1 Subdivision of simplicial sets

Let F be a simplicial set, and let F_n denote its n-simplices, with face and degeneracy operators d_i and s_i.

Define simplicial set sdF as follows. Let K_n represent the simplicial complex consisting of an n-simplex and its faces. Let K_n' be its first barycentric subdivision (as a complex). Let Δ_n and Δ_n' be the simplicial sets associated with K_n and K_n', respectively - that is, the non-degenerate simplices of Δ_n are the faces of K_n, etc.

Note that the (nondegenerate) q-simplices of Δ_n correspond to the (injective) maps $[q] \to [n]$ in $\mathbf{\Delta}$.

In K_n', the q-simplices are the chains $x_0 \overset{f_1}{\mapsto} x_1 \ldots x_{q-1} \overset{f_q}{\mapsto} x_q$ of faces $x_0, \ldots x_q$ and such that for each i, x_i is a proper face of x_{i-1} via some composition of face operations f_i. And, face operations on these q-simplices can be expressed as composing adjacent pairs of arrows in this chain, or deleting the first or last arrows. Therefore these are the non-degenerate of Δ_n', and the degenerate simplices can be expressed by allowing x_i to be any face of x_{i-1} rather than necessarily a proper face.

Note that another way to express this is to say that the q-simplices of Δ_n' are chains $\xi_0 \overset{l_1^*}{\mapsto} \xi_1 \ldots \xi_{q-1} \overset{l_q^*}{\mapsto} \xi_q$ where ξ_i is some injective map $[m_i] \to [n]$ and for each i, $\xi_i = l_i^* \left(\xi_{i-1}\right) = \xi_{i-1} \circ l_i$.

Then $\Delta_{[-]}$ and $\Delta_{[]}'$ are covariant functors $\mathbf{\Delta} \to \mathcal{S}$. (The same construction works for nth subdivision, for any n).

Now for a simplicial set F and a covariant functor $\mathbf{\Delta} \xrightarrow{G} \mathcal{S}$, we define simplicial set $F \otimes G$ as follows. $(F \otimes G)_n = \bigcup_n F_n \otimes G\left([n]\right)_q$, under equivalence relation $\left(F\left(\alpha\right)\left(x\right), \sigma\right) \equiv \left(x, G\left(\alpha\right)\left(\sigma\right)\right)$ for any $[m] \xrightarrow{\alpha} [n]$. And for a morphism $[m] \xrightarrow{\alpha} [n]$, the map $\left(F \otimes G\right)\left(\alpha\right)$ will map $(x, \sigma) \mapsto \left(F\left(\alpha\right)\left(x\right), \sigma\right) = \left(x, G\left(\alpha\right)\left(\sigma\right)\right)$.

Note that for any F, $F \otimes \Delta_{[-]} \cong F$, as any $(x, \sigma) \equiv \left(F\left(\sigma\right)\left(x\right), id_{[q]}\right)$ (where the domain of σ is $[q]$).

Define $sdF := F \otimes \Delta'_{[-]}$. Note that by this definition, $sd^n(F) \cong F \otimes \Delta^{(n)}_{[-]}$, and the face and degeneracy operations correspond as well.

Define $\widetilde{sd}F$ by setting $\widetilde{sd}F_n := \{x_0 \overset{Fl_1}{\mapsto} x_1 \overset{Fl_2}{\mapsto} x_2 \ldots \overset{Fl_n}{\mapsto} x_n\}$. Here the l_i's are maps in Δ and the x_i's are non-degenerate simplices of F such that $x_i = l_i(x_{i-1})$ for each i. Note that this forces the l_i's to be injective, i.e. compositions of face operations (or identity maps). Furthermore this shows $\deg(x_{i-1}) \geq \deg(x_i)\ \forall i$. The face and degeneracy operations are given by composing adjacent maps, or inserting identity maps, in the appropriate location.

I will prove that the two definitions are equivalent, in order to use the second (nonstandard) definition in these notes.

Claim 0: $sdF \cong \widetilde{sd}F$.

Proof: A q-simplex in sdF is an element $\left(x, \xi_0 \overset{l_1^*}{\mapsto} \xi_1 \ldots \xi_{q-1} \overset{l_q^*}{\mapsto} \xi_q\right)$, for $x \in F_n$ and each ξ_i a map with target $[n]$. This element corresponds to the q-simplex in $\widetilde{sd}F$, $F(\xi_0)(x) \overset{F(l_1)}{\mapsto} F(\xi_1)(x) \ldots F(\xi_{q-1})(x) \overset{F(l_q)}{\mapsto} F(\xi_q)(x)$. The correspondence preserves the equivalence relation and face and degeneracy operations (by the definition of the two simplicial sets).

