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A FIBRED HOMOTOPY EQUIVALENCE AND HOMOLOGY THEORIES FOR THE CATEGORY OF SMALL CATEGORIES

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1. Introduction

Watts [23], Laudal [12], Oberst [19], André [1], and more recently Quillen [20], have asserted that the derived functors of colimit of an appropriate diagram define a homology theory for $\mathscr{C}at$, the category of small categories, with arbitrary coefficients. In this paper, we develop a variation of the Eilenberg–Steenrod–Milnor axioms ([3], [17]) for such homology theories, give a constructive definition of these homology theories, and show, up to natural isomorphism, that such theories are uniquely determined by their coefficient system.

In [11], the homotopic (as different from homotopy; see Section 3) category of the functor category $\mathscr{K}$ of (semi-) simplicial sets was shown to be equivalent to the "corresponding" homotopic category of $\mathscr{C}at$, by constructing a weak homotopy (WH) inverse $\Gamma : \mathscr{K} \to \mathscr{C}at$ of the nerve functor $N : \mathscr{C}at \to \mathscr{K}$. This was done by defining natural transformations

$$\eta'' : N\Gamma \dot{\to} 1\mathrm{d}_{\mathscr{K}} \quad \text{and} \quad \eta' : \Gamma N \dot{\to} 1\mathrm{d}_{\mathscr{C}at}$$

such that the morphisms corresponding to each object are equivalences in the respective homotopic categories. This enabled us to use the uniqueness of homology in $\mathscr{K}$, developed by Milnor [17], to demonstrate the uniqueness of homology for $\mathscr{C}at$ with constant coefficients.

A homology theory for $\mathscr{C}at$ with arbitrary coefficients is defined by a functor

$$h : A(\mathscr{C}at \downarrow \mathbf{B}) \to \mathscr{A}b^{\mathbf{Z}}$$

from a category of pairs of the comma category $\mathscr{C}at \downarrow \mathbf{B}$ of small categories over a fixed small category $\mathbf{B}$, to the category of graded abelian groups. The homology theory is then determined uniquely by a coefficient functor

$$A : \Gamma N\mathbf{B} \to \mathscr{A}b.$$

When $\mathbf{B}$ is the trivial one pointed category, $\mathcal{C}at \downarrow \mathbf{B} \simeq \mathcal{C}at$, and the homology is just the homology for $\mathcal{C}at$ with constant coefficients.

The definition of weak homotopy equivalence (WHE) is extended to the corresponding fibred categories; and the fibred homotopic category of $\mathcal{K} \downarrow N\mathbf{B}$ is shown to be equivalent to the fibred homotopic category of $\mathcal{C}at \downarrow \mathbf{B}$, by extending the natural transformations η'' and η' to weak fibred homotopy equivalences (WFHE). This enables us to use the uniqueness of homology in $\mathcal{K} \downarrow N\mathbf{B}$ with varying coefficients developed by Chen [2], to prove the uniqueness of homology for $\mathcal{C}at \downarrow \mathbf{B}$.

The paper is organized as follows. In Section 2, we develop the notion of a b-fibre as a pullback in $\mathcal{K}$, and indicate why general pullback functors are bicontinuous (preserve limits and colimits). The strong homotopy and weak homotopy relations for $\mathcal{C}at$ and $\mathcal{K}$ are extended to corresponding fibred relations for $\mathcal{C}at \downarrow \mathbf{B}$ and $\mathcal{K} \downarrow B$ in Section 3. Also in Section 3, we prove a technical lemma (Lemma 3.8), using a fibred gluing lemma (Lemma 3.7), which builds naturally global weak fibre homotopy equivalences from weak homotopy equivalences on each of the fibres. In Section 4, we review the construction of η' and η'', and use the technical lemma (Lemma 3.8) to show that they induce weak fibre homotopy equivalences. Section 5 contains an argument showing that the unique homology theories commute with a larger homotopy relation in $\mathcal{K} \downarrow B$: If $\langle H, \partial \rangle$ is a homology theory for $\mathcal{K} \downarrow B$, then the image under H of a weak fibre homotopy equivalence is an isomorphism in $\mathcal{A}b^{\mathbf{Z}}$.

In Section 6, we completely define and describe homology for $\mathcal{C}at \downarrow \mathbf{B}$ with arbitrary coefficients. The axioms, which are similar to those for $\mathcal{K} \downarrow NB$, are stated first. The definition of homology $\langle H, \partial \rangle$ for $\mathcal{C}at \downarrow \mathbf{B}$ is given in terms of homology for $\mathcal{K} \downarrow N\mathbf{B}$, and then $\langle H, \partial \rangle$ is shown to satisfy the axioms for homology in $\mathcal{C}at \downarrow \mathbf{B}$. Furthermore, the well-known connection between such homology theories and derived functors of colimit of an appropriate diagram is detailed (see [18], [21], [19], [1], etc.). Lastly, by demonstrating that a homology for $\mathcal{C}at \downarrow \mathbf{B}$ generates naturally a homology theory for $\mathcal{K} \downarrow N\mathbf{B}$, homology for $\mathcal{C}at \downarrow \mathbf{B}$ with arbitrary coefficients is shown to be unique.

2. Preliminaries

The reader is referred to Sections 2 and 3 of [11] for the details that define and describe the category $\mathcal{K}$ of simplicial sets and the following functors: nerve $N : \mathcal{C}at \to \mathcal{K}$, category of simplices $\Gamma : \mathcal{K} \to \mathcal{C}at$, categorical realization $c : \mathcal{K} \to \mathcal{C}at$, subdivision $N\Gamma : \mathcal{K} \to \mathcal{K}$ and Milnor geometric realization $|-| : \mathcal{K} \to \mathcal{T}op$ [16]. Also, the notation used in the present paper is the same as that employed in [11].

For each fixed simplicial set B, let $\mathcal{K} \downarrow B$ denote the comma category of simplicial sets over B [14; II, 6]: an object of $\mathcal{K} \downarrow B$ is a pair (X, ϕ) where $X \in \operatorname{ob} \mathcal{K}$ and $\phi : X \to B$ in $\mathcal{K}$; a morphism $f : (X, \phi) \to (Y, \psi)$ is a map $f : X \to Y$ in $\mathcal{K}$ such that $\psi \circ f = \phi$. Colimits in $\mathcal{K} \downarrow B$ are just colimits in $\mathcal{K}$ mapped "naturally" into B.

Suppose $f: B' \to B$ is a simplicial map. Let $f^*: \mathcal{K}\downarrow B' \to \mathcal{K}\downarrow B$ denote the functor *composition with* f, i.e. $f^*(X', \phi') = (X', f \circ \phi')$. It has a right adjoint, $P(f): \mathcal{K}\downarrow B \to \mathcal{K}\downarrow B'$, called *pullback along* f. For each (X, ϕ) in $\mathcal{K}\downarrow B$, $Pf(X)(X, \phi) \equiv (X', \phi')$, where

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "\phi'"] & X \arrow[d, "\phi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd} \tag{1}$$

is a pullback diagram in $\mathcal{K}$. Thus $P(f)$ preserves all limits, products, and equalizers. In addition, because $\mathcal{K} = [\mathbf{\Delta}^{\text{op}}, \mathcal{S}]$ is a locally cartesian closed category [4], $P(f)$ has a right adjoint as well; and hence, it commutes with colimits.

Lemma 2.1. *For each* $f: B' \to B$, *the pullback along* f, $P(f): \mathcal{K}\downarrow B \to \mathcal{K}\downarrow B'$ *is bicontinuous, i.e.* $P(f)$ *commutes with both limits and colimits.*

Because of the extensive use of this pullback construction for specific types of maps $f: B' \to B$, we include a catalogue of notation. If $\hat{b}: \Delta[k] \to B$ is the representing map for $b \in B_k$, then $P(\hat{b})(X, \phi) \equiv \Delta(b, \phi)$ denotes the *b-fibre* of (x, ϕ). Similarly, if $\dot{\Delta}[k]$ is the boundary subsimplicial set of $\Delta[k]$ [6; II, 3], then $P(\hat{b}/\dot{\Delta}[k]) \equiv \dot{\Delta}(b, \phi)$ is the *boundary of the b-fibre*. Let $\mathrm{Sk}^r B$ denote the r-skeleton of B, the subsimplicial set of B generated by simplices of dimension at most r, and $u^r: \mathrm{Sk}^r B \to B$ be the inclusion map. Then $P(u^r)(X, \phi) \equiv (X^r, \phi)$. Because $B = \mathrm{colim}_r\, \mathrm{Sk}^r B$, and for every $r \geq 0$, $P(u^r)(X, \phi) \equiv P(\phi)(\mathrm{Sk}^r B, u^r)$, $\{(X^r, \phi) \mid r \geq 0\}$ is a directed filtration of (X, ϕ) in $\mathcal{K}\downarrow B$; i.e.

$$(X, \phi) \simeq \mathrm{colim}_r (X^r, \phi). \tag{2}$$

Remark 2.2. In general, X^r and $\mathrm{Sk}^r X$, the r-skeleton of X, are not the same simplicial set. The difference comes from the fact that X^r is generated by all those simplices of X which map under ϕ to nondegenerate r-simplices of B, while $\mathrm{Sk}^r X$ is generated by the nondegenerate r-simplices of X and $\mathrm{Sk}^{r-1} X$ (see (19)). It is possible that X^r may contain higher dimensional nondegenerate simplices of X.

A variant of the above construction will be used in Theorem 3.9. For each $f: B' \to B$, let

$$\bar{P}(f): \mathcal{K}\downarrow B \to \mathcal{K} \tag{3}$$

be defined by the composition of $P(f): \mathcal{K}\downarrow B \to \mathcal{K}\downarrow B'$ with the forgetful functor $U: \mathcal{K}\downarrow B' \to \mathcal{K}$, where $U(X', \phi') = X'$. Then for each $m: (B, f) \to (C, g)$ in $\mathcal{K}\downarrow B$, it follows from the universality of the definition of pullback, that there exists a natural transformation

$$\bar{m}: \bar{P}(f) \dot{\to} \bar{P}(g). \tag{4}$$

Since colimits in $\mathcal{K}\downarrow B'$ are colimits in $\mathcal{K}$ naturally mapped to B', $\bar{P}(f):\mathcal{K}\downarrow B\to\mathcal{K}$ preserves colimits. Furthermore $\bar{P}(f)$ also commutes with inclusions.

