any $[m] \xrightarrow{\phi} [n]$ we have $\phi^* r(x \times S) = \phi^* x = r((\phi^* x) \times (\phi^* S)) = r(\phi^*(x \times S))$. Therefore r is a map of simplicial sets.

Trivially, $r \circ i = id_K$.

4 L is not the nerve of a category

Define two distinct maps $\Delta_2 \to L$ via $\iota_2 \mapsto y \times \emptyset$ and $\iota_2 \mapsto y \times \{id_{[2]}\}$. (Here ι_2 is the generating 2-simplex of Δ_2, i.e. the map $id_{[2]}$). These two maps are equal when restricted to Λ_2^1, therefore fillers of inner horns are not unique and so L is not the nerve of a category.

5 L is not a Kan complex

By assumption, K is not a Kan complex. Therefore there is some $\Lambda_n^k \xrightarrow{f} K$, for $k \in \{0, n\}$, such that there is no extension of f, $\Delta_n \xrightarrow{\tilde{f}} K$.

Suppose $g := i \circ f : \Lambda_n^k \to L$ has some extension $\Delta_n \xrightarrow{\tilde{g}} L$. Then set $\tilde{f} := r \circ \tilde{g} : \Delta_n \to K$. It follows that $\tilde{f}|_{\Lambda_n^k} = r \circ \left(\tilde{g}|_{\Lambda_n^k}\right) = r \circ g = r \circ i \circ f = f$.

This contradicts the assumption; therefore no such $\tilde{g}$ can exist. Therefore L is not a Kan complex.

6 L is a quasi-category

Given $\Lambda_n^k \xrightarrow{f} L$ with $0 < k < n$, we want to find an extension $\Delta_n \xrightarrow{\tilde{f}} L$. This is equivalent to finding some n-simplex $y \times S$ such that for any $[m] \overset{\beta}{\hookrightarrow} [n]$ in Λ_n^k, $\beta^* (y \times S) = f(\beta)$.

(Here the notation is that the m-simplices, or m-faces, of Δ_n are maps $[m] \overset{\beta}{\hookrightarrow} [n]$, and so Δ_n is generated by its unique non-degenerate n-simplex, $\iota_n = id_{[n]}$).

Case 1: $n = 2$**.**
Given $\Lambda_2^1 \xrightarrow{f} L$, let $g := r \circ f$, let $\Lambda_2^1 \xrightarrow{\tilde{g}} K$ be an extension of g in K, and let $\tilde{f} := i \circ \tilde{g}$.

For any m-face β in Λ_2^1, $\tilde{f}(\beta) = g(\beta) \times \emptyset$ (since $\tilde{g}|_{\Lambda_2^1} = g$ and $\tilde{f} = i \circ \tilde{g}$), and $f(\beta) = g(\beta) \times S$ for some $S \in \mathcal{P}(\mathcal{M}_m)$ (since $g = r \circ f$)).

But $m < 2$, so $M_m = \emptyset$, and therefore $S = \emptyset$ and $\tilde{f}(\beta) = f(\beta)$.

Therefore $\tilde{f}$ is an extension of f as desired.

Case 2: $n = 3$**.**
I will show that a map from the inner horn can be extended to a map $\partial\Delta_3 \to L$; the extension to $\Delta_3 \to L$ will then proceed exactly as in Case 3 below.

Given $\Lambda_3^k \xrightarrow{f} L$, let $g := r \circ f$, let $\Lambda_3^k \xrightarrow{\tilde{g}} K$ be an extension of g in K, and let $y = \tilde{g}(\alpha) \in K_2$, where $[2] \xrightarrow{\alpha} [3]$ is the kth 2-face of Δ_3.

Define $\partial\Delta_3 \xrightarrow{\tilde{f}} L$ by setting $\tilde{f}|_{\Lambda_3^k} = f$ and $\tilde{f}(\alpha) = y \times \emptyset$. We need to check that $\tilde{f}$ is well-defined, i.e. need to check that for any proper face $[m] \xrightarrow{\beta} [3]$of α (i.e. $\beta = \gamma \circ \alpha$), $f(\beta) = \gamma^* (y \times \emptyset)$.

We know that $r \circ f(\beta) = g(\beta) = \tilde{g}(\beta) = \gamma^* \tilde{g}(\alpha) = \gamma^* y$, and that $r(\gamma^* (y \times \emptyset)) = \gamma^* y$ also. Furthermore since $m < 2$, $\mathcal{P}(\mathcal{M}_m) = \{\emptyset\}$, so therefore $f(\beta) =$

$\gamma^* (y \times \emptyset) = \gamma^* y \times \emptyset$, giving the desired equality.

Case 3: $n \geq 4$**.**

Given $\Lambda_n^k \xrightarrow{f} L$, let $g := r \circ f$, let $\Lambda_n^k \xrightarrow{\tilde{g}} K$ be an extension of g in K, and let $y = \tilde{g} (\iota_n) \in K_n$.

For any $[2] \overset{\alpha}{\hookrightarrow} [n] \in M_n$, note that α is a face of Λ_n^k. Since $M_2 = \{id_{[2]}\}$, either $f (\alpha) = g (\alpha) \times \emptyset$ or $f (\alpha) = g (\alpha) \times \{id_{[2]}\}$; in the first case set $z_\alpha := 0$, in the second case set $z_\alpha := 1$.

Define $S := \{[2] \overset{\alpha}{\hookrightarrow} [n] \in M_n \mid z_\alpha = 1\}$.

I claim that $y \times S$ is the desired n-simplex, i.e. that defining $\tilde{f} (\iota_n) := y \times S$ gives an extension of f.

To prove this, we need to check that for any $[m] \overset{\beta}{\hookrightarrow} [n]$ in Λ_n^k, $\beta^* (y \times S) = f (\beta)$. By definition, $\beta^* (y \times S) = (\beta^* y) \times (\beta^* S) = g (\beta) \times \beta^* S$. Write $f (\beta) = g (\beta) \times T$ for some $T \in \mathcal{P} (\mathcal{M}_m)$. We want to show that $T = \beta^* S$.

Suppose $[2] \overset{\alpha}{\hookrightarrow} [m] \in T$. Then since $f \left([2] \overset{\alpha}{\hookrightarrow} [n] \overset{\beta}{\to} [n] \right) = \alpha^* f \left([m] \overset{\beta}{\hookrightarrow} [n] \right)$, we have $z_{\beta \circ \alpha} = 1$. Therefore $\beta \circ \alpha \in S$ and so $\alpha \in \beta^* S$.

Conversely suppose $\alpha \in \beta^* S$. Then $\beta \circ \alpha \in S$, therefore $z_{\beta \circ \alpha} = 1$, and therefore $[2] \overset{\alpha}{\hookrightarrow} [m] \in T$.

Therefore $T = \beta^* S$ for any $[m] \overset{\beta}{\hookrightarrow} [n] \in \Lambda_n^k$, and so $\tilde{f}$ is an extension of f as desired.