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THE UNIQUENESS OF HOMOLOGY FOR THE CATEGORY OF SMALL CATEGORIES

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1. Introduction

Oberst [15], Laudal [8], Watts [20] and André [1] have shown that derived functors of colimit define a homology theory for $\mathscr{C}at$, the category of small categories. In this paper, we show that, up to natural isomorphism, there is only one such homology theory.

The homotopic (as different from homotopy; see Section 2) category of the functor category $\mathscr{K}$ of (semi-) simplicial sets is shown to be equivalent to the "corresponding" homotopic category of $\mathscr{C}at$, by constructing a weak homotopy (WH) inverse $\Gamma : \mathscr{K} \to \mathscr{C}at$ of the nerve functor $N : \mathscr{C}at \to \mathscr{K}$. This is done by defining natural transformations

$$\eta'' : N\Gamma \dot{\to} 1_{\mathscr{K}}$$
$$\eta' : \Gamma N \dot{\to} 1_{\mathscr{C}at}$$

such that the morphisms corresponding to each object are equivalences in the respective homotopic categories. This enables us to use the uniqueness of homology in $\mathscr{K}$ (see [2], [12], [13], [5; Appendix II]) to prove uniqueness of homology for $\mathscr{C}at$. The functor $\Gamma^{op} : \mathscr{K} \to \mathscr{C}at$ occurs in a construction of Segal [19].

The paper is organized as follows. In Section 2, we review some basic constructions and definitions, including the notions of weak homotopy in $\mathscr{C}at$ and the homotopic category of $\mathscr{C}at$. Γ is shown to be the WH inverse of N in Section 3: that $N\Gamma$ is weak homotopy equivalent (WHE) to 1 follows from the fact that $N\Gamma$ satisfies the hypothesis of the subdivision theorem (Theorem 1), and that ΓN is WHE to 1 follows easily from the definition of WHE in $\mathscr{C}at$. Section 4 contains axioms for a homology theory on $\mathscr{C}at$ and an existence theorem for such a homology theory, while Section 5 does precisely the same thing for homology in $\mathscr{K}$. In Section 6, a homology theory on $\mathscr{C}at$ is shown to be unique by comparing such a theory with *the* unique homology theory on $\mathscr{K}$ [2]. The comparison uses both functors $N : \mathscr{C}at \to \mathscr{K}$ and $\Gamma : \mathscr{K} \to \mathscr{C}at$. Appendix I contains an outline of the proof

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of the subdivision theorem, and Appendix II consists of a comparison of WHE Axiom for the unique homology theory on $\mathscr{K}$ with the usual form of the Homotopy Axiom.

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2. Preliminaries

The following Kan-type construction is used in several contexts (see [5; II, 1.3]).

Lemma A. *Let $\mathscr{S}$ be the category of sets, $\mathscr{C}$ a cocomplete (i.e., arbitrary colimits exist) category, $\mathbf{D}$ a small category, and $\theta : \mathbf{D} \to \mathscr{C}$ a functor. Then there exists an adjoint pair $\hat{\theta} \dashv S_\theta$, where*

$$S_\theta : \mathscr{C} \to (\mathbf{D}^{\mathrm{op}}, \mathscr{S})$$

is the θ-singular functor *from $\mathscr{C}$ to the functor category $(\mathbf{D}^{\mathrm{op}}, \mathscr{S})$ defined by $S_\theta(A) = \mathscr{C}(\theta_-, A)$ for $A \in \mathrm{ob}\mathscr{C}$; and the* θ-realization functor

$$\hat{\theta} : (\mathbf{D}^{\mathrm{op}}, \mathscr{S}) \to \mathscr{C}$$

is its left adjoint.

Let Δ be the small skeletal category whose objects are the finite ordinals and morphisms are order-preserving functions [5; II, 2]. The functor category $(\Delta^{\mathrm{op}}, \mathscr{S})$ of simplicial sets is denoted by $\mathscr{K}$.

In particular, when $\mathscr{C} = \mathscr{K}$ and $\mathbf{D} = \Delta$ in Lemma A, $\hat{\theta}$ has the following explicit description [7]: Let X be a simplicial set. The k-simplices of $\hat{\theta}X$ are equivalence classes $[x, u]$ of pairs (x, u) with $x \in X$ and $u \in (\theta[\dim x])_k$, where the equivalence relation is generated by

$$(0) \qquad (x, (\theta\alpha)v) \sim (X(\alpha)x, v).$$

The boundaries and degeneracies act on the second component.

The Milnor geometric realization functor $|\ | : \mathscr{K} \to \mathscr{T}\mathrm{op}$ [12], $\mathscr{T}\mathrm{op}$ the cocomplete category of CW complexes, can be viewed as a realization functor in a "Lemma A" situation. If $\tau : \Delta \to \mathscr{T}\mathrm{op}$ is given by $\tau([k]) = \Delta^k$, the standard k-dimensional affine simplex in $\mathbf{R}^{k+1}$, for each $[k] \in \mathrm{ob}\Delta$, then $|\ | : \mathscr{K} \to \mathscr{T}\mathrm{op}$ is the left adjoint of the singular complex functor $S_\tau : \mathscr{T}\mathrm{op} \to \mathscr{K}$ [12].

Another "Lemma A" situation arises from the full inclusion functor $\iota : \Delta \to \mathscr{C}\mathrm{at}$ from Δ to cocomplete category $\mathscr{C}\mathrm{at}$, where each finite ordinal $[k]$ is itself considered as the small category with objects $\{0, 1, 2, \ldots k\}$ and a unique morphism $u \to v$ for each pair $u \leqslant v$. The *nerve functor* $N : \mathscr{C}\mathrm{at} \to \mathscr{K}$ is the ι-singular adjoint of categorical realization $c : \mathscr{K} \to \mathscr{C}\mathrm{at}$. In fact, $cN \equiv 1$ [5; II, 4]. For each small category $\mathbf{C}$, $N\mathbf{C}$ is the simplicial set whose k-simplices, $(N\mathbf{C})_k$, are diagrams in $\mathbf{C}$ of the form

(1) $$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \longrightarrow \cdots \longrightarrow p_{k-1} \xrightarrow{a_k} p_k.$$

The ith-face (resp. degeneracy) of this k-dimensional simplex is obtained by deleting the objects p_i (resp. replacing p_i by $1 : p_i \to p_i$) in the evident way. In particular, since $\iota : \Delta \to \mathscr{C}\mathrm{at}$ is full and faithful,