$\mathcal{C}at\downarrow\mathbf{B}$ denotes the analogous comma category, for each fixed small category $\mathbf{B}$. As in the case for $\mathcal{K}\downarrow B$, colimits in $\mathcal{C}at\downarrow\mathbf{B}$ can be considered as colimits in $\mathcal{C}at$ "naturally" mapped to $\mathbf{B}$. Each functor can be extended to a functor between corresponding comma categories. For example, $N:\mathcal{C}at\downarrow\mathbf{B}\to\mathcal{K}\downarrow N\mathbf{B}$ is defined by

$$N(\mathbf{C},\Phi)\equiv(N\mathbf{C},N\Phi), \tag{5}$$

for each $(\mathbf{C},\Phi)$ in $\mathcal{C}at\downarrow\mathbf{B}$. In a natural way, the proofs of the properties of these functors in the absolute case, given in [11], can be extended to show that the analogous properties hold for functors between comma categories. For completeness, the following lemmas are listed.

Lemma 2.3. *$\Gamma:\mathcal{K}\downarrow B\to\mathcal{C}at\downarrow\Gamma B$ commutes with pullbacks; and hence $\Gamma:\mathcal{K}\downarrow B\to\mathcal{C}at\downarrow\Gamma B$ preserves bicartesian squares and inclusions.*

Lemma 2.4. *$N\Gamma:\mathcal{K}\downarrow B\to\mathcal{K}\downarrow N\Gamma B$ commutes with colimits.*

3. Homotopy for $\mathcal{K}\downarrow B$ and $\mathcal{C}at\downarrow\mathbf{B}$

Strong homotopy (SH) in $\mathcal{K}$ is the equivalence relation generated as follows [15]. Let the inclusions $u_i:X\to X\times\Delta[1]$ correspond to the simplicial maps

$$1\times N(\delta^i):X\times\Delta[0]\to X\times\Delta[1],\quad i=0,1.$$

If $f,\ g\in\mathcal{K}(X,\ Y)$, $f\sim g$ iff there is a simplicial map $h:X\times\Delta[1]\to Y$ such that $h\circ u_1=f$ and $h\circ u_0=g$. In [6; IV, 1], Gabriel and Zizman showed that the strong homotopy equivalence relation could be described so that symmetry and transitivity were "naturally" included in the following manner: Let $\mathbf{W}$ be the small category whose set of objects is $\{p_k\mid k\geqslant 0\}$, and whose nonidentity morphisms are

$$a_{2m}:p_{2m}\to p_{2m+1},\quad m\geqslant 0,$$

$$b_{2m}:p_{2m}\to p_{2m-1},\quad m\geqslant 1.$$

Then $\mathbf{W}$ is a small category having no nontrivial compositions. It can be pictured as an infinite "zig-zag." Let $\varepsilon^k:[0]\to\mathbf{W}$ denote the injection with $\varepsilon^k(0)=p_k$. Then $W\equiv N(\mathbf{W})$ corresponds to the "simplicial half-line" since $|W|=|N(\mathbf{W})|\simeq\mathbf{R}^+$. f and g in $\mathcal{K}(X,\ Y)$ are strongly homotopic (SH), $f\sim g$, iff there exists $h:X\times W\to Y$ and positive integers $m\leqslant n$ (or $n\leqslant m$) such that

$$h\circ u_k=\begin{cases} f, & k\leqslant m\ \ (\text{or } k\geqslant m)\\ g, & k\geqslant n\ \ (\text{or } k\leqslant n),\end{cases}$$

where $u_k:X\to X\times W$ is the inclusion corresponding to $1\times N(\varepsilon^k):X\times\Delta[0]\to X\times W$.

Since the Milnor geometric realization functor is compatible with products, the canonical map $|X\times W|\to|X|\times\mathbf{R}^+$ is a homeomorphism; and $|-|:\mathcal{K}\to\mathcal{T}op$ preserves strong homotopies. However, Milnor's geometric realization is *not* full, even when factored through the *homotopy category* $\mathcal{K}/\sim$ of simplicial sets; i.e., $|f|:|X|\to|Y|$ may be a homotopy equivalence (HE) in $\mathcal{T}op$ without having a simplicial homotopy inverse in K (see [5], [20]). Gabriel and Zizman [6; IV] define the homotopic category $\mathcal{K}'$ of simplicial sets to be the category of fractions in $\mathcal{K}/\sim$ of the set of anodyne extensions (See [6; IV] or [11; Appendix II]), and show that $\mathcal{K}$ is isomorphic, via Milnor's geometric realization, to a *full* subcategory of the homotopy category $\mathcal{T}op/\sim$ [6; VII, 1]. We use this expanded homotopy relation: $f, g\in\mathcal{K}(X, Y)$ are *weakly homotopic* (WH) iff $|f|$ is homotopic to $|g|$ in $\mathcal{T}op$.

The following well known "gluing" lemma for weak homotopy equivalences (WHE) in $\mathcal{K}$ will be extended below to an analogous theorem for $\mathcal{K}\downarrow B$. It appeared implicitly in a paper of Heller [7] and its proof is outlined in [11; Appendix I].

Lemma 3.1 (Heller). *If*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & P \arrow[d, "r"] & Z \arrow[l, hook'] \arrow[d, "h"] & X \arrow[l] \arrow[d, "f"] \\
X' \arrow[r, hook] & Y' \arrow[r] & P' & Z' \arrow[l, hook'] & X' \arrow[l]
\end{tikzcd}$$

is a commutative diagram in $\mathcal{K}$ in which the rows are pushouts, with $X\to Y$ and $X'\to Y'$ inclusions, and with f, g, h WHE; *then $r:P\to P'$ is also a* WHE. *The pushout of a* WHE *is itself a* WHE.

The extension of these definitions of homotopy to the comma category $\mathcal{K}\downarrow B$, is straightforward for the SH relation, but is more complicated for the WH relation. The maps $f, g\in\mathcal{K}\downarrow B((X,\phi),(Y,\psi))$ are *strongly fibre homotopic* (SFH) in $\mathcal{K}\downarrow B$ iff there exist an $h:(X\times W,\phi\circ p)\to(Y,\psi)$ and integers $m\leqslant n$ (or $n\leqslant m$) such that

$$h\circ u_k=\begin{cases} f, & k\leqslant m \ (\text{or } k\geqslant m)\\ g, & k\geqslant n \ (\text{or } k\leqslant n),\end{cases}$$

where $u_k:(X,\phi)\to(X\times W,\phi\circ p)$ is the inclusion corresponding to $1\times N(\varepsilon^k)$. Correspondingly, a morphism $m:(X,\phi)\to(Y,\psi)$ in $\mathcal{K}\downarrow B$ is said to be a *weak fibre homotopy equivalence* (WFHE) iff for each $b\in B$, the corresponding morphism of b-fibres, $m(b):\Delta(b,\phi)\to\Delta(b,\psi)$ is a WHE in $\mathcal{K}$.

Remark 3.2. Because $\mathcal{K}\downarrow\Delta[0]\simeq\mathcal{K}$, WFHE in $\mathcal{K}\downarrow\Delta[0]$ corresponds to WHE in $\mathcal{K}$, and SFH in $\mathcal{K}\downarrow\Delta[0]$ is simply SH in $\mathcal{K}$.

Lemma 3.3. *If* $h:(X\times W,\phi\circ P)\to(Y,\psi)$ is a SFH *between* $f,g\in\mathcal{K}\downarrow B((X,\phi),(Y,\psi))$ in $\mathcal{K}\downarrow B$, *then for each* $b\in B$, *the corresponding simplicial map of b-fibres*

$$h(b):\Delta(b,\phi\circ p)\to\Delta(b,\psi)$$

is a homotopy in $\mathcal{K}$ *between* $f(b),g(b)\in\mathcal{K}(\Delta(b,\phi),\Delta(b,\psi))$. *Furthermore, if* $m:(X,\phi)\to(Y,\psi)$ *is a* SFHE *in* $\mathcal{K}\downarrow B$, *then* $m:(X,\phi)\to(Y,\psi)$ *is a* WFHE *in* $\mathcal{K}\downarrow B$.

Proof. The first part of the lemma follows immediately from the fact that the pullback functor $P(\hat{b}):\mathcal{K}\downarrow B\to\mathcal{K}\downarrow\Delta[k]$ commutes with products. The second from the first part and from the fact that the Milnor geometric realization functor preserves homotopies. □

Remark 3.4. The converse of Lemma 3.3 is not true, because $|-|:\mathcal{K}\downarrow B\to\mathcal{T}op\downarrow|B|$ is not full even on the corresponding homotopy categories.

The next lemma follows from the gluing lemma, Lemma 3.1, and the definition of WFHE.

Lemma 3.5. *If* $m:(X,\phi)\to(Y,\psi)$ *is a* WFHE, *then for each* $b\in B$,

$$m/\dot{\Delta}(b,\phi):\dot{\Delta}(b,\phi)\to\dot{\Delta}(b,\psi)$$

is a WHE *in* $\mathcal{K}$.

Remark 3.6. The above lemma is a key step in "building" WHE (see [11, Appendix I]). However, the next lemma ensures that the "gluing" can be accomplished so that the complete fibre structure be preserved.

Lemma 3.7 (Fibred Gluing Lemma). *Let f be a map of pushouts in* $\mathcal{K}\downarrow B$, *i.e. the diagram*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & Y \arrow[d] \arrow[r, bend left, "f"] & X' \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\
Z \arrow[r, hook] & P \arrow[dr, "\phi"'] & Z' \arrow[r, hook] & P' \arrow[dl, "\psi"] \\
 & & B &
\end{tikzcd}$$

commutes with each square a pushout and with $X\to Y$ *and* $X'\to Y'$ *inclusions. If the corresponding simplicial maps* f_X,f_Y,f_Z *are* WFHE *in* $\mathcal{K}\downarrow B$, *then* $f:(P,\phi)\to(P',\psi)$ *is a* WFHE *in* $\mathcal{K}\downarrow B$.

Proof. Since the pullback functor, $P(b):\mathcal{K}\downarrow B\to\mathcal{K}\downarrow\Delta[k]$ is continuous (by Lemma 2.1), $P(b)(f)$ is a map of pushouts in $\mathcal{K}\downarrow\Delta[k]$. Because f_X,f_Y,f_Z are WFHE in $\mathcal{K}\downarrow B$, the maps of fibres $f_X(b),f_Y(b),f_Z(b)$ are all WHE in $\mathcal{K}$. Furthermore, since pushouts in $\mathcal{K}\downarrow\Delta[k]$ are also pushouts in $\mathcal{K}$, the hypotheses of the gluing lemma, Lemma 3.1, are satisfied. Hence $f_p(b):\Delta(b,\phi)\to\Delta(b,\psi)$, the map between the pushout b-fibres, is also a WHE and thus $f_p:(P,\phi)\to(P',\psi)$ is a WFHE in $\mathcal{K}\downarrow B$. □

The following lemma is technical; however, it will be applied easily in a number of situations: to compare WFHE in $\mathcal{K}\downarrow B$ with WHE in $\mathcal{K}$, to develop generalized subdivisions for $\mathcal{K}\downarrow B$, and to show the equivalence between the homotopic categories in $\mathcal{K}\downarrow N\mathbf{B}$ and $\mathcal{C}at\downarrow\mathbf{B}$ (see below).

