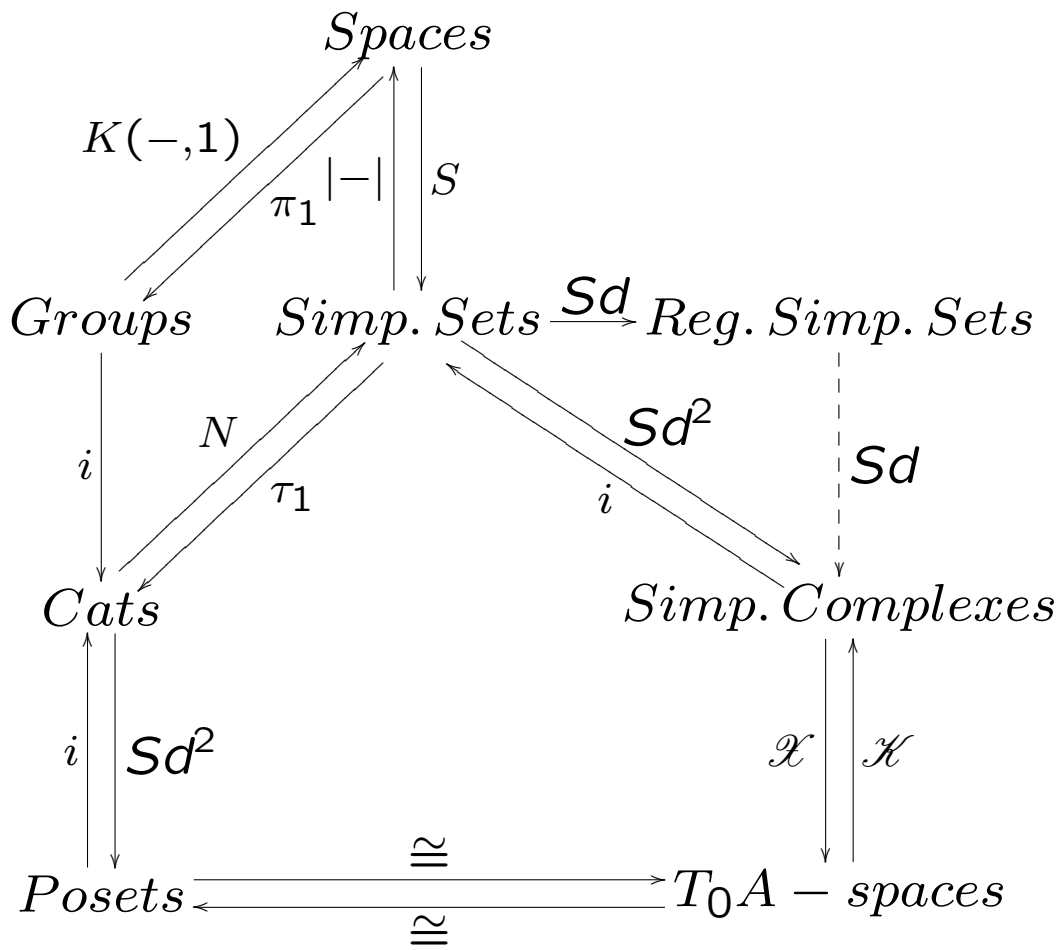


# Categories, posets, Alexandrov spaces, and simplicial complexes

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## Simplicial sets and subdivision

(Any new results are due to Rina Foygel)

$\underline{\Delta} \equiv$  standard simplicial category.

$\Delta[n]$  is represented on  $\underline{\Delta}$  by  $\mathbf{n}$ .

It is  $N\underline{\mathbf{n}}$ , where  $\underline{\mathbf{n}}$  is the poset  $\{0, 1, \dots, n\}$ .

$Sd\Delta[n] \equiv \Delta[n]' \equiv NSd\underline{\mathbf{n}}$ , where

$Sd\underline{\mathbf{n}} \equiv \mathbf{n}' \equiv \text{monos}/\mathbf{n}$ .

$SdK \equiv K \otimes_{\underline{\Delta}} \Delta'$ .

**Lemma 1**  $SdK \cong SdL$  does not imply  $K \cong L$  but does imply  $K_n \cong L_n$  as sets, with corresponding simplices having corresponding faces.

## Regular simplicial complexes

A nondegenerate  $x \in K_n$  is regular if the subcomplex  $[x]$  it generates is the pushout of

$$\Delta[n] \xleftarrow{\delta^n} \Delta[n-1] \xrightarrow{d_n x} [d_n x]$$

$K$  is regular if all  $x$  are so.

**Theorem 1** *For any  $K$ ,  $SdK$  is regular.*

**Theorem 2** *If  $K$  is regular, then  $|K|$  is a regular CW complex:  $(e^n, \partial e^n) \cong (D^n, S^{n-1})$  for all closed  $n$ -cells  $e$ .*

**Theorem 3** *If  $X$  is a regular CW complex, then  $X$  is triangulable; that is  $X$  is homeomorphic to some  $|i(K)|$ .*

## Properties of simplicial sets $K$

Let  $x \in K_n$  be a nondegenerate simplex of  $K$ .

A: For all  $x$ , all faces of  $x$  are nondegenerate.

B: For all  $x$ ,  $x$  has  $n + 1$  distinct vertices.

C: Any  $n + 1$  distinct vertices are the vertices of at most one  $x$ .

**Lemma 2**  $K$  has B iff for all  $x$  and all monos

$$\alpha, \beta: \mathbf{m} \longrightarrow \mathbf{n}, \alpha^*x = \beta^*x \text{ implies } \alpha = \beta.$$

**Lemma 3** If  $K$  has B, then  $K$  has A.

No other general implications among A, B, C.