(2) $$N[k] = \mathscr{C}\mathrm{at}\,(\iota\_, [k]) = \Delta\,(\_, [k]) = \Delta\,[k],$$

where $\Delta\,[k] = \Delta(\_, [k])$ is the standard k-dimensional representable simplicial set having one non-degenerate k-simplex and having all faces and degeneracies generated freely. Since $\mathscr{K}$ is the functor category $(\Delta^{\mathrm{op}}, \mathscr{S})$, the Yoneda Lemma insures that each simplicial set can be realized as a colimit, or specifically as a coend, of these representable simplicial sets [10; IX, 6], i.e., for each $X \in \mathrm{ob}\mathscr{K}$,

(3) $$X = \int^k \Delta\,[k] \cdot X_k.$$

Define the *representable functor* $R : \Delta \to \mathscr{K}$ by $R[k] = \Delta[k]$. Let $B = |N\_| : \mathscr{C}\mathrm{at} \to \mathscr{T}\mathrm{op}$ denote the Segal classifying space functor [18]; then $B\mathbf{C}$ is the CW complex whose k-cells are in one-to-one correspondence with nondegenerate simplices of $N\mathbf{C}$, i.e., those for which no arrow of (1) is an identity.

Similarly, the functor $\gamma : \Delta \to \mathscr{C}\mathrm{at}$ defined as a "comma category" functor $\gamma([k]) = (\Delta \downarrow [k])^{\mathrm{op}}$, gives rise to another pair of adjoint functors; $S = S_\gamma : \mathscr{C}\mathrm{at} \to \mathscr{K}$ and $\Gamma = \hat{\gamma} : \mathscr{K} \to \mathscr{C}\mathrm{at}$. For $X \in \mathrm{ob}\mathscr{K}$, ΓX is the small category of simplices of X : an object is a pair $([k], x)$ with $x \in X_k$, and a morphism $([k], x) \to ([m], y)$ is a map $\mu : [m] \to [k]$ in Δ such that $X(\mu)x = y$.

The two functors $\gamma : \Delta \to \mathscr{C}\mathrm{at}$ and $\iota : \Delta \to \mathscr{C}\mathrm{at}$ are related by a natural transformation "first", $\eta : \gamma \dot{\to} \iota$; i.e., for each $[k]$, $\eta_k : \gamma([k]) \to \iota([k])$ is the functor defined by $\eta_k(\alpha : [p] \to [k]) = \alpha(0) \in [k]$. Using adjoint functor theory and using the theory of coends, we describe in Section 3 how η induces natural transformations $\eta' : \Gamma N \dot{\to} cN \equiv 1$ and $\eta'' : N\Gamma \dot{\to} 1$. These natural transformations will be used in showing that $\Gamma : \mathscr{K} \dot{\to} \mathscr{C}\mathrm{at}$ is a WH inverse of $N : \mathscr{C}\mathrm{at} \to \mathscr{K}$.

Strong homotopy (SH) in $\mathscr{K}$ is the equivalence relation generated by: $f \sim g$ iff there is a simplicial map $h : X \times \Delta\,[1] \to Y$ such that $h \circ u_0 = f$ and $h \circ u_1 = g$, where $u_i = 1 \times N(\delta^i)$. ($\delta^i : [0] \to [1]$ is the injection which skips the ith place, $i = 0, 1$.) Since the Milnor geometric realization is compatible with products [12], the canonical map $|X \times \Delta\,[1]| \to |X| \times \mathbf{I}$ is a homeomorphism because $|\Delta\,[1]| = \mathbf{I}$ is a finite complex. Hence $|\ | : \mathscr{K} \to \mathscr{T}\mathrm{op}$ preserves strong homotopies. However, Milnor's geometric realization is *not* full, even when factored through the *homotopy category* $\mathscr{K}/\sim$ of simplicial sets; i.e., $|f| : |X| \to |Y|$ may be a homotopy equivalence (HE) in $\mathcal{T}$op, but no simplicial homotopy inverse exists (see [4], [16]). Gabriel and Zisman [5; IV] define the *homotopic category* $\mathcal{K}'$ of simplicial sets to be the category of fractions in $\mathcal{K}/\sim$ of the set of anodyne extensions (see [5; IV] or Appendix II) and show that $\mathcal{K}'$ is isomorphic, via the Milnor geometric realization functor, to a *full* subcategory of the homotopy category $\mathcal{T}$op$/\sim$ of CW complexes [5; VII, 1]. We use this expanded homotopy relation: $f, g \in \mathcal{K}(X, Y)$ are *weakly homotopic* (WH) iff $|f|$ is homotopic to $|g|$ in $\mathcal{T}$op.

Suppose $F, G \in \mathcal{C}at(\mathbf{C}, \mathbf{D})$. Each natural transformation $\omega : F \dot{\to} G$ corresponds to a functor $\bar{\omega} : \mathbf{C} \times [1] \to \mathbf{D}$ such that $\bar{\omega} \circ (1 \times \delta^0) = F$ and $\bar{\omega} \circ (1 \times \delta^1) = G$. Since $N : \mathcal{C}at \to \mathcal{K}$ commutes with products, Lee [9] shows that $NF \sim NG$ whenever such a natural transformation exists. Furthermore, because $N : \mathcal{C}at \to \mathcal{K}$ is fully faithful, $NF \sim NG$ in $\mathcal{K}$ implies the existence of a natural transformation $\omega : F \dot{\to} G$. Hence, the *strong homotopy* relation in $\mathcal{C}at$, i.e., symmetric transitive closure of natural transformation, corresponds to SH in $\mathcal{K}$. For example, whenever a small category $\mathbf{C}$ has an initial or terminal object t, it is *strongly contractible* (SC); i.e., $\mathbf{C}$ is SHE to $[0]$, the terminal category of $\mathcal{C}at$ which has one object and only the identity morphism. If $J : [0] \to \mathbf{C}$ is the inclusion functor such that $J(0) = t$ and $T : \mathbf{C} \to [0]$ is the terminal functor, then $JT = 1$ and there is a natural transformation $\omega : 1 \dot{\to} JT$.

As in the case of simplicial sets, $BF : B\mathbf{C} \to B\mathbf{D}$ may be a HE in $\mathcal{T}$op without $F : \mathbf{C} \to \mathbf{D}$ being a SHE in $\mathcal{C}at$ (see [16]). The corresponding *homotopic category* $(\mathcal{C}at)'$ is defined to reflect the expanded weak homotopy relation in $\mathcal{K}$: $F, G \in \mathcal{C}at$ are *weakly homotopic* iff NF and NG are equal in $\mathcal{K}'$, or equivalently, iff BF and BG are homotopic in $\mathcal{T}$op.