Lemma 3.8. *Let $S, T:\mathcal{K}\downarrow B\to\mathcal{K}\downarrow B'$ be functors which preserve inclusions, pushouts, and sequential colimits. Furthermore, suppose $m:S\dashrightarrow T$ is a natural transformation such that for any representable $\hat{c}:\Delta[k]\to B$*

$$m(\hat{c}):S(\Delta[k],\hat{c})\to T(\Delta[k],\hat{c})$$

is a WFHE *in $\mathcal{K}\downarrow B'$. Then for any $d:D\to B$,*

$$m(d):S(D,d)\to T(D,d)$$

is a WFHE *in $\mathcal{K}\downarrow B'$.*

Proof. The proof follows immediately from the fibred gluing lemma, Lemma 3.7, using an argument similar to the one appearing in [11; Appendix I]. □

Theorem 3.9. If $m:(X,\phi)\to(Y,\psi)$ *is a* WFHE *in $\mathcal{K}\downarrow B$,, then $m:X\to Y$ is a* WHE *in $\mathcal{K}$.*

Proof. By (3), there exist functors

$$\bar{P}(\phi),\bar{P}(\psi):\mathcal{K}\downarrow B\to\mathcal{K}\downarrow\Delta[0]\simeq\mathcal{K}$$

which preserve inclusions, pushouts, and sequential colimits. From (4) and the fact that m is a WFHE, there is a natural transformation

$$\bar{m}:\bar{P}(\phi)\dashrightarrow\bar{P}(\psi)$$

such that for each $\hat{b}:\Delta[k]\to B$

$$\bar{m}(\hat{b})\equiv m(b):\Delta(b,\phi)\to\Delta(b,\psi)$$

is a WHE in $\mathcal{K}$, or a WFHE in $\mathcal{K}\downarrow\Delta[0]$. Thus $\bar{m}:\bar{P}(\phi)\dashrightarrow\bar{P}(\psi)$ satisfies the hypotheses of Lemma 3.8 with $B'=\Delta[0]$, and the result follows. □

Remark 3.10. The converse of Theorem 3.9 is not true, i.e. $m:X\to Y$ a WHE in $\mathcal{K}$ does not necessarily imply that $m:(X,\phi)\to(Y,\psi)$ is a WFHE in $\mathcal{K}\downarrow B$.

Suppose $F, G\in\mathcal{C}at(\mathbf{C},\mathbf{D})$. Each natural transformation $\omega:F\dashrightarrow G$ corresponds to a functor $\bar{\omega}:\mathbf{C}\times[1]\to\mathbf{D}$ such that $\bar{\omega}\circ(1\times\delta^1)=F$ and $\bar{\omega}\circ(1\times\delta^0)=G$. Because $N:\mathcal{C}at\to\mathcal{K}$ commutes with products, Lee [13] was able to show that NF $\sim$ NG in $\mathcal{K}$, whenever such a natural transformation exists. Furthermore, since $N:\mathcal{C}at\to\mathcal{K}$ is fully faithful, $NF \sim NG$ in $\mathscr{K}$ implies the existence of a functor $\bar{\omega} : \mathbf{C} \times [1] \to \mathbf{D}$, and thus the existence of a natural transformation $\omega : F \dot{\to} G$. Hence, the *strong homotopy* (SH) *relation* in $\mathscr{C}at$, i.e. symmetric transitive closure of natural transformation, corresponds via N to the SH relation in $\mathscr{K}$.

As in the case of simplicial sets, $|NF| : |N\mathbf{C}| \to |N\mathbf{D}|$ may be a HE in $\mathscr{T}op$ without $F : \mathbf{C} \to \mathbf{D}$ being a SHE in $\mathscr{C}at$ (see [20]). The corresponding *homotopic category* $(\mathscr{C}at)'$ is defined to reflect the expanded weak homotopy relation in $\mathscr{K}$: $F, G \in \mathscr{C}at(\mathbf{C}, \mathbf{D})$ are *weakly homotopic* (WH) iff NF and NG are equal in $\mathscr{K}'$, or equivalently, iff $|NF|$ and $|NG|$ are homotopic in $\mathscr{T}op$.

The extension of these definitions of homotopy to the comma category $\mathscr{C}at \downarrow \mathbf{B}$ is motivated by the desire that $N : \mathscr{C}at \downarrow \mathbf{B} \to \mathscr{K} \downarrow N\mathbf{B}$ preserve both the SFH relation and the WFHE definition: $F, G : (\mathbf{C}, \Phi) \to (\mathbf{D}, \Psi)$ are SFH in $\mathscr{C}at \downarrow \mathbf{B}$ iff there exist an $H : (\mathbf{C} \times \mathbf{W}, \Phi \circ P) \to (\mathbf{D}, \Psi)$ and integers $m \leqslant n$ (or $n \leqslant m$) such that

$$H \circ (1 \times \varepsilon^k) = \begin{cases} F, & k \leqslant m \ (\text{or } k \geqslant m) \\ G, & k \geqslant n \ (\text{or } k \leqslant n) \end{cases}$$

where $P : \mathbf{C} \times \mathbf{W} \to \mathbf{C}$ is the projection functor. Hence F and G are SFH in $\mathscr{C}at \downarrow \mathbf{B}$ iff NF and NG are SFH in $\mathscr{K} \downarrow N\mathbf{B}$. Similarly $M : (\mathbf{C}, \Phi) \to (\mathbf{D}, \Psi)$ is defined to be a WFHE in $\mathscr{C}at \downarrow \mathbf{B}$ iff $NM : (N\mathbf{C}, N\Phi) \to (N\mathbf{D}, N\psi)$ is a WFHE in $\mathscr{K} \downarrow N\mathbf{B}$.

4. WFH inverse for the functor nerve

In [11], the functor $\Gamma : \mathscr{K} \to \mathscr{C}at$ was shown to be a WH inverse for the functor $N : \mathscr{C}at \to \mathscr{K}$, i.e. natural transformations

$$\eta'' : N\Gamma \dot{\to} 1\mathrm{d}_{\mathscr{K}}, \qquad \eta' : \Gamma N \dot{\to} 1\mathrm{d}_{\mathscr{C}at}$$

were given so that $\eta''(X) : N\Gamma(X) \to X$ and $\eta'(\mathbf{C}) : \Gamma N\mathbf{C} \to \mathbf{C}$, for each simplicial set X and small category $\mathbf{C}$, are WHE in the respective homotopic categories. After a brief review of this process, a similar theorem for the fibred categories is given in this section.

The two functors $\gamma : \mathbf{\Delta} \to \mathscr{C}at$ and $\iota : \mathbf{\Delta} \to \mathscr{C}at$ are related by a natural transformation "first",

$$\eta : \gamma \dot{\to} \iota; \tag{6}$$

i.e. for each $k \geqslant 0$, $\eta_k : \gamma([k]) \to \iota([k])$ is the functor defined by $\eta_k(\alpha : [p] \to [k]) \equiv \alpha(0) \in [k]$. Composition with "first" gives a natural transformation between singular functors

$$\eta_1 : N \dot{\to} S.$$

The universality of adjoint functors [14; IV, 7], ensures the existence of a natural transformation between corresponding left adjoints

$$\eta_2 : \Gamma \dot{\to} c.$$

Next, composition on the right with N yields the natural transformation

$$\eta' \equiv \eta_2 \circ N : \Gamma N \dot{\rightarrow} cN \simeq \mathrm{Id}_{\mathscr{C}at}. \tag{7}$$

Similarly, composition on the left with N and on the right with the Yoneda representing functor $R : \mathbf{\Delta} \rightarrow \mathscr{K}$ gives the natural transformation

$$\eta_3 \equiv N \circ \eta_2 \circ R : N\Gamma R \dot{\rightarrow} NcR \simeq R,$$

since $NcR[k] \equiv NcN[k] \simeq N[k] \equiv R[k]$ by (2) of [11]. Lastly, because $N\Gamma$ preserves colimits, η_3 is a natural transformation between representing functors, and each simplicial set is a colimit of representables; η_3 extends to a natural transformation

$$\eta'' : N\Gamma \dot{\rightarrow} \mathrm{Id}_{\mathscr{K}}. \tag{8}$$

In addition, from the uniqueness of η'' and the fact that $cN \simeq \mathrm{Id}_{\mathscr{C}at}$,

$$N(\eta'(\mathbf{C})) \simeq \eta''(N\mathbf{C}) : N\Gamma N\mathbf{C} \rightarrow N\mathbf{C}, \tag{9}$$

for each small category $\mathbf{C}$. Whenever the context is clear, we denote both η' and η'' simply by η. For example, since

$$\Gamma(\Delta[k]) \simeq \Gamma N([k]) \simeq (\mathbf{\Delta}\downarrow[k])^{\mathrm{op}} \equiv \gamma[k]$$

by Lemma A of [11],

$$\eta([k]) \simeq \eta_k : (\mathbf{\Delta}\downarrow[k])^{\mathrm{op}} \rightarrow [k]$$

for every $k \geqslant 0$. More generally, $\eta(\mathbf{B}) : \Gamma N\mathbf{B} \rightarrow \mathbf{B}$, is defined by

$$\eta(\mathbf{B})(\langle b_0 \xrightarrow{\beta_1} b_1 \longrightarrow \cdots \xrightarrow{\beta_k} b_k\rangle, [k]) \equiv b_0 \tag{10}$$

on objects of $\Gamma N\mathbf{B}$, and is given by

$$\begin{aligned} &\eta(\mathbf{B})(\delta^0) \equiv b_0 \xrightarrow{\beta_1} b_1, \\ &\eta(\mathbf{B})(\delta^i) \equiv b_0 \xrightarrow{1} b_0, \quad 0 < i \leqslant k, \\ &\eta(\mathbf{B})(\sigma^i) \equiv b_0 \xrightarrow{1} b_0, \quad 0 \leqslant i \leqslant k \end{aligned} \tag{11}$$

on generating morphisms of $\Gamma N\mathbf{B}$.