Properties A, B, C and subdivision

**Lemma 4**  *$K$  has A iff  $SdK$  has A.*

**Lemma 5**  *$K$  has A iff  $SdK$  has B.*

**Lemma 6**  *$K$  has B iff  $SdK$  has C.*

Characterization of simplicial complexes

**Lemma 7**  *$K$  has A iff  $Sd^2K$  has C,  
and then  $K$  also has B.*

**Lemma 8**  *$K$  has B and C iff  $K \in Im(i)$ .*

**Theorem 4**  *$K$  has A iff  $Sd^2K \in Im(i)$ .*

## Subdivision and horn-filling

**Lemma 9** *If  $SdK$  is a Kan complex, then  $K$  is discrete.*

**Lemma 10** *If  $K$  does not have  $A$ , then  $SdK$  cannot be a quasicategory.*

## Relationship of the properties to categories

**Theorem 5** *If  $K$  has  $A$ , then  $SdK \in \text{Im}(N)$ .*

Proof: Check the Segal maps criterion.

**Definition 1** A category  $\mathcal{C}$  satisfies *A*, *B*, or *C* if  $N\mathcal{C}$  satisfies *A*, *B*, or *C*.

**Lemma 11**  $\mathcal{C}$  has *A* iff for any  $i: C \longrightarrow D$  and  $r: D \longrightarrow C$  such that  $r \circ i = id$ ,  $C = D$  and  $i = r = id$ . (Retracts are identities.)

**Lemma 12**  $\mathcal{C}$  has *B* iff for any  $i: C \longrightarrow D$  and  $r: D \longrightarrow C$ ,  $C = D$  and  $i = r = id$ .

**Lemma 13**  $\mathcal{C}$  has *B* and *C* iff  $\mathcal{C}$  is a poset.

**Definition 2** Define a category  $T\mathcal{C}$ :

Objects: nondegenerate simplices of  $N\mathcal{C}$ . e.g.

$$\underline{C} = C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_q$$

$$\underline{D} = D_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow D_r$$

Morphisms: maps  $\underline{C} \longrightarrow \underline{D}$  are maps  $\alpha: \mathbf{r} \longrightarrow \mathbf{q}$  in  $\underline{\Delta}$  such that  $\alpha^*\mathbf{D} = \mathbf{C}$  (implying  $\alpha$  is mono).

Quotient category  $sd\mathcal{C}$  with the same objects:

$$\alpha \circ \beta_1 \sim \alpha \circ \beta_2: \underline{C} \longrightarrow \underline{D}$$

if  $\sigma \circ \beta_1 = \sigma \circ \beta_2$  for a surjection  $\sigma: \mathbf{q} \longrightarrow \mathbf{p}$  such that  $\alpha^*\mathbf{D} = \sigma^*\mathbf{C}$  (so that  $\beta_i: \mathbf{q} \longrightarrow \mathbf{p}$ ).

$$(\beta_i^* \alpha^* \underline{D} = \beta_i^* \sigma^* \underline{C} = \underline{C}, \quad i = 1, 2)$$

(Anderson, Thomason, Fritsch-Latch, del Hoyo)

**Lemma 14** *For any  $\mathcal{C}$ ,  $T\mathcal{C}$  has B.*

**Corollary 1** *For any  $\mathcal{C}$ ,  $sd\mathcal{C}$  has B.*

**Lemma 15**  *$\mathcal{C}$  has B iff  $sd\mathcal{C}$  is a poset.*

**Theorem 6** *For any  $\mathcal{C}$ ,  $sd^2\mathcal{C}$  is a poset.*

Contrast with  $K$  has A iff  $Sd^2K \in Im(i)$ .

Del Hoyo: Equivalence  $\varepsilon: sd\mathcal{C} \longrightarrow \mathcal{C}$ .

(Implied by equivalence  $\varepsilon: SdK \longrightarrow K!$ )

Left adjoint  $\tau_1$  to  $N$  (Gabriel–Zisman)

Objects of  $\tau_1 K$  are the vertices.

Think of 1-simplices  $y$  as maps

$$d_1 y \longrightarrow d_0 y,$$

form the free category they generate, and impose the relations

$$s_0 x = id_x \quad \text{for } x \in K_0$$

$$d_1 z = d_0 z \circ d_2 z \quad \text{for } z \in K_2.$$

The counit  $\varepsilon: \tau_1 N \mathcal{A} \longrightarrow \mathcal{A}$  is an isomorphism.

$\tau_1 K$  depends only on the 2-skeleton of  $K$ . When

$K = \partial \Delta[n]$  for  $n > 2$ , the unit  $\eta: K \longrightarrow \tau_1 N K$

is the inclusion  $\partial \Delta[n] \longrightarrow \Delta[n]$ .

Direct combinatorial proof:

**Theorem 7** *For any  $\mathcal{C}$ ,  $sd\mathcal{C} \cong \tau_1 SdN\mathcal{C}$ .*

**Corollary 2**  $\varepsilon: sd\mathcal{C} \longrightarrow \tau_1 N\mathcal{C} \cong \mathcal{C}$ .

**Corollary 3**  $\mathcal{C}$  has A iff  $SdN\mathcal{C} \cong Nsd\mathcal{C}$ .

**Remark 1** *Even for posets  $P$  and  $Q$ ,  $sdP \cong sdQ$  does not imply  $P \cong Q$ .*

In the development above, there is a counterexample to the converse of each implication that is not stated to be iff.