3. Weak homotopy inverse

Let X be a simplicial set. Then $N\Gamma X$ is the simplicial set having k-dimensional simplices

$$(N\Gamma X)_k = \{\langle([n], x); [n_k] \xrightarrow{\mu_k} [n_{k-1}] \longrightarrow \cdots \longrightarrow [n_1] \xrightarrow{\mu_1} [n]\rangle\},$$

where $\langle([n], x); [n_k] \xrightarrow{\mu_k} [n_{k-1}] \to \cdots \to [n_1] \xrightarrow{\mu_1} [n]\rangle$ represents the diagram in ΓX

$$([n], x) \xrightarrow{\mu_1} ([n_1], X(\mu_1)x) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_k} ([n_k], X(\mu_1\mu_2\cdots\mu_k)x).$$

Boundaries are defined by composition or deletion, and degeneracy maps are given by "insertion" of identities, in the second component

$$[n_k] \xrightarrow{\mu_k} [n_{k-1}] \longrightarrow \cdots \longrightarrow [n_1] \xrightarrow{\mu_1} [n].$$

If $f: X \to Y$ in $\mathcal{K}$, then $N\Gamma f: N\Gamma X \to N\Gamma Y$ is the simplicial map defined by

$$(N\Gamma f)_k \left(\langle([n], x); [n_k] \xrightarrow{\mu_k} [n_{k-1}] \longrightarrow \cdots \longrightarrow [n_1] \xrightarrow{\mu_1} [n]\rangle\right)$$

$$= \langle([n], f_n(x)); [n_k] \xrightarrow{\mu_k} [n_{k-1}] \longrightarrow \cdots \longrightarrow [n_1] \xrightarrow{\mu_1} [n]\rangle.$$

From the explicit description of $N\Gamma: \mathcal{K} \to \mathcal{K}$, it is easily seen that $N\Gamma$ is the θ-realization functor of a "Lemma A" situation when $\mathbf{D} = \Delta$ and $\mathscr{C} = \mathcal{K}$: Define $\theta: \Delta \to \mathcal{K}$ to the composition of $N\Gamma$ with the representable functor $R: \Delta \to \mathcal{K}$. Then $u \in (N\Gamma\Delta[n])_k$ is represented by a sequence $u = (\mu_0, \mu_1, \ldots, \mu_k)$ of morphisms in Δ such that $\mu_0 \circ \mu_1 \circ \mu_2 \circ \cdots \circ \mu_k$ is defined and $\mu_0 \in \Delta[n]$. The equivalence class $[x, u]$ in $\hat{\theta}X$ of the pair (x, u) (see (0)) corresponds to the simplex

$$\langle([n_0], X(\mu_0)x); [n_k] \xrightarrow{\mu_k} [n_{k-1}] \longrightarrow \cdots \longrightarrow [n_1] \xrightarrow{\mu_1} [n_0]\rangle$$

in $N\Gamma X$. The next lemma follows immediately from the fact that left adjoints, in particular θ-realization functors, commute with colimits.

Lemma B. *$N\Gamma: \mathcal{K} \to \mathcal{K}$ is cocontinuous, i.e., $N\Gamma$ commutes with colimits.*

We note that $\Gamma(\Delta[k]) = \Gamma(\Delta(-, [k]))$ is the small category whose objects are $\{\alpha: [p] \to [k] \mid p \geq 0\}$. The morphisms of $\Gamma(\Delta[k])$ can be described as triples $\langle\alpha', \mu, \alpha\rangle$ such that $\alpha'\mu = \Delta(\mu, [k])(\alpha') = \alpha$. Hence $\Gamma(\Delta[k]) = (\Delta \downarrow [k])^{op}$. Clearly, $1 \in \Delta([k], [k])$ is an initial object in $\Gamma(\Delta[k])$ since $\langle 1, \alpha, \alpha\rangle$ is the *only* possible morphism between 1 and α. Thus $\Gamma(\Delta[k])$ is a SC category; and since $N: \mathscr{C}\text{at} \to \mathcal{K}$ preserves strong homotopies, $N\Gamma(\Delta[k])$ is SC as a simplicial set. Hence the following lemma holds.

Lemma C. *For each $k \geq 0$, $\Gamma(\Delta[k])$ is SC in $\mathscr{C}$at and $N\Gamma(\Delta[k])$ is SC in $\mathcal{K}$.*

Although $N\Gamma: \mathcal{K} \to \mathcal{K}$ is a left adjoint, $N\Gamma$ preserves monomorphisms in $\mathcal{K}$. This follows easily from the definition of $\Gamma: \mathcal{K} \to \mathscr{C}\text{at}$ and from the fact that $N: \mathscr{C}\text{at} \to \mathcal{K}$ is a right adjoint.

Lemma D. *$N\Gamma: \mathcal{K} \to \mathcal{K}$ preserves inclusions in $\mathcal{K}$; i.e., if X' is a subsimplicial set of X, then $N\Gamma X'$ is a subsimplicial set of $N\Gamma X$.*

In order to prove that $N\Gamma X$ and X are WHE for every simplicial set X, and that this WHE is natural, we show that there exists a natural transformation $\eta'': N \dot{\to} 1$. Recall that $\langle N, c\rangle$ and $\langle S, \Gamma\rangle$ are adjoint pairs arising from the Lemma A situation using the inclusion functor $\iota: \Delta \to \mathscr{C}\text{at}$ and a "comma category" functor $\gamma: \Delta \to \mathscr{C}\text{at}$, respectively. The natural transformation "first" $\eta: \gamma \dot{\to} \iota$ yields by composition a natural transformation between singular functors

(4) $\eta_1 : N \dot{\to} S.$

Using naturality of adjoint functor theory [10; IV, 7], there exists an induced natural transformation between corresponding left adjoints

(5) $\eta_2 : \Gamma \dot{\to} c.$

Next, composition with $N : \mathscr{C}\mathrm{at} \to \mathscr{K}$ yields a natural transformation

(6) $\eta_3 = N\eta_2 : N\Gamma \dot{\to} Nc$

of endofunctors of $\mathscr{K}$. Composition of η_3 with the representable functor $R : \Delta \to \mathscr{K}$ gives

(7) $\eta_4 : N\Gamma R \dot{\to} R,$

since $NcR([k]) = Nc(N[k]) = N[k] = R[k]$ by (2). Lastly, because $N\Gamma : \mathscr{K} \to \mathscr{K}$ commutes with colimits, η_4 is a natural transformation between representing functors, and each simplicial set is a colimit of representable simplicial sets (see (3)), η_4 extends to the natural transformation

(8) $\eta'' : N\Gamma \dot{\to} 1_{\mathscr{K}}.$

Hence $N\Gamma : \mathscr{K} \to \mathscr{K}$ satisfies the hypotheses of the following theorem.