Lemma 4.1. *For each $k \geqslant 0$ and each functor $b : [k] \rightarrow \mathbf{B}$,*

$$\eta([k]) : ((\mathbf{\Delta}\downarrow[k])^{\mathrm{op}}, \eta(\mathbf{B}) \circ \Gamma Nb) \rightarrow ([k], b)$$

is an SFHE *in $\mathscr{C}at \downarrow \mathbf{B}$.*

Proof. For each $k \geqslant 0$, define the functor

$$\mu([k]) : [k] \rightarrow (\mathbf{\Delta}\downarrow[k])^{\mathrm{op}} \tag{12}$$

by

$$\mu([k])(t) \equiv \underbrace{\delta^0\,\delta^0 \cdots \delta^0}_{t-1} : [k-t] \rightarrowtail [k]$$

for each $t \in [k]$. Since $(\delta^0\,\delta^0 \cdots \delta^0)$ $(i) = i + t$, denote $\mu([k])(t)$ by $a(t)$. By the naturality of $\eta : \Gamma N \dot{\rightarrow} 1\mathrm{d}_{\mathscr{C}at}$,

$$\mu([k]) : ([k], b) \rightarrow ((\Delta\!\downarrow\![k])^{\mathrm{op}}, \eta(\mathbf{B}) \circ \Gamma N b)$$

in $\mathscr{C}at \downarrow \mathbf{B}$. From the definitions,

$$\eta([k]) \circ \mu([k]) \simeq 1 : ([k], b) \rightarrow ([k], b).$$

Define the SFH

$$H : ((\Delta\!\downarrow\![k])^{\mathrm{op}} \times [1], \eta(\mathbf{B}) \circ \Gamma \mathrm{Nb} \circ \mathrm{P}) \rightarrow ((\Delta\!\downarrow\![k])^{\mathrm{op}}, \eta(\mathbf{B}) \circ \Gamma B b)$$

in $\mathscr{C}at \downarrow \mathbf{B}$ by

$$H(\alpha, 1) \equiv \alpha : [p] \rightarrow [k],$$

$$H(\alpha, 0) \equiv a(\alpha(0)) : [k - \alpha(0)] \rightarrowtail [k],$$

$$H((1, \rightarrow) : (\alpha, 0) \rightarrow (\alpha, 1)) \equiv (S(\alpha) : a(\alpha(0)) \rightarrow \alpha)$$

where $S(\alpha) : [p] \rightarrow [k - \alpha(0)]$ is given by $S(\alpha)(i) = \alpha(i) - \alpha(0)$. Clearly, H is an SFH between $\mu([k]) \circ \eta([k])$ and $1d$. Hence $\eta([k])$ is an SFHE. □

Remark 4.2. Although $\mu([k]) : [k] \rightarrow (\Delta\!\downarrow\![k])^{\mathrm{op}}$ is defined for each $k \geq 0$, μ does not define a natural transformation between the functors ι and γ. In particular, for each epimorphism $\varepsilon : [k] \rightarrow [m]$ in Δ, $\mu([m]) \circ \varepsilon \neq (\Delta\!\downarrow\!\varepsilon)^{\mathrm{op}} \circ \mu([k])$. In contrast with $\eta : \gamma \dot{\rightarrow} \iota$, μ cannot be extended to a natural transformation from $1\mathrm{d}_{\mathscr{K}}$ to $N\Gamma$.

By definition, $\eta([k])$ is a SFHE in $\mathscr{C}at \downarrow \mathbf{B}$ iff $N(\eta([k]))$ is a SFHE in $\mathscr{K} \downarrow N\mathbf{B}$. From (9), $N(\eta([k])) \simeq \eta(\Delta[k])$. Hence, the next corollary follows from Lemma 4.1.

Corollary 4.3. *For each* $\hat{b} : \Delta[k] \rightarrow N\mathbf{B}$,

$$\eta(\Delta[k]) : (N\Gamma\Delta[k], \eta(N\mathbf{B}) \circ N\Gamma\hat{b}) \rightarrow (\Delta[k], \hat{b})$$

is a SFHE *in* $\mathscr{K} \downarrow N\mathbf{B}$.

Theorem 4.4. $\eta : N\Gamma \dot{\rightarrow} 1\mathrm{d} : \mathscr{K} \downarrow N\mathbf{B} \rightarrow \mathscr{K} \downarrow N\mathbf{B}$ *is a* WFHE, *i.e.*

$$\eta(X) : (N\Gamma X, \eta(N\mathbf{B}) \circ N\Gamma\phi) \rightarrow (X, \phi)$$

is a WFHE, *for every* $(X, \phi) \in \mathscr{K} \downarrow N\mathbf{B}$.

Proof. $N\Gamma : \mathscr{K} \downarrow N\mathbf{B} \rightarrow \mathscr{K} \downarrow N\mathbf{B}$ preserves colimits and inclusions, since $N\Gamma : \mathscr{K} \rightarrow \mathscr{K}$ does. Clearly, $1\mathrm{d} : \mathscr{K} \downarrow N\mathbf{B} \rightarrow \mathscr{K} \downarrow N\mathbf{B}$ also satisfies the same properties. By Corollary 4.3, the natural transformation $\eta : N\Gamma \dot{\rightarrow} 1\mathrm{d}$ has the property

$$\eta(\Delta[k]) : (N\Gamma\Delta[k], \eta(N\mathbf{B}) \circ N\Gamma\hat{b}) \rightarrow (\Delta[k], \hat{b})$$

is a SFHE, and hence a WFHE by Lemma 3.3. Thus the hypotheses of Lemma 3.8 are satisfied with $B = N\mathbf{B} = B'$, and the theorem follows. □

The definition of WFHE in $\mathscr{C}at\downarrow\mathbf{B}$ and formula (9) immediately imply the following corollary.

Corollary 4.5. *For each* $(\mathbf{C}, \Phi)$ *in* $\mathscr{C}at\downarrow\mathbf{B}$,

$$\eta(\mathbf{C}) : (\Gamma N C, \eta(\boldsymbol{B})\circ \Gamma N\Phi) \to (\mathbf{C}, \Phi)$$

is a WFHE.

5. Homology for $\mathscr{K}\downarrow B$

In [2], Chen developed axioms for homology in $\mathscr{K}\downarrow B$ with arbitrary coefficients (called costacks), showed that such homology theories exist, and then proved that these homology theories are unique up to natural isomorphism. For completeness, this section includes a constructive definition of these unique homology theories. Lastly, the unique homology theories are shown to commute with the larger homotopy relation in $\mathscr{K}\downarrow B$: Let $\langle H, \partial\rangle$ be a homology theory. The image under H of a WFHE is an isomorphism in $\mathscr{A}b^{\mathbf{Z}}$, the category of graded abelian groups.

Let $A(\mathscr{K}\downarrow B)$ denote the category of admissible pairs in $\mathscr{K}\downarrow B$: an object of $A(\mathscr{K}\downarrow B)$ is a triple $(X, X'; \phi)$ with X' a subsimplicial set of X and $\phi : X \to B$; a morphism $f : (X, X'; \phi) \to (Y, Y'; \psi)$ is a simplicial map $f : X \to Y$ such that $\psi \circ f = \phi$ and $f(X') \subseteq Y'$. The pair $(X; \phi)$ is identified with the admissible pair $(X, E; \phi)$, where E is the empty simplicial set. When there is no ambiguity, the inclusion map is omitted from the notation; e.g. $(X'; \phi)$ will denote $(X'; \phi\circ i)$.

A *homology theory* for $\mathscr{K}\downarrow B$ is a pair $\langle h, \partial\rangle$, where $h : A(\mathscr{K}\downarrow B) \to \mathscr{A}b^{\mathbf{Z}}$ is a functor and ∂ is a natural transformation of degree -1, i.e.

$$\partial_* : h_*(X, X'; \phi) \to h_{*-1}(X'; \phi).$$

These satisfy a variation of the Eilenberg–Steenrod–Milnor axioms which can be found in [2]. However, for Chen's uniqueness theorem (Theorem 5.1) to hold, the Dimension Axiom of [2] needs to be modified by requiring:

For every nondegenerate $x \in X_k$ with $\phi x = b$, let $\Delta(x)$ denote subsimplicial set generated by x and $\dot{\Delta}(x)$ denote the corresponding subsimplicial set generated by the boundaries of x. The representable map

$$\hat{x} : (\Delta[k], \dot{\Delta}[k]; \hat{b}) \to (\Delta(x), \dot{\Delta}(x); \phi)$$

induces a corresponding isomorphism of graded homology groups.

A functor $A : \Gamma B \to \mathscr{A}b$ is called a *costack* whenever $A : \Gamma B \to \mathscr{A}b$ satisfies the normalization condition; i.e., for each degeneracy $\sigma^i : (b, [k]) \to (\sigma^i b, [k+1])$ in ΓB, $A(\sigma^i) : A(b, [k]) \to A(\sigma^i b, [k+1])$ is an isomorphism. For each $(X, X'; \phi)$ in

$A(\mathcal{K}\downarrow B)$, consider the short exact sequence

$$0 \to A(X';\phi) \xrightarrow{\bar{i}} A(X;\phi) \to A(X,X';\phi) \to 0 \tag{13}$$

in the functor category $[\Gamma X, \mathscr{Ab}]$, where

$$A(X;\phi) \equiv A \circ \Gamma\phi : \Gamma X \to \mathscr{Ab}; \tag{14}$$

$A(X';\phi) : \Gamma X \to \mathscr{Ab}$ is given by

$$A(X';\phi)(x,[k]) \equiv \begin{cases} A(\phi x,[k]), & \text{if } x \in X'_k, \\ 0, & \text{if } x \notin X'_k; \end{cases} \tag{15}$$

$\bar{i}$ is the inclusion in $[\Gamma X, \mathscr{Ab}]$; and $A(X,X';\phi)$ is coker$(\bar{i})$ in $[\Gamma X, \mathscr{Ab}]$. Hence $A(X,X';\phi)$ is defined by the dual formula

$$A(X,X';\phi)(x,[k]) \equiv \begin{cases} 0, & \text{if } x \in X'_k, \\ A(\phi x,[k]), & \text{if } x \notin X'_k. \end{cases} \tag{16}$$

Let $C((X,X';\phi),A)$ denote the chain complex of abelian groups given by:

$$C_k((X,X';\phi),A) \equiv \bigoplus_{x \in X_k} A(X,X';\phi)(x,[k])$$

with boundary

$$d_k : C_k((X,X';\phi),A) \to C_{k-1}((X,X';\phi),A)$$

given by the usual alternating sum of the face maps. Defining C in the evident way on simplicial maps of $A(\mathcal{K}\downarrow B)$,

$$C : A(\mathcal{K}\downarrow B) \to \mathscr{C}(\mathscr{Ab})$$

is a functor from $A(\mathcal{K}\downarrow B)$ to the category of chain complexes of abelian groups. Let $H : \mathscr{C}(\mathscr{Ab}) \to \mathscr{Ab}^{\mathbf{Z}}$ be the stand homology functor [14; VIII, 4] and let H also denote the composition $H \circ C$; i.e.,

$$H((X,X';\phi),A) \equiv H(C(X,X';\phi),A). \tag{17}$$

In [2], Chen showed that $\langle H, \partial\rangle$ is the unique *homology theory for* $\mathcal{K}\downarrow B$ *defined by* $A : \Gamma B \to \mathscr{Ab}$.