Theorem 1 (Subdivision Theorem). *Suppose* $\mathrm{Sd} : \mathscr{K} \to \mathscr{K}$ *is a functor such that*

(i) *there is a natural transformation* $\rho : \mathrm{Sd} \cdot R \dot{\to} R$;

(ii) $\mathrm{Sd}(\Delta[k])$ *is contractible for every* $k \geq 0$;

(iii) Sd *commutes with colimits*;

(iv) Sd *preserves inclusions in* $\mathscr{K}$.

Then there exists a unique extension of ρ *to a natural transformation*

$$\rho'' : \mathrm{Sd} \dot{\to} 1_{\mathscr{K}}$$

such that $\rho''_X : \mathrm{Sd}X \to X$ *is a WHE for every simplicial set* X.

Proof. See Appendix I.

Theorem 2. $\eta''_X : N\Gamma X \to X$ *is a WHE for every* $X \in \mathrm{ob}\mathscr{K}$.

Thus $\Gamma : \mathscr{K} \to \mathscr{C}\mathrm{at}$ is a right WH inverse.

Next, by composing $\eta_2 : \Gamma \dot{\to} c$ with $N : \mathscr{C}\mathrm{at} \to \mathscr{K}$ on the right, we get the natural transformation

(9) $\eta' : \Gamma N \dot{\to} cN \equiv 1_{\mathscr{C}\mathrm{at}}.$

Theorem 3. $\eta'_{\mathbf{C}} : \Gamma N\mathbf{C} \to \mathbf{C}$ *is a WHE in* $\mathscr{C}\mathrm{at}$, *for each small category* $\mathbf{C}$.

Proof. By the definition of WHE in $\mathscr{C}\mathrm{at}$, it suffices to show that

$$N(\eta'_c): N\Gamma NC \to NC$$

is a WHE in $\mathscr{K}$. But, from (6), (9), the uniqueness of η'', and the fact that $cN \equiv 1$,

$$N(\eta'_c) \equiv N(\eta_2 N)_c \simeq (N\eta_2)_{NC} \equiv \eta''_{NC}.$$

But Theorem 2 insures that

$$\eta''_{NC}: N\Gamma NC \to NcNC \simeq NC$$

is a WHE in $\mathscr{K}$.
Hence Γ is a WH inverse of N.

Remark. There exist other functors from $\mathscr{K}$ to $\mathscr{C}$at which are WH inverses of N. For example, parallel constructions and arguments for the functor $\rho : \Delta \to \mathscr{C}\text{at}$ defined by $\rho([k]) = \Delta \downarrow [k]$ show that $P \equiv \hat{\rho} : \mathscr{K} \to \mathscr{C}\text{at}$ is also a WH inverse for N. In this case, the original natural transformation $\phi : \rho \to \iota$ picks out the "last" element; i.e., $\phi_k : \rho([k]) \to \iota([k])$ is given by $\phi_k(\alpha : [p] \to [k]) = \alpha(p)$.

4. $\mathscr{C}$at homology axioms and existence

A subcategory $\mathbf{C}'$ of the small category $\mathbf{C}$ is *admissible* if all morphisms of $\mathbf{C}$ with domain in $\mathbf{C}'$ are in $\mathbf{C}'$. Such a $\mathbf{C}'$ is necessarily a full subcategory of $\mathbf{C}$. The category of admissible pairs and obvious morphisms is denoted by $A(\mathscr{C}\text{at})$. A small category $\mathbf{C}$ is identified with the admissible pair $(\mathbf{C}, \emptyset)$, where $\emptyset$ is the empty small category.

Remark. When $\mathbf{P}$ is the category of a poset, an admissible subcategory corresponds to a subposet which is the union of "terminal" segments of $\mathbf{P}$. If $\mathbf{G}$ is a group, then the only admissible subcategories of $\mathbf{G}$ are $\emptyset$ or $\mathbf{G}$ itself.

A *homology theory* for $\mathscr{C}$at is a functor $h : A(\mathscr{C}\text{at}) \to \mathscr{A}\text{b}^Z$ from the category of admissible pairs to the category of graded abelian groups together with a natural transformation ∂ of degree -1, i.e.,

$$\partial_* : h_*(\mathbf{C}, \mathbf{C}') \to h_{*-1}(\mathbf{C}')$$

for every admissible pair $(\mathbf{C}, \mathbf{C}')$, such that a variation of the standard Eilenberg–Steenrod–Milnor axioms ([3], [13]) are satisfied:

(i) *Dimension Axiom.* If $[0]$ denotes the trivial one point category, then

$$h_k([0]) = \begin{cases} A, & \text{if } k = 0 \\ 0, & \text{if } k > 0 \end{cases}$$

where $h_0([0]) = A$ is called the *coefficient group* for the homology theory $\langle h, \partial \rangle$.

(ii) *Exactness Axiom.* Let $i:\mathbf{C}'\to\mathbf{C}$ and $j:\mathbf{C}\to(\mathbf{C},\mathbf{C}')$ be the obvious inclusion morphisms in $A(\mathscr{C}\mathrm{at})$. Then there is a long exact sequence

$$\cdots\longrightarrow h_q(\mathbf{C}')\xrightarrow{h_q(i)}h_q(\mathbf{C})\xrightarrow{h_q(j)}h_q(\mathbf{C},\mathbf{C}')\xrightarrow{\partial_q}h_{q-1}(\mathbf{C}')\longrightarrow\cdots.$$

(iii) *Homotopy Equivalence Axiom* (*WHE Axiom*). If $F:(\mathbf{C},\mathbf{C}')\to(\mathbf{D},\mathbf{D}')$ is a WHE, i.e., BF and $BF/B\mathbf{C}'$ are both HE in $\mathscr{T}\mathrm{op}$, then

$$hF:h(\mathbf{C},\mathbf{C}')\to h(\mathbf{D},\mathbf{D}')$$

is an isomorphism in $\mathscr{A}\mathrm{b}^Z$.