Theorem 5.1 (Chen). (A) (*Existence*). *If* $A : \Gamma B \to \mathscr{Ab}$ *is a costack and if* $\langle H, \partial\rangle$ *is the homology theory for* $\mathcal{K}\downarrow B$ *defined by* $A : \Gamma B \to \mathscr{Ab}$, *then* $\langle H, \partial\rangle$ *satisfies the axioms for a homology theory for* $\mathcal{K}\downarrow B$.

(B) (*Uniqueness*). *If* $\langle h, \partial\rangle$ *is a homology theory for* $\mathcal{K}\downarrow B$ *with coefficient costack* $A : \Gamma B \to \mathscr{Ab}$ *defined by the Dimension Axiom, then* $\langle h, \partial\rangle$ *is naturally isomorphic to* $\langle H, \partial\rangle$; *i.e.,*

$$h(X,X';\phi) \simeq H((X,X';\phi),A)$$

for each $(X,X';\phi)$ *in* $A(\mathcal{K}\downarrow B)$.

Example 5.2. If $B = \Delta[0]$, the terminal object of $\mathscr{K}$, then $\mathscr{K}\downarrow B$ is equivalent to $\mathscr{K}$. In this case, $\langle H, \partial\rangle$ is the unique homology for $\mathscr{K}$ with constant coefficients defined by Milnor [17]; i.e., $H(X, X'; \phi), A) \equiv H((X, X'), G)$.

Example 5.3. A costack $L : \Gamma B \to \mathscr{A}b$ is called a *local coefficient system on B* with values in $\mathscr{A}b$ whenever the image $L(\mu)$ of *any* $\mu : (b, [k]) \to (\mu b, [m])$ in ΓB is invertible. Then the homology theory for $\mathscr{K}\downarrow B$ defined by $L : \Gamma B \to \mathscr{A}b$ is the usual homology theory for $\mathscr{K}\downarrow B$ with local coefficient system L (see [6; Appendix II], [20]).

In [6; Appendix II, 1], Gabriel and Zisman outlined a proof of the fact that the WHE Axiom and the SH Axiom are equivalent when the defining costack is *constant* or equivalently, when $B = \Delta[0]$. In this section, we also show that a more general homotopy axiom holds for homology $\langle h, \partial\rangle$ in $\mathscr{K}\downarrow B$.

(iii)′ *Weak Fibre Homotopy Equivalence Axiom* (WFHE *Axiom*). If $m : (X, X'; \phi) \to (Y, Y'; \psi)$ is a WFHE in $A(\mathscr{K}\downarrow B)$; i.e., m and m/X' are both WFHE in $\mathscr{K}\downarrow B$, then $hm : h(X, X'; \phi) \xrightarrow{\simeq} h(Y, Y'; \psi)$ is an isomorphism in $\mathscr{A}b^{\mathbf{Z}}$.

Theorem 5.4. *Suppose $A : \Gamma B \to \mathscr{A}b$ is a costack. If $\langle H, \partial\rangle$ is the homology theory for $\mathscr{K}\downarrow B$ defined by A, then the* WFHE *Axiom holds for $\langle H, \partial\rangle$.*

Proof. Let $m : (X, X'; \phi) \to (Y, Y'; \psi)$ be a WFHE in $A(\mathscr{K}\downarrow B)$. From the Exactness Axiom and the Five Lemma [14; VII, 4], it suffices to prove that the image Hm of a WFHE, in the absolute case, is an isomorphism; i.e.,

$$Hm : H((X; \phi), A) \xrightarrow{\simeq} H((Y; \psi), A).$$

From the definition of the b-fibre $\Delta(b, \phi)$ and its boundary $\dot{\Delta}(b, \phi)$,

$$A(\Delta(b, \phi), \dot{\Delta}(b, \phi); \phi) : \Gamma\Delta(b, \phi) \to \mathscr{A}b$$

is a relatively *constant* functor with value $A(b)$. The definition of WFHE and Lemma 3.5 insure that

$$m/\Delta(b, \phi) : (\Delta(b, \phi), \dot{\Delta}(b, \phi)) \to (\Delta(b, \psi), \dot{\Delta}(b, \psi))$$

is a WHE in $A(\mathscr{K})$. Hence the WHE Axiom applies, and

$$\begin{aligned} &H(m/\Delta(b, \phi)) : H((\Delta(b, \phi), \dot{\Delta}(b, \phi); \phi), A) \xrightarrow{\simeq} \\ &\xrightarrow{\simeq} H((\Delta(b, \psi), \dot{\Delta}(b, \psi); \psi), A) \end{aligned} \tag{18}$$

is an isomorphism of graded abelian groups.

The r-skeleton of B, $\mathrm{Sk}^r(B)$, can be described by the bicartesian square

$$\begin{tikzcd}
\coprod_{B'_r} \dot{\Delta}[r] \arrow[r, hook] \arrow[d, "\coprod \hat{b}/\Delta[r]"] & \coprod_{B'_r} \Delta[r] \arrow[d, "\coprod \hat{b}"] \\
\mathrm{Sk}^{r-1}(B) \arrow[r, hook] & \mathrm{Sk}^{r}(B)
\end{tikzcd} \tag{19}$$

where B'_r is the set of nondegenerate r-simplices of B (see [6; II, 3]). Applying $\bar{P}(\phi)$ (see (3)) to diagram (19) yields the bicartesian square

$$\begin{tikzcd}
\coprod_{B'_r} \dot{\Delta}(b, \phi) \arrow[r, hook] \arrow[d] & \coprod_{B'_r} \Delta(b, \phi) \arrow[d] \\
(X^{r-1}, \phi) \arrow[r, hook] & (X^{r}, \phi)
\end{tikzcd} \tag{20}$$

Because (20) satisfies the hypotheses of the Strong Additivity Axiom [2; p. 110]

$$\bigoplus H(u_b) : \bigoplus H((\Delta(b,\phi), \dot{\Delta}(b,\phi); \phi), A) \xrightarrow{\simeq} H((X^r, X^{r-1}; \phi), A) \tag{21}$$

is an isomorphism of $\mathscr{Ab}^{\mathbf{Z}}$. Similarly, pulling back along $\psi : Y \to B$, yields the isomorphism

$$\bigoplus H(u_b) : \bigoplus H((\Delta(b,\psi), \dot{\Delta}(b,\psi); \psi), A) \xrightarrow{\simeq} H((Y^r, Y^{r-1}; \psi), A). \tag{22}$$

Because the direct sum of isomorphisms is an isomorphism in $\mathscr{Ab}^{\mathbf{Z}}$, an isomorphism of type (18), in conjunction with (21) and (22), ensure

$$H(m/X^r) : H((X^r, X^{r-1}; \phi), A) \xrightarrow{\simeq} H((Y^r, Y^{r-1}; \psi), A) \tag{23}$$

is also an isomorphism. A standard inductive argument using the Five Lemma and the Exactness Axiom (see [3]) yields the absolute isomorphism:

$$H(m/X^r) : H((X^r; \phi), A) \xrightarrow{\simeq} H((Y^r; \psi), A). \tag{24}$$

Lastly, since $\langle H, \partial \rangle$ commutes with the exact directed colimit functor, (2) ensures that the isomorphisms (24) extend to

$$H(m) : H((X; \phi), A) \xrightarrow{\simeq} H((Y; \psi), A);$$

and the theorem follows. □

Theorem 5.5. *Let $\langle h, \partial \rangle$ be a homology theory for $\mathscr{K}\downarrow B$ satisfying all the axioms except possibly the SFH Axiom. If $\langle h, \partial \rangle$ satisfies the WFHE Axiom, then $\langle h, \partial \rangle$ also satisfies the SFH Axiom.*

Proof. The theorem follows from the functoriality of $h:\mathscr{K}\downarrow B\to\mathscr{A}b^{\mathbf{Z}}$ and the fact that the projection $p:(X\times W, X'\times W;\phi op)\to(X,X';\phi)$ is a WFHE. The argument parallels the well-known one due to Eilenberg and Steenrod [3; p. 12]. □

6. Homology for $\mathscr{C}at\downarrow\mathbf{B}$

In this section, we completely define and describe homology for $\mathscr{C}at\downarrow\mathbf{B}$ with arbitrary coefficients. The axioms, which are similar to those for $\mathscr{K}\downarrow N\mathbf{B}$, are stated first. The definition of homology $\langle H,\partial\rangle$ for $\mathscr{C}at\downarrow\mathbf{B}$ is given in terms of homology for $\mathscr{K}\downarrow N\mathbf{B}$, and then $\langle H,\partial\rangle$ is shown to satisfy the axioms for homology in $\mathscr{C}at\downarrow\mathbf{B}$. Furthermore, the well known connection between such homology thories and derived functors of colimit of an appropriate diagram is detailed (see [21, 18, 12, 23, 1, 6, 19]). Lastly, by demonstrating that a homology theory for $\mathscr{C}at\downarrow\mathbf{B}$ generates naturally a homology theory for $\mathscr{K}\downarrow N\mathbf{B}$, homology for $\mathscr{C}at\downarrow\mathbf{B}$ with arbitrary coefficients is shown to be unique.

A subcategory $\mathbf{C}'$ of the small category $\mathbf{C}$ is *admissible* if all morphisms of $\mathbf{C}$ with domain in $\mathbf{C}'$ are in $\mathbf{C}'$. Such a $\mathbf{C}'$ is necessarily a full subcategory of $\mathbf{C}$. In particular, when $\mathbf{P}$ is the category of a poset, an admissible subcategory corresponds to a subposet which is the union of "terminal" segments of $\mathbf{P}$. If $\mathbf{G}$ is a group, then the only admissible subcategories of $\mathbf{G}$ are the *empty category* $\mathbf{E}$ or $\mathbf{G}$ itself. The definition of $\Gamma:\mathscr{K}\to\mathscr{C}at$ guarantees that if X' is a subsimplicial set of X in K, then $\Gamma X'$ is an admissible subcategory of ΓX in $\mathscr{C}at$.

Let $A(\mathscr{C}at\downarrow\mathbf{B})$ denote the category of admissible pairs in $\mathscr{C}at\downarrow\mathbf{B}$: an object is a triple $(\mathbf{C},\mathbf{C}';\Phi)$ with $\mathbf{C}'$ admissible in $\mathbf{C}$ and $\Phi:\mathbf{C}\to\mathbf{B}$; a morphism $F:(\mathbf{C},\mathbf{C}';\Phi)\to(\mathbf{D},\mathbf{D}';\Psi)$ is a functor $F:(\mathbf{C},\Phi)\to(\mathbf{D},\Psi)$ in $\mathscr{C}at\downarrow\mathbf{B}$ with $F/\mathbf{C}':\mathbf{C}'\to\mathbf{D}'$. $(\mathbf{C},\Phi)$ in $\mathscr{C}at\downarrow\mathbf{B}$ is identified with $(\mathbf{C},\mathbf{E};\Phi)\equiv(\mathbf{C};\Phi)$ in $A(\mathscr{C}at\downarrow\mathbf{B})$. When there is little ambiguity, the inclusion functor is omitted from the notation; e.g. $(\mathbf{C}';\Phi)$ will denote $(\mathbf{C}';\Phi\circ I)$.

A *homology theory for* $\mathscr{C}at\downarrow\mathbf{B}$ is a pair $\langle h,\partial\rangle$, where $h:A(\mathscr{C}at\downarrow\mathbf{B})\to\mathscr{A}b^{\mathbf{Z}}$ is a functor and ∂ is a natural transformation of degree -1; i.e.

$$\partial_*:h_*(\mathbf{C},\mathbf{C}';\Phi)\to h_{*-1}(\mathbf{C}';\Phi).$$

These satisfy the following variation of the standard Eilenberg–Steenrod–Milnor axioms ([3], [17]).

(i) *Exactness Axiom.* For each admissible pair $(\mathbf{C},\mathbf{C}';\Phi)$ with inclusion functors $I:(\mathbf{C}';\Phi)\to(\mathbf{G};\Phi)$ and $J:(\mathbf{C};\Phi)\to(\mathbf{C},\mathbf{C}';\Phi)$, there exists a long exact sequence of abelian groups

$$\cdots\to h_q(\mathbf{C}';\Phi)\xrightarrow{I_q}h_q(\mathbf{C};\Phi)\xrightarrow{J_q}h_1(\mathbf{C},\mathbf{C}';\Phi)\xrightarrow{\partial_q}h_{q-1}(\mathbf{C}';\Phi)\to\cdots$$

(ii) *Excision Axiom.* If $\mathbf{C}_1$ and $\mathbf{C}_2$ are admissible in $\mathbf{C}$ and $\Phi:\mathbf{C}\to\mathbf{B}$, then the inclusion functor in $A(\mathscr{C}at\downarrow\mathbf{B})$

$$I:(\mathbf{C}_2,\mathbf{C}_1\cap\mathbf{C}_2;\Phi)\to(\mathbf{C}_1\cup\mathbf{C}_2,\mathbf{C}_1;\Phi)$$

induces the corresponding isomorphism of graded homology groups; where $\mathbf{C}_1 \cap \mathbf{C}_2$ and $\mathbf{C}_1 \cap \mathbf{C}_2$ are the subcategories of $\mathbf{C}$ making the following square bicartesian in $\mathscr{C}at{\downarrow}\mathbf{B}$ (via Φ),

$$\begin{tikzcd}
\mathbf{C}_1 \cap \mathbf{C}_2 \arrow[r, hook] \arrow[d, hook] & \mathbf{C}_1 \arrow[d, hook] \\
\mathbf{C}_2 \arrow[r, hook] & \mathbf{C}_1 \cup \mathbf{C}_2.
\end{tikzcd} \tag{25}$$

(iii) *Weak Fibre Homotopy Equivalence Axiom* (*WFHE Axiom*). If $M:(\mathbf{C}, \mathbf{C}'; \Phi) \to (\mathbf{D}, \mathbf{D}'; \Psi)$ is a WFHE in $A(\mathscr{C}at{\downarrow}\mathbf{B})$ i.e.

$$NM:(N\mathbf{C}, N\mathbf{C}'; N\Phi) \to (N\mathbf{D}, N\mathbf{D}'; N\Psi)$$

is a WFHE in $A(\mathscr{K}{\downarrow}N\mathbf{B})$, then

$$hM:h(\mathbf{C}, \mathbf{C}'; \Phi) \xrightarrow{\simeq} h(\mathbf{D}, \mathbf{D}'; \Psi)$$

is an isomorphism of graded abelian groups.

(iv) *Milnor Additivity Axiom.* Let $\{(\mathbf{C}_\alpha, \mathbf{C}'_\alpha; \Phi_\alpha)\}$ be a collection of admissible pairs in $\mathscr{C}at{\downarrow}\mathbf{B}$; then the inclusion functors $\{U_\alpha:\mathbf{C}_\alpha \to \bigsqcup \mathbf{C}_\alpha\}$ induce the isomorphism

$$\bigoplus hU_\alpha : \bigoplus h(\mathbf{C}_\alpha, \mathbf{C}'_\alpha; \Phi_\alpha) \xrightarrow{\simeq} h(\bigsqcup \mathbf{C}_\alpha, \bigsqcup \mathbf{C}'_\alpha; \bigsqcup \Phi_\alpha)$$

in $\mathscr{A}b^{\mathbf{Z}}$.

The Dimension Axiom below is defined to reflect the Dimension Axiom for $\mathscr{K}{\downarrow}N\mathbf{B}$.

(v) *Dimension Axiom.* Assume $(X, X'; \phi)$ is an admissible pair in $\mathscr{K}{\downarrow}N\mathbf{B}$.

(a) For every nondegenerate $x \in X_k$ with $\phi x = b$, the representable map

$$\hat{x}:(\Delta[k], \dot{\Delta}[k]; \hat{b}) \to (\Delta(x), \dot{\Delta}(x); \phi)$$

induces the natural isomorphism

$$h(\Gamma\hat{x}):h(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \xrightarrow{\simeq} h(\Gamma\Delta(x), \Gamma\dot{\Delta}(x), \eta(\mathbf{B})\circ\Gamma\phi).$$

(b) For each $b \in B_k$, $h_r(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) = 0$, whenever $r \neq k$.

(c) Normalization Requirement: Because $\Gamma:\mathscr{K} \to \mathscr{C}at$ preserves bicartesian squares (Lemma 2.3), the image of the bicartesian square [6; IV, 2]

$$\begin{tikzcd}
\dot{\Delta}[k-1] \arrow[r, hook] \arrow[d, hook] & \Delta[k-1] \arrow[d, hook, "\Gamma N(\delta^i)"] \\
\Lambda^i[k] \arrow[r, hook] & \dot{\Delta}[k]
\end{tikzcd}$$

is bicartesian in $\mathscr{C}at{\downarrow}\mathbf{B}$. The Excision Axiom ensures that, for each $b \in (N\mathbf{B})_k$

$$hu:h(\Gamma\Delta[k-1], \Gamma\dot{\Delta}[k-1]; \eta(\mathbf{B})\circ\Gamma\widehat{\delta^i b}) \xrightarrow{\simeq} h(\Gamma\dot{\Delta}[k], \Gamma\Lambda^i[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \tag{26}$$

is an isomorphism. The Exactness Axiom in conjunction with isomorphism (26) yields the commutative diagram

$$\begin{tikzcd}
h_k(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \arrow[r, "\partial_k"] \arrow[ddr, "F^i(b)"'] & h_{k-1}(\Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \arrow[d, "(\Gamma j)_{k-1}"] \\
 & h_{k-1}(\Gamma\dot{\Delta}[k], \Gamma\Lambda^i[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \arrow[d, "\alpha_{k-1}"] \\
 & h_{k-1}(\Gamma\Delta[k-1], \Gamma\dot{\Delta}[k-1]; \eta(\mathbf{B})\circ\Gamma\widehat{\delta_i b})
\end{tikzcd}$$

with $j:(\dot{\Delta}[k-1]; \hat{b})\to(\dot{\Delta}[k], \Lambda^i[k]; \hat{b})$ the inclusion map, $\alpha_{k-1}\equiv(h_{k-1}u)^{-1}$ and $F^i(b)\equiv\alpha_{k-1}\circ h_{k-1}(\Gamma j)\circ\partial_k$. The normalization requirement is that whenever $b\in B_k$ is degenerate,

$$F^i(b): h_k(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b})\to h_{k-1}(\Gamma\Delta[k-1], \Gamma\dot{\Delta}[k-1]\circ\Gamma\widehat{\delta^i b}) \tag{28}$$

be an isomorphism.

Definie the *coefficient system* $A:\Gamma N\mathbf{B}\to\mathscr{A}b$ for $\langle h, \partial\rangle$ as follows. For each object $(b, [k])$ of $\Gamma N\mathbf{B}$,

$$A(b, [k])\equiv h_k(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}). \tag{29}$$

The images of the generating morphisms of $\Gamma N\mathbf{B}$,

$$\delta^i:(b, [k])\to(\delta^i b, [k-1]), \qquad \sigma^i:(b, [k])\to(\sigma^i b, [k+1])$$

are defined respectively by

$$\begin{aligned}
A(\delta^i)\equiv F^i(b): &\, h_k(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \\
&\to h_{k-1}(\Gamma\Delta[k-1], \Gamma\dot{\Delta}[k-1]; \eta(\mathbf{B})\circ\Gamma\widehat{\delta^i b}) \\
A(\sigma^i)\equiv(F^i(\sigma^i b))^{-1}: &\, h_k(\Gamma\Delta[k], \Gamma\dot{\Delta}[k]; \eta(\mathbf{B})\circ\Gamma\hat{b}) \\
&\to h_{k+1}(\Gamma\Delta[k+1], \Gamma\dot{\Delta}[k+1]; \eta(\mathbf{B})\circ\Gamma\widehat{\sigma^i n}).
\end{aligned} \tag{30}$$

Since $\delta^i\sigma^i b=b$, the normalization requirement guarantees that $F^i(\sigma^i b)$ is an isomorphism, and hence that $A(\sigma^i)$ is well defined.

For each costack $A:\Gamma N\mathbf{B}\to\mathscr{A}b$, define $H:A(\mathscr{C}at\downarrow\mathbf{B})\to\mathscr{A}b$ by

$$H((\mathbf{C}, \mathbf{C}'; \Phi), A)\equiv H((N\mathbf{C}, N\mathbf{C}'; N\Phi), A), \tag{31}$$

where $H:A(\mathscr{K}\downarrow N\mathbf{B})\to\mathscr{A}b$ is the unique homology theory for $\mathscr{K}\downarrow N\mathbf{B}$ defined by $A:\Gamma N\mathbf{B}\to\mathscr{A}b$ (see (17)).