(iv) *Excision Axiom.* Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be admissible in $\mathbf{C}$. Then the inclusion morphism

$$i:(\mathbf{C}_2,\mathbf{C}_1\cap\mathbf{C}_2)\to(\mathbf{C}_1\cup\mathbf{C}_2,\mathbf{C}_1)$$

induces a corresponding isomorphism of graded homology groups, where $\mathbf{C}_1\cap\mathbf{C}_2$ and $\mathbf{C}_1\cup\mathbf{C}_2$ are admissible subcategories of $\mathbf{C}$ making the following square bicartesion in $\mathscr{C}\mathrm{at}$:

$$\begin{array}{ccc}\mathbf{C}_1\cap\mathbf{C}_2 & \longrightarrow & \mathbf{C}_1\\ \downarrow & & \downarrow\\ \mathbf{C}_2 & \longrightarrow & \mathbf{C}_1\cup\mathbf{C}_2.\end{array}$$

(v) *Strong Additivity* (*Milnor*) *Axiom.* Suppose $\{\mathbf{C}_\alpha \mid \alpha\in I\}$ is a collection of small categories. Then the inclusions $\{u_\alpha:\mathbf{C}_\alpha\to\sqcup\,\mathbf{C}_\alpha\}$ induce the isomorphism

$$\oplus h(u_\alpha):\oplus h(\mathbf{C}_\alpha)\to h(\sqcup\mathbf{C}_\alpha)$$

of graded abelian groups.

Remark. The usual Strong Homotopy Axiom (SH Axiom) can be derived from the WHE Axiom [3]. Although there do not appear to be enough strong homotopies, i.e., natural transformations, in $\mathscr{C}\mathrm{at}$, an "abstract nonsense" argument (see Appendix II) shows that the SH Axiom implies the WHE Axiom for the (unique) homology theory defined below.

For each $A\in\mathrm{ob}\mathscr{A}\mathrm{b}$, define $A_{\mathbf{C},\mathbf{C}'}:\mathbf{C}\to\mathscr{A}\mathrm{b}$ by

$$(A_{\mathbf{C},\mathbf{C}'})p=\begin{cases}A, & \text{if } p\notin\mathrm{ob}\mathbf{C}'\\ 0, & \text{otherwise}\end{cases}$$

on objects and in the obvious fashion on morphisms. $\mathbf{C}'$ admissible in $\mathbf{C}$ guarantees that $A_{\mathbf{C},\mathbf{C}'}$ is a well-defined functor.

The following well-known theorem shows that there exists a homology theory for $\mathscr{C}\mathrm{at}$.

Theorem 4 (Existence). *$\langle H,\partial\rangle$ is a homology theory for* $\mathscr{C}\mathrm{at}$, *where*

$$H_*(\mathbf{C},\mathbf{C}';A)=L_*\operatorname{colim}_{\mathbf{C}}(A_{\mathbf{C},\mathbf{C}'}),$$

$L_* \operatorname{colim}_{\mathbf{C}}: \mathscr{A}b^{\mathbf{C}} \to \mathscr{A}b$ *being the left derived functors of* $\operatorname{colim}_{\mathbf{C}}: \mathscr{A}b^{\mathbf{C}} \to \mathscr{A}b$.

Proof. See [8], [15], [20].

5. Homology for simplicial sets

Similarly, $A(\mathscr{K})$ denotes the category of admissible pairs of simplicial sets: an object (X, X') is a pair with X' a subsimplicial set of X and a morphism $f:(X,X')\to(Y,Y')$ is a simplicial map $f: X\to Y$ such that $f(X')\subseteq Y'$. The simplicial set X is identified with the admissible pair $(X,\emptyset)$. A *homology theory* for $\mathscr{K}$ is a pair $\langle h, \partial\rangle$, where $h: A(\mathscr{K})\to \mathscr{A}b^{\mathbf{Z}}$ is a functor and ∂ is a natural transformation of degree -1, i.e.,

$$\partial_*: h_*(X,X')\to h_{*-1}(X')$$

satisfying the Eilenberg–Steenrod–Milnor axioms ([3], [13]):

(i) *Dimension Axiom.* If $\Delta[0]$ is the "one-pointed" representable simplicial set then

$$h_k(\Delta[0])=\begin{cases}A, & \text{if } k=0\\ 0, & \text{if } k>0\end{cases}$$

where $h_0(\Delta[0])=A$ is called the *coefficient group* for $\langle h,\partial\rangle$.

(ii) *Exactness Axiom.* For (X,X') admissible, there is a long exact sequence

$$\cdots\longrightarrow h_q(X')\xrightarrow{h_q(i)} h_q(X)\xrightarrow{h_q(j)} h_q(X,X')\xrightarrow{\partial_q} h_{q-1}(X')\longrightarrow\cdots$$

where i, j are the obvious inclusions.

(iii) *Excision Axiom.* If X', X'' are subcomplexes of X, then the inclusion map

$$i:(X', X''\cap X')\to(X'\cup X'', X'')$$

induces a corresponding isomorphism of graded homology groups.

(iv) *Strong Homotopy Axiom (SH Axiom).* If $f,g:(X,X')\to(Y,Y')$ are SH, i.e., f is SH to g and f/X' is SH to g/X', then

$$h(f)=h(g): h(X,X')\to h(Y,Y').$$

(v) *Strong Additivity (Milnor) Axiom.* Suppose X is the disjoint union of $\{X_\alpha \mid \alpha\in I\}$ with inclusions $u_\alpha: X_\alpha\to X$. Then

$$\oplus\, h(u_\alpha) : \oplus\, h(X_\alpha) \to h(X)$$

is an isomorphism in $\mathscr{A}\mathrm{b}^{\mathbf{Z}}$.

As in the case of the homotopy axiom for $\mathscr{C}$at, the SH Axiom is equivalent to a WHE Axiom (see Appendix II).

(iv)′ *Homotopy Equivalence Axiom* (*WHE Axiom*). If $f:(X,X')\to(Y,Y')$ is a WHE, i.e., both $|f|$ and $|f/X'|$ are HE in $\mathscr{T}$op, then

$$h(f): h(X,X') \to h(Y,Y')$$

is an isomorphism in $\mathscr{A}\mathrm{b}^{\mathbf{Z}}$.