Theorem 6.1 (Existence). *If* $A : \Gamma N\mathbf{B} \to \mathscr{A}b$ *is a costack, then* $\langle H, \partial\rangle$ *given by* (31) *defines a homology theory for* $\mathscr{C}at{\downarrow}\mathbf{B}$.

Proof. It suffices to show that $\langle H, \partial\rangle$ satisfies the axioms for homology in $\mathscr{C}at{\downarrow}\mathbf{B}$.

(i) The *Exactness Axiom* is direct consequence of the Exactness Axiom for $H : A(\mathscr{K}{\downarrow}N\mathbf{B}) \to \mathscr{A}b$ and the fact that $N : \mathscr{C}at{\downarrow}\mathbf{B} \to \mathscr{K}{\downarrow}N\mathbf{B}$ preserves inclusions.

(ii) Similarly, the *Excision Axiom* follows from the Excision Axiom for $H : A(\mathscr{K}{\downarrow}N\mathbf{B}) \to \mathscr{A}b$. Although, $N : \mathscr{C}at{\downarrow}\mathbf{B} \to \mathscr{K}{\downarrow}N\mathbf{B}$ preserves all limits and hence intersections, it does not necessarily preserve unions. The requirement of admissibility guarantees that the image of the bicartesian square (25) is a bicartesian square in $\mathscr{K}{\downarrow}N\mathbf{B}$. Thus the Excision Axiom holds.

(iii) The *WFHE Axiom* is an immediate corollary of Theorem 5.4. and the definition of WFHE in $\mathscr{C}at{\downarrow}\mathbf{B}$.

(iv) *Milnor Additivity Axiom.* Since $N : A(\mathscr{C}at{\downarrow}\mathbf{B}) \to A(\mathscr{K}{\downarrow}N\mathbf{B})$ commutes with disjoint union (coproducts in $\mathscr{C}at{\downarrow}\mathbf{B}$), additivity follows from the corresponding additivity of $H : A(\mathscr{K}{\downarrow}N\mathbf{B}) \to \mathscr{A}b$.

(v) *Dimension Axiom.* Assume $(X, X'; \phi)$ is an admissible pair in $\mathscr{K}{\downarrow}N\mathbf{B}$.

(a) For each representable map

$$\hat{x} : (\Delta[k], \dot{\Delta}[k]; \hat{b}) \to (\Delta(x), \dot{\Delta}(x); \phi)$$

of a nondegenerate $x \in X_k$ with $\phi(x) = b$, it suffices to show, from the definition of $\langle H, \partial\rangle$ for $\mathscr{C}at{\downarrow}\mathbf{B}$, that

$$H(N\Gamma\hat{x}) : H((N\Gamma\Delta[k], N\Gamma\dot{\Delta}[k]; \eta(N\mathbf{B}) \circ N\Gamma\hat{b}), A)$$

$$\xrightarrow{\simeq} H((N\Gamma\Delta(x), N\Gamma\dot{\Delta}(x), \eta(N\mathbf{B}) \circ \mathrm{N}\Gamma\phi), \mathrm{A})$$

is an isomorphism in $\mathscr{A}b^{\mathbf{Z}}$. Consider the commutative diagram in $A(\mathscr{K}{\downarrow}N\mathbf{B})$

$$\begin{tikzcd}
(N\Gamma\Delta[k], N\Gamma\dot{\Delta}[k]; \eta(N\mathbf{B}) \circ N\Gamma\hat{b}) \arrow[r, "N\Gamma x"] \arrow[d, "\eta(\Delta[k])"] & (N\Gamma\Delta(x), N\Gamma\dot{\Delta}(x); \eta(N\mathbf{B}) \circ N\Gamma\phi) \arrow[d, "\eta(\Delta(x))"] \\
(\Delta[k], \dot{\Delta}[k]; \hat{b}) \arrow[r, "\hat{x}"] & (\Delta(x), \dot{\Delta}(x); \phi)
\end{tikzcd} \tag{32}$$

Theorem 4.4, insures that $\eta(\Delta[k])$ and $\eta(\Delta(x))$ are WFHE; and hence by Theorem 5.4, $H(\eta(\Delta[k]))$ and $H(\eta(\Delta(x)))$ are both isomorphisms in $\mathscr{A}b^{\mathbf{Z}}$. Since $\langle H, \partial\rangle$ is the unique homology for $\mathscr{K}{\downarrow}N\mathbf{B}$ determined by $A : \Gamma N\mathbf{B} \to \mathscr{A}b$, $\langle H, \partial\rangle$ satisfies the Dimension Axiom and $H(\hat{x})$ is also an isomorphism in $\mathscr{A}b^{\mathbf{Z}}$. Thus three sides of (32) have isomorphisms as images under $\langle H, \partial\rangle$ and $H(\Gamma N\hat{x})$ is therefore also an isomorphism.

(b) For each $b \in B_k$,

$$\eta(\Delta[k]):(N\Gamma\Delta[k], N\Gamma\dot{\Delta}[k]; \eta(N\mathbf{B})\circ N\Gamma\hat{b}) \to (\Delta[k], \dot{\Delta}[k]; \hat{b})$$

is a WFHE by Corollary 4.3. Thus, as above, $H(\eta(\Delta[k]))$ induces an isomorphism of homology groups. The Dimension Axiom for $\langle H, \partial\rangle$ in $\mathcal{K}\downarrow N\mathbf{B}$ ensures that

$$H_r((N\Gamma\Delta[k], N\Gamma\dot{\Delta}[k]; \eta(N\mathbf{B})\circ N\Gamma\hat{b}), \mathrm{A}) = 0$$

for $r \neq k$. Thus condition (b) holds for $\langle H, \partial\rangle$ in $\mathcal{C}at\downarrow\mathbf{B}$.

(c) A similar argument to that of part (a), using both Theorem 4.4 and Theorem 5.4, as well as diagrams of type (32), demonstrates that $\langle H, \partial\rangle$ in $\mathcal{C}at\downarrow\mathbf{B}$ satisfies the normalization requirement.

Thus $\langle H, \partial\rangle$ is a homology theory for $\mathcal{C}at\downarrow\mathbf{B}$. □

Roos [21], Milnor [17], Nöbeling [18], Watts [23], André [1], Laudal [12], and Quillen [20] all have asserted that left derived functors of colimit define a homology theory for $\mathcal{C}at$. The next corollary indicates the connection between derived functors and *some* homology theories for $\mathcal{C}at\downarrow\mathbf{B}$.

Corollary 6.2. *If the costack* $A:\Gamma N\mathbf{B} \to \mathcal{A}b$ *can be factored as*

$$\begin{tikzcd}
\Gamma N\mathbf{B} \arrow[rr, "\eta(\mathbf{B})"] \arrow[dr, "A"'] & & \mathbf{B} \arrow[dl, dashed, "M"] \\
& \mathcal{A}b &
\end{tikzcd}$$

for some $M:\mathbf{B} \to \mathcal{A}b$, *then*

$$H_*((\mathbf{C}, \mathbf{C}'; \Phi), A) \simeq (L_* \operatorname{colim}_{\mathbf{C}})M(\mathbf{C}, \mathbf{C}'; \Phi),$$

where $M(\mathbf{C}, \mathbf{C}'; \Phi):\mathbf{C} \to \mathcal{A}b$ *is given by*

$$M(\mathbf{C}, \mathbf{C}', \Phi)p = \begin{cases} M\Phi(p), & p \notin \operatorname{ob}\mathbf{C}' \\ 0, & p \in \operatorname{ob}\mathbf{C}' \end{cases}$$

Proof. The corollary follows from the fact that the chain complex used to determine homology is the colimit of a canonical coflabby resolution used to calculate the left derived functors of colimit. For details see [6; Appendix II, 3], [18] or [19]. □

Corollary 6.3. *If* $A:\Gamma N\mathbf{B} \to \mathcal{A}b$ *is a costack, then*

$$H_*((\mathbf{C}, \mathbf{C}'; \Phi), A) \simeq (L_* \operatorname{colim}_{\Gamma N\mathbf{C}})A(\Gamma N\mathbf{C}, \Gamma N\mathbf{C}'; \Gamma N\Phi).$$

Proof. By Corollary 4.5,

$$\eta(\mathbf{C}):(\Gamma N\mathbf{C}, \Gamma N\mathbf{C}'; \eta(\mathbf{B})\circ\Gamma N\Phi) \to (\mathbf{C}, \mathbf{C}'; \Phi)$$

is a WFHE in $A(\mathscr{C}at\downarrow\mathbf{B})$. Since $\langle H,\partial\rangle$ is a homology theory for $\mathscr{C}at\downarrow\mathbf{B}$, the WFHE Axiom insure that

$$H(\eta(\mathbf{C})):H((\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi),A)\xrightarrow{\cong}H((\mathbf{C},\mathbf{C}';\Phi),A)$$

is an isomorphism. However,

$$H((\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi),A)=H((\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\Gamma N\Phi),A\circ\eta(\Gamma N\mathbf{B}))$$

because the chain complexes, used to calculate homology in $\mathscr{C}at\downarrow\mathbf{B}$ and $\mathscr{C}at\downarrow\Gamma N\mathbf{B}$ respectively, are identical. By Corollary 6.2

$$H((\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\Gamma N\Phi),A\circ\eta(\Gamma N\mathbf{B}))\simeq(L_*\operatorname{colim}_{\Gamma N\mathbf{C}})A(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\Gamma N\Phi)$$

and the corollary follows. □

Remark 6.4. Not all costacks $A:\Gamma N\mathbf{B}\to\mathscr{A}b$ can be factored through $\mathbf{B}$, i.e. there may *not* exist a module $M:\mathbf{B}\to\mathscr{A}b$ such that $A=M\circ\eta(\mathbf{B})$. For example, let $\mathbf{B}=\mathbf{Z}_2$, considered as a small category: $\mathbf{B}$ has one object p with morphism set $\mathbf{B}(p,p)=\{1,a\mid a^2=1\}$. Then any module $M:\mathbf{B}\to\mathscr{A}b$ can be described as a pair $\langle G,\alpha:G\to G\rangle$, where $G=M(p)$ and $\alpha=M(a)$ is an automorphism of G. By definition of $\eta(\mathbf{B}):\Gamma N\mathbf{B}\to\mathbf{B}$ (see (10) and (11))

$$(M\circ\eta(\mathbf{B}))(\langle p\xrightarrow{b_1}p\longrightarrow\cdots\xrightarrow{b_k}p\rangle,[k])=M(p)=G$$