Let A be a fixed abelian group. If $X \in \mathrm{ob}\mathscr{K}$, then we obtain a chain complex $C(X;A)$ of abelian groups:

$$C_k(X;A) = \bigoplus_{x\in X_k} A_x,$$

where $A_x = A$; with boundary

$$d_k : C_k(X;A) \to C_{k-1}(X;A)$$

given by the usual alternating sum of the face maps. Defining C in the evident way on simplicial maps, we obtain a functor $C:\mathscr{K}\to C(\mathscr{A}\mathrm{b})$, from simplicial sets to the category of chain complexes of abelian groups. This induces in the usual way a functor $C: A(\mathscr{K}) \to C(\mathscr{A}\mathrm{b})$, i.e., $C(X,X';A)$ is the quotient of $C(X)$ by $C(X')$. Let $H : C(\mathscr{A}\mathrm{b}) \to \mathscr{A}\mathrm{b}^{\mathbf{Z}}$ be the standard homology functor and H also denote the composition $H\circ C : A(\mathscr{K}) \to \mathscr{A}\mathrm{b}^{\mathbf{Z}}$. It is well known that $\langle H,\partial\rangle$ is *the* homology theory for $\mathscr{K}$.

Theorem 5 (Uniqueness). *If $\langle h,\partial\rangle$ is a homology theory for $\mathscr{K}$, then $h(X,X')$ and $H(X,X';A)$ are naturally isomorphic, where $A = h_0(\Delta[0])$.*

Proof. See [12], [13], [2].

A calculation using the canonical coflabby resolution of $A_{c,c'}$ (see [17], [14], [15], [8]) shows that the following holds.

Theorem 6. *Let $\langle H,\partial\rangle$ be the unique homology theory in $\mathscr{K}$ for the abelian group A then*

$$\begin{array}{ccc} A(\mathscr{C}\mathrm{at}) & \xrightarrow{N} & A(\mathscr{K}) \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle H} & \\ & \mathscr{A}\mathrm{b}^{\mathbf{Z}} & \end{array}$$

commutes up to natural isomorphism.

Proof. See [17], [14], [15], [20].

6. Uniqueness of homology for $\mathscr{C}$at

In order to prove the uniqueness of homology for $\mathscr{C}$at, we develop some of the properties of $\Gamma : \mathscr{K} \to \mathscr{C}\mathrm{at}$.

Lemma E. *Γ commutes with pullbacks.*

Proof. Consider

$$Z \xleftarrow{f} X \leftarrow P \rightarrow Y \xrightarrow{g} Z$$

a pullback diagram in $\mathscr{K}$. Then $P_k = \{(x, y) \mid f_k(x) = g_k(y)\} \subseteq X_k \times Y_k$. Let $\mathbf{P}$ be a small category such that

$$\Gamma Z \xleftarrow{\Gamma f} \Gamma X \longleftarrow \mathbf{P} \longrightarrow \Gamma Y \xrightarrow{\Gamma g} \Gamma Z$$

is a pullback diagram in $\mathscr{C}$at. Clearly ΓP is included in $\mathbf{P}$. Suppose $\langle([k], x), ([m], y)\rangle \in \mathrm{ob}\,\mathbf{P}$, i.e.,

$$\langle([k], x), ([m], y)\rangle \in (\mathrm{ob}\Gamma X \times \mathrm{ob}\Gamma Y)$$

such that

$$(\Gamma f)([k], x) = ([k], f_k(x)) = ([m], g_m(y)) = (\Gamma g)([m], y)$$

in ΓZ. Hence, $k = m$ and $f_k(x) = g_k(y)$. Thus $\langle([k], x), ([m], y)\rangle$ in $\mathbf{P}$ corresponds uniquely to $([k], (x, y))$ in ΓP and $\Gamma P = \mathbf{P}$.

Remark. Although Γ commutes with pullbacks, Γ does not commute with products since Γ does not preserve terminal objects, i.e., $\Gamma(\Delta[0])$ is equivalent to Δ, not $[0]$.

Lemma G. *Γ extends to a functor between admissible categories, i.e., $(\Gamma X, \Gamma X') \in \mathrm{ob}A(\mathscr{C}\mathrm{at})$.*

The lemma follows easily from the fact that X' is a subsimplicial set of X and the definition of morphisms in ΓX. This functor between admissible categories is also denoted by Γ.

Theorem 7. *If $\langle h, \partial\rangle$ is a homology theory for $\mathscr{C}$at, then*

$$\begin{array}{ccc} A(\mathscr{K}) & \xrightarrow{\Gamma} & \mathscr{A}(\mathscr{C}\mathrm{at}) \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle h} & \\ & \mathscr{A}\mathrm{b}^{Z} & \end{array}$$

commutes up to a natural isomorphism.

Proof. We verify that $\langle h \circ \Gamma, \partial \rangle$ satisfies the axioms for a homology theory in $\mathscr{K}$:
(i) *Dimension Axiom.* By Lemma C, $\Gamma(\Delta[0])$ is SC, i.e., $\Gamma(\Delta[0])$ is SHE $[0]$. Because the WHE Axiom implies the SH Axiom (see Appendix II), the homotopy and dimension axioms for h imply the dimension axiom holds for $h \circ \Gamma$.
(ii) *Exactness Axiom* is a direct consequence of Lemma G and the exactness axiom of h.
(iii) *Excision Axiom.* Since Γ is a left adjoint, it commutes with pushouts. Lemma E implies that it commutes with pullbacks and hence with the bicartesian square

$$X \cap Y \to X \to X \cup Y \leftarrow Y \leftarrow X \cap Y.$$

Excision for h yields excision for $h \circ \Gamma$.
(iv)′ *WHE Axiom.* Since the SH Axiom for $\mathscr{K}$ is equivalent to the WHE Axiom (see Appendix II), we verify that the WHE Axiom holds for $h \circ \Gamma$. Suppose $f:(X, X') \to (Y, Y')$ is a WHE in $A(\mathscr{K})$. By definition of WHE in $\mathscr{C}$at and $\mathscr{K}$, Γf is a WHE in $\mathscr{C}$at iff $B\Gamma f = |N\Gamma f|$ is a HE in $\mathscr{T}$op iff $N\Gamma f$ is a WHE in $\mathscr{K}$. By Theorem 2, the natural transformation $\eta'': N \dot{\to} 1$ yields the following commutative diagram

$$\begin{array}{ccc} N\Gamma X & \xrightarrow{N\Gamma f} & N\Gamma Y \\ \downarrow{\scriptstyle \eta''_X} & & \downarrow{\scriptstyle \eta''_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

with three sides WHE; and thus, $N\Gamma f$ is a WHE. Similarly $N\Gamma f / N\Gamma X' : N\Gamma X' \to N\Gamma Y'$ is a WHE. Hence $N\Gamma f : (N\Gamma X, N\Gamma X') \to (N\Gamma Y, N\Gamma Y')$ is a WHE in $A(\mathscr{K})$, as required.
(v) *Strong Additivity Axiom.* Since Γ commutes with colimits, strong additivity for $h \circ \Gamma$ follows from the corresponding strong additivity for h. Thus $\langle h \circ \Gamma, \partial \rangle$ is a homology theory for $\mathscr{K}$; and by Theorem 5, $\langle h \circ \Gamma, \partial \rangle$ and $\langle H, \partial \rangle$ are naturally isomorphic.