on *all* objects of $\Gamma N\mathbf{B}$, and is given by

$$(M\circ\eta(\mathbf{B}))(\delta^0)=\begin{cases}\alpha, & \text{if } b_1=a,\\ 1, & \text{if } b_1=1,\end{cases}$$

$$(M\circ\eta(\mathbf{B}))(\delta^i)=1,\quad\text{for } 0<i\leqslant k,$$

$$(M\circ\eta(\mathbf{B}))(\sigma^i)=1,\quad\text{for } 0\leqslant i\leqslant k$$

on generating morphisms of $\Gamma N\mathbf{B}$. Thus none of the morphisms is 0 *unless* $G=0$. Let

$$((q))\equiv\langle p\xrightarrow{a}p\xrightarrow{a}\cdots\xrightarrow{a}p\rangle$$

denote the unique nondegenerate q-simplex and $\Delta((q))$ be the subcomplex of $N\mathbf{B}$ generated by $((q))$. Then

$$A=\Delta\mathbf{Z}(N\mathbf{B},\Delta((q));1):\Gamma N\mathbf{B}\to\mathscr{A}b$$

is a costack with many 0 morphisms, where $\Delta\mathbf{Z}:\Gamma N\mathbf{B}\to\mathscr{A}b$ is the constant costack with value $\mathbf{Z}$. Hence $A\neq\eta(\mathbf{B})\circ M$ for any $M\in[\mathbf{B},\mathscr{A}b]$. Thus there exist more homology theories for $\mathscr{C}at\downarrow\mathbf{B}$ than those "generated" by modules $M:\mathbf{B}\to\mathscr{A}b$.

Theorem 6.5 (Comparison). *Let $\langle h,\partial\rangle$ be a homology theory for $\mathscr{C}at\downarrow\mathbf{B}$ and define $\langle\tilde{h},\partial\rangle$ by*

$$\tilde{h}(X,X';\phi)\equiv h(\Gamma X,\Gamma X';\eta(\mathbf{B})\circ\Gamma\phi)\tag{33}$$

for each $(X, X'; \phi) \in A(\mathscr{K} \downarrow N\mathbf{B})$. *Then* $\langle \tilde{h}, \partial \rangle$ *is a homology theory for* $\mathscr{K} \downarrow N\mathbf{B}$, *and*

$$\tilde{h}(X, X'; \phi) \simeq H((X, X'; \phi), A) \tag{34}$$

where $A : \Gamma N\mathbf{B} \to \mathscr{A}b$ *is the coefficient costack of* $\langle \tilde{h}, \partial \rangle$.

Proof. It suffices to verify that $\langle \tilde{h}, \partial \rangle$ satisfies the axioms for homology in $\mathscr{K} \downarrow N\mathbf{B}$. Formula (34) follows immediately from the uniqueness of such homology theories (Theorem 5.1). Furthermore

$$\tilde{h} \equiv h \circ (\eta(\mathbf{B}))^* \circ \Gamma : \mathscr{K} \downarrow N\mathbf{B} \to \mathscr{A}b, \tag{35}$$

where $\eta(\mathbf{B})^* : \mathscr{C}at \downarrow \Gamma N\mathbf{B} \to \mathscr{C}at \downarrow \mathbf{B}$ is defined by composition with $\eta(\mathbf{B})$ and $\Gamma : \mathscr{K} \downarrow N\mathbf{B} \to \mathscr{C}at \downarrow \Gamma N\mathbf{B}$.

(i) The *Exactness Axiom* is a direct consequence of the Exactness Axiom for $\langle h, \partial \rangle$ and that X' a subsimplicial set of X ensures that $\Gamma X'$ is admissible in ΓX.

(ii) Similarly the *Excision Axiom* follows from the Excision Axiom for $\langle h, \partial \rangle$. By Lemma 2.3, $\Gamma : \mathscr{K} \downarrow N\mathbf{B} \to \mathscr{C}at \downarrow \Gamma N\mathbf{B}$ preserves bicartesian squares. Because $(\eta(\mathbf{B}))^* : \mathscr{C}at \downarrow \Gamma N\mathbf{B} \to \mathscr{C}at \downarrow \mathbf{B}$ has both left and right adjoints, it is bicontinuous; and hence, it also preserves bicartesian squares. Thus $(\eta(\mathbf{B}))^* \circ \Gamma : \mathscr{K} \downarrow N\mathbf{B} \to \mathscr{C}at \downarrow \mathbf{B}$ preserves bicartesian squares and the Excision Axiom follows.

(iii) *SFH Axiom*. From Theorem 5.5, it suffices to verify that the WFHE Axiom holds for $\langle \tilde{h}, \partial \rangle$. Suppose $m : (X, X'; \phi) \to (Y, Y'; \psi)$ is a WFHE in $a(\mathscr{K} \downarrow N\mathbf{B})$. If

$$\Gamma m : (\Gamma X, \Gamma X'; \eta(\mathbf{B}) \circ \Gamma\phi) \to (\Gamma Y, \Gamma Y'; \eta(\mathbf{B}) \circ \Gamma\psi)$$

is a WFHE in $A(\mathscr{C}at \downarrow \mathbf{B})$, then the WFHE Axiom for $\mathscr{C}at \downarrow \mathbf{B}$ ensures that the WFHE Axiom holds for $\mathscr{K} \downarrow N\mathbf{B}$. By definition, Γm is a WFHE in $A(\mathscr{C}at \downarrow \mathbf{B})$ iff $N\Gamma m$ is a WFHE in $A(\mathscr{K} \downarrow N\mathbf{B})$. By Theorem 4.4, the natural transformation $\eta : N\Gamma \dashrightarrow 1d$ yields the following commutative diagram

$$\begin{tikzcd}
(N\Gamma X, N\Gamma X'; \eta(N\mathbf{B}) \circ N\Gamma\phi) \arrow[r, "N\Gamma m"] \arrow[d, "\eta(X)"] & (N\Gamma Y, N\Gamma Y'; \eta(N\mathbf{B}) \circ N\Gamma\psi) \arrow[d, "\eta(Y)"] \\
(X, X'; \phi) \arrow[r, "m"] & (Y, Y'; \psi)
\end{tikzcd}$$

of $A(\mathscr{K} \downarrow N\mathbf{B})$, with three sides WFHE; and thus $N\Gamma m$ *is also a WFHE in* $A(\mathscr{K} \downarrow N\mathbf{B})$, as required.

(iv) *Milnor Additivity Axiom*. Since $(\eta(\mathbf{B}))^* \circ \Gamma : (\mathscr{K} \downarrow N\mathbf{B}) \to \mathscr{C}at \downarrow \mathbf{B}$ is cocontinuous, additivity follows from the corresponding additivity of $h : A(\mathscr{C}at \downarrow \mathbf{B}) \to \mathscr{A}b$.

(v) The *Dimension Axiom* follows immediately from the Dimension Axiom of $\langle h, \partial \rangle$ for $\mathscr{C}at \downarrow \mathbf{B}$; the Dimension Axiom of $\langle h, \partial \rangle$ was defined so that this occurred. ☐

Theorem 6.6 (Uniqueness). *If* $\langle h, \partial\rangle$ *is a homology theory for* $\mathscr{C}at\downarrow\mathbf{B}$ *with coefficient system* $A:\Gamma N\mathbf{B}\to\mathscr{A}b$, *then*

$$h_*(\mathbf{C},\mathbf{C}';\Phi)\simeq H_*((\mathbf{C},\mathbf{C}';\Phi)\simeq(L_*\operatorname{colim}_{\Gamma N\mathbf{C}})A(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\Gamma N\Phi)$$

for every $(\mathbf{C},\mathbf{C}',\Phi)$ *in* $A(\mathscr{C}at\downarrow\mathbf{B})$, *and these equivalences are natural.*

Proof. By Corollary 4.5

$$\eta(\mathbf{C}):(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi)\to(\mathbf{C},\mathbf{C}';\Phi)$$

is a WFHE in $a(\mathscr{C}at\downarrow\mathbf{B})$. Hence, the WFHE Axiom implies

$$h(\eta(\mathbf{C})):h(\mathbf{C},\mathbf{C}';\Phi)\xleftarrow{\simeq}h(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi),$$

naturally. By the comparison theorem, Theorem 6.5,

$$h(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi)\simeq H((N\mathbf{C},N\mathbf{C}';N\Phi),A),$$

where $A:\Gamma N\mathbf{B}\to\mathscr{A}b$ is the coefficient costack for $\langle h\circ(\eta(\mathbf{B}))^*\circ\Gamma,\partial\rangle$. But the existence theorem, Theorem 6.1, ensures

$$H((N\mathbf{C},N\mathbf{C}';N\Phi),A)\equiv H((\mathbf{C},\mathbf{C}';\Phi),A).$$

Corollary 6.3 yields the last equivalence. ☐

Remark 6.7. It is well known (see [23, 19, 12, 18]) that more general coefficient categories $\mathscr{A}$ for $\langle h,\partial\rangle$ can be chosen – any $AB4$ abelian category with enough projectives suffices. In this case, if

$$-\otimes_{\Gamma N\mathbf{C}}-:[(\Gamma N\mathbf{C})^{\mathrm{op}},\mathscr{A}b]\times[\Gamma N\mathbf{C},\mathscr{A}]\to\mathscr{A}$$

is a generalized tensor product, then the functors $\Delta\mathbf{Z}\otimes_{\Gamma N\mathbf{C}}-$ and $\operatorname{colim}_{\Gamma N\mathbf{C}}-$ from $[\Gamma N\mathbf{C},\mathscr{A}]$ to $\mathscr{A}$ are isomorphic; where $\Delta\mathbf{Z}$ is the constant diagram of type $(\Gamma N\mathbf{C})^{\mathrm{op}}$ with value $\mathbf{Z}$. Thus their respective left derived functors, $\operatorname{Tor}_*^{\Gamma N\mathbf{C}}(\Delta\mathbf{Z},-)$ and $L_*\operatorname{colim}_{\Gamma N\mathbf{C}}$ are isomorphic. Hence parallel definitions and arguments as those used in the case for $\mathscr{A}b$, yield the corresponding uniqueness theorem, where

$$H_*((\mathbf{C},\mathbf{C}';\Phi),A)\simeq(L_*\operatorname{colim}_{\Gamma N\mathbf{C}})A(\Gamma N\mathbf{C},\Gamma N\mathbf{C}';\eta(\mathbf{B})\circ\Gamma N\Phi)$$

and $A:\Gamma N\mathbf{B}\to\mathscr{A}$ is a coefficient system.

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