Theorem 8 (Uniqueness). *If $\langle h, \partial \rangle$ is a homology theory for* $\mathscr{C}$at, *then $h_*(\mathbf{C}, \mathbf{C}')$ is naturally isomorphic to $\mathbf{H}_*(\mathbf{C}, \mathbf{C}'; A)$, where $A = h_0([0])$.*

Proof. By Theorem 3, $\eta'_{\mathbf{C}} : (\Gamma N\mathbf{C}, \Gamma N\mathbf{C}') \to (\mathbf{C}, \mathbf{C}')$ is a WHE in $A(\mathscr{C}\mathrm{at})$. Hence the

WHE Axiom implies $h(\mathbf{C},\mathbf{C}') \simeq h(\Gamma N\mathbf{C},\Gamma N\mathbf{C}')$. By Theorem 7, $h(\Gamma N\mathbf{C},\Gamma N\mathbf{C}') \simeq H(N\mathbf{C},N\mathbf{C}';A)$. But Theorem 6 insures that $H(N\mathbf{C},N\mathbf{C}';A) \simeq H(\mathbf{C},\mathbf{C}';A)$.

Remark. It is well known ([20], [15], [8], [14]) that more general coefficient categories $\mathscr{A}$ for $\langle h,\partial\rangle$ can be chosen — any AB4 abelian category with enough projectives suffices. In this case, if

$$\_\otimes_{\mathbf{C}}\_ : (\mathbf{C}^{\mathrm{op}},\mathscr{A}\mathrm{b}) \times (\mathbf{C},\mathscr{A}) \to \mathscr{A}$$

is a generalized tensor product, then $\Delta Z \otimes_{\mathbf{C}}\_ : (\mathbf{C},\mathscr{A}) \to \mathscr{A}$ and $\mathrm{colim}_{\mathbf{C}} : (\mathbf{C},\mathscr{A}) \to \mathscr{A}$ are isomorphic as functors, where ΔZ is the constant diagram of type $\mathbf{C}^{\mathrm{op}}$ and value Z. Thus their respective left derived functors, $\mathrm{Tor}_*^{\mathbf{C}}(\Delta Z,\_)$ and $L_*\mathrm{colim}_{\mathbf{C}}$ are isomorphic. Hence parallel definitions and arguments to those used in the case for $\mathscr{A}\mathrm{b}$ yield the corresponding Uniqueness Theorem, where $H_*(\mathbf{C},\mathbf{C}';A) \simeq L_*\mathrm{colim}_{\mathbf{C}}(A_{\mathbf{C},\mathbf{C}'})$ and $A \in \mathrm{ob}\mathscr{A}$.

Remark. If $\mathscr{B}$ is a complete abelian category, then a cohomology theory for $\mathscr{C}\mathrm{at}$ with coefficients in $\mathscr{B}$ is defined by "dualizing" the definition of a homology theory: a cohomology theory is a contravariant functor

$$h : A^*(\mathscr{C}\mathrm{at}) \to \mathscr{B}$$

from a "corresponding" category of admissible pairs to the category of graded $\mathscr{B}$-objects, along with a natural transformation ∂ of degree $+1$, i.e.,

$$\partial^* : h^*(\mathbf{C}') \to h^{*+1}(\mathbf{C},\mathbf{C}')$$

for each admissible pair $(\mathbf{C},\mathbf{C}')$, such that the dual axioms of those for a homology theory for $\mathscr{C}\mathrm{at}$ (see Section 4) are satisfied. Thus a cohomology theory with coefficients in $\mathscr{B}$ is simply a homology theory with coefficients in the dual category $\mathscr{B}^{\mathrm{op}}$. Hence, if $\mathscr{B}$ is any AB4* abelian category, dualization shows that a cohomology theory for $\mathscr{C}\mathrm{at}$ with coefficients in $\mathscr{B}$ is unique up to a natural isomorphism, and is, in fact, isomorphic to the right derived functors of the (inverse) limit functor,

$$\lim_{\mathbf{C}} : (\mathbf{C}^{\mathrm{op}},\mathscr{B}) \to \mathscr{B},$$

of an appropriate diagram over $\mathbf{C}^{\mathrm{op}}$ ([13], [14], [15]).

Appendix I. The Subdivision Theorem

In order to prove the Subdivision Theorem (Theorem 1), we use repeatedly the following result in the "folklore" which appears implicitly in a paper of Heller [6].

Theorem 9 (Heller). *If*

$$\begin{array}{ccccccccc} X & \longrightarrow & Y & \longrightarrow & P & \longleftarrow & Z & \longleftarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & Y' & \longrightarrow & P' & \longleftarrow & Z' & \longleftarrow & X' \end{array}$$

is a commutative diagram in $\mathcal{K}$ *with the rows pushouts, with* $X \to Y$ *and* $X' \to Y'$ *inclusions, and with* f, g, h *WHE, then* $r : P \to P'$ *is also a WHE; i.e., the pushout of a WHE is itself a WHE.*

For the remainder of this section, we assume that $\mathrm{Sd} : \mathcal{K} \to \mathcal{K}$ is a functor satisfying the hypotheses of Theorem 1; i.e.,

(i) there is a natural transformation $\rho : \mathrm{Sd} \circ R \dot{\to} R$ between representing functors;

(ii) $\mathrm{Sd}(\Delta[n])$ is contractible for every $n \geqslant 0$;

(iii) Sd commutes with colimits;

(iv) Sd preserves inclusions in $\mathcal{K}$.

Lemma H. *There is a unique extension of* $\rho : \mathrm{Sd} \circ R \dot{\to} R$ *to a natural transformation* $\rho'' : \mathrm{Sd} \dot{\to} 1$.

Proof. The proof follows immediately from the facts that Sd commutes with colimits and each simplicial set X is equivalent to a colimit $\int^k \Delta[k] \circ X_k$ of representable simplicial sets (see (3)).

Lemma I. $\rho''_{\Delta[k]} : \mathrm{Sd}(\Delta[k]) \to \Delta[k]$ *is a WHE for every* $k \geqslant 0$.

Proof. Since both $\mathrm{Sd}(\Delta[k])$ and $\Delta[k]$ are weakly contractible, any simplicial map, and in particular $\rho_{\Delta[k]}$, yields a WHE.

A subcomplex V of $\Delta[k]$ is said to be *starlike* if there is a collection of faces (i.e., monomorphisms of Δ) $A = \{\alpha : [p] \to [k]\}$ such that all the faces have a vertex in common, such that A generates V, and such that A is maximal in V (i.e., A is maximal if whenever $\beta \in A$ is a face of $\alpha \in A$, then $\beta = \alpha$). For example, the starlike subcomplex Λ^r on $A = \{\delta^i : [k-1] \rightarrowtail [k] \mid i \neq r\}$ is called the *r-horn.*

Lemma J. *If* V *is a starlike subcomplex of* $\Delta[k]$ *on* A, *then* $\rho''_V : \mathrm{Sd}\, V \to V$ *is a WHE.*

Proof. The proof is done by induction on the number n of faces in A, using a standard representation of V as a starlike subcomplex V' on $n-1$ faces of A and a representable simplicial set $\Delta[m]$. The hypotheses of Theorem 9 are satisfied because Sd commutes with pushouts, Sd preserves subobjects, and Lemma I holds. Hence ρ''_V is a WHE.

Using analogous arguments for the standard pushout diagrams which represent the boundary of $\Delta[k]$ and the k-skeletons of X, we get the following results.

Lemma K. *If $\dot{\Delta}[k]$ is the boundary of the standard simplicial set $\Delta[k]$, then*

$$\rho''_{\dot{\Delta}[k]} : \mathrm{Sd}(\dot{\Delta}[k]) \to \dot{\Delta}[k]$$

is a WHE.

Lemma L. *If X^k is the k-skeleton of the simplicial set X, then*

$$\rho''_{X^k} : \mathrm{Sd}\, X^k \to X^k$$

is a WHE.

Since $\rho''_X : \mathrm{Sd}X \to X$ is a WHE when restricted to each skeleton, it follows from Whitehead's theorem that $\rho''_X : \mathrm{Sd}X \to X$ is a WHE. This completes the proof of the Subdivision Theorem.

Appendix II. Equivalence of the WHE Axiom and the SH Axiom

In this section we outline a proof of the fact that the WHE Axiom and the SH Axiom of the unique homology theory $\langle H, \partial \rangle$ for $\mathcal{K}$, and hence, by definition, for the unique homology theory $\langle H, \partial \rangle$ in $\mathscr{C}$at, are equivalent. That the WHE Axiom implies the SH Axiom is well known and follows from an argument due to Eilenberg and Steenrod [3; p. 12] and the fact that the WH relation properly includes the SH relation.

Theorem 10. *The WHE Axiom implies the SH Axiom for $\langle H, \partial \rangle$.*

Although there do not appear to be enough strong homotopies in $\mathcal{K}$, the following argument, due to Gabriel and Zissman [5; Appendix Two, 1], shows that the SH Axiom implies the WHE axiom.

A set S of monomorphisms of $\mathcal{K}$ is called *saturated* if the following four conditions are satisfied:

(i) All isomorphisms belong to S.

(ii) If

$$X \xrightarrow{f} Y \to P \xleftarrow{g} Z \leftarrow X$$

is a pushout diagram in $\mathcal{K}$ with $f \in S$, then $g \in S$; i.e., S is stable under pushouts.

(iii) If there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{u'} & Y & \xrightarrow{v'} & X' \end{array}$$

such that $vu = 1$, $v'u' = 1$, and $f \in S$, then $g \in S$; i.e., each retract of a monomorphism in S is in S.

(iv) S is stable under countable compositions and arbitrary direct sums: if $f_i : X_i \to X_{i+1}$ is in S for every $i \geq 1$, then $u_1 : X_1 \to \operatorname{colim} X_i$ is in S; if $\{f_\alpha : X_\alpha \to Y_\alpha\}$ is a family of morphisms in S, then $\sqcup f_\alpha : \sqcup X_\alpha \to \sqcup Y_\alpha$ is in S.

Clearly, the intersection of all saturated sets containing a given set of monomorphisms M is saturated. It is called the *saturated set generated* by M. In particular the set A of *anodyne extensions* of $\mathscr{K}$ is defined to be the saturated set generated by the set of all inclusions of r-horns into standard representable simplicial sets,

$$B = \{u_{r,k} : \Lambda^r \to \Delta[k] \mid 1 \leq k, 0 \leq r \leq k\}.$$

Note that each $u_{r,k}$ is a SHE since Λ^r is SC onto the rth vertex and $\Delta[k]$ is SC by Lemma C.

Recall that the homotopic category $\mathscr{K}'$ is the category of fractions of the homotopy category $\mathscr{K}/\sim$ of this set A of anodyne extensions and that Gabriel and Zisman [5; VII, 1] showed that $\mathscr{K}'$ was equivalent to $\mathscr{K}$ modulo the WH relation. The SH Axiom for $\langle H, \partial \rangle$ insures that $H : A(\mathscr{K}) \to \mathscr{A}\mathrm{b}^{\mathbf{Z}}$ factors through the homotopy category $\mathscr{K}/\sim$. Hence, in order to prove that the WHE Axiom holds, it suffices to show $Hf : HX \to HY$ is invertible in $\mathscr{A}\mathrm{b}^{\mathbf{Z}}$ whenever $f \in A$.

Let Σ denote the collection of all monomorphisms $s : X \to Y$ in $\mathscr{K}$ such that Hs is an isomorphism of $\mathscr{A}\mathrm{b}^{\mathbf{Z}}$. The functoriality of $H : A(\mathscr{K}) \to \mathscr{A}\mathrm{b}^{\mathbf{Z}}$ and the Strong Additivity Axiom trivially insure that Σ satisfies conditions (i), (iii) and (iv) above. That Σ satisfies condition (ii) follows from an "abstract nonsense" lemma on pushouts in the category $C(\mathscr{A}\mathrm{b})$ of chain complexes of abelian groups (see [5; Appendix Two, lemma 1.3] for detailed statement). Hence Σ is a saturated set. B is contained in Σ by the SH Axiom, because $u_{r,k}$ is a SHE for all $k \geq 1$ and $0 \leq r \leq k$. Thus $A \subseteq \Sigma$ and the WHE Axiom holds.

Theorem 11. *The SH Axiom implies the WHE Axiom for* $\langle H, \partial \rangle$.

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