$\square$.

2 Property α

Say F has property α if for any non-degenerate $x \in F_n$, and for l_1, l_2 any two injective maps $[k] \to [n]$, $Fl_1(x) = Fl_2(x) \iff l_1 = l_2$. Equivalently, any n-simplex has $\binom{n+1}{k+1}$ distinct non-degenerate k-faces (this is the number of injective maps $[k] \to [n]$).

Claim 1: For any F, sdF has property α.

Proof: Take non-degenerate $X \in sdF_n$ and let L_1, L_2 be injective maps $[k] \to [n]$. Write X as $x_0 \overset{Fl_1}{\mapsto} x_1 \overset{Fl_2}{\mapsto} x_2 \ldots \overset{Fl_n}{\mapsto} x_n$. Note that since X is nondegenerate and for any m, the only injective map $[m] \to [m]$ is the identity, the degrees of the x_i's must strictly decrease.

Write $FL_1(X)$ as $y_0 \overset{Fr_1}{\mapsto} y_1 \overset{Fr_2}{\mapsto} y_2 \ldots \overset{Fr_k}{\mapsto} y_k$. Note that $\{y_0, \ldots y_k\}$ must be a subset of $\{x_0, \ldots x_n\}$ with strictly decreasing degree (since X was non-degenerate) and therefore can find $0 \leq i_0 < \ldots < i_k \leq n$ such that $y_j = x_{i_j}$ for each j. Note that because the degrees of the x_i's strictly decrease, the i_j's are unique. Therefore that this uniquely determines $[k] \overset{L_1}{\to} [n]$ - specifically, have $j \mapsto i_j$.

Therefore if $FL_1(X) = FL_2(X)$, then $L_1 = L_2$.
$\square$.

Claim 2: $F \in \mathcal{S}$ has α if and only if any non-degenerate n-simplex in F has $n+1$ distinct vertices.
Proof: Clearly if F has α then the second statement is true.

The converse is true by an unenlightening combinatorial argument. The details are as follows:

Suppose the second statement is true. Fix some non-degenerate n-simplex x in F. I will show by induction on k that x has $\binom{n+1}{k+1}$ distinct k-faces. By the assumption, this is true for $k = 0$. Assume that it is true for $k-1$; will prove that it's true for k.

Let d_i be the ith face operation. For an m-simplex y, one can apply d_i to y for $0 \leq i \leq m$.

If $j \geq i$, $d_i \circ d_{j+1} = d_j \circ d_i$. Therefore any k-face of x can be obtained by a composition of face operations $d_{i_0} \circ \ldots \circ d_{i_{n-k-1}}$, with $0 \leq i_0 < \ldots < i_{n-k-1} \leq n$. Written in this form, these compositions are distinct.

Observe that for a composition of face operations $d_{i_0} \circ \ldots \circ d_{i_s}$, if $i_a > r \ \forall a$, then in reduced form $d_{j_0} \circ \ldots \circ d_{j_s}$, will have $j_a > r \ \forall a$. This is true because the reduction step (replacing $d_j \circ d_i$ with $d_i \circ d_{j+1}$ if $j \geq i$) does not lower any indices.

Now take any two distinct compositions of face operations of length $n-k$, $l_1 = d_{i_0} \circ \ldots \circ d_{i_{n-k-1}}$ and $l_2 = d_{j_0} \circ \ldots \circ d_{j_{n-k-1}}$. We want to show $l_1(x) \neq l_2(x)$.

Let r be the unique integer such that for $a < r$, $i_a = j_a$, and also $i_r \neq j_r$. Assume $i_r < j_r$. Note that as the indices must strictly increase (reading from left to right), $i_r \geq r$.

Case 1: $i_r = r$.
Then $i_0 = 0, \ldots i_{r-1} = r-1$. Then apply d_1 to each map. $d_1 \circ l_1 = d_1 \circ d_0 \circ \ldots \circ d_r \circ d_{i_{r+1}} \circ \ldots \circ d_{i_{n-k-1}} = d_0 \circ d_1 \circ \ldots \circ d_r \circ d_{r+2} \circ d_{i_{r+1}} \circ \ldots \circ d_{i_{n-k-1}}$. And, $d_1 \circ l_2 = d_1 \circ d_0 \circ \ldots \circ d_{j_r} \circ d_{j_{r+1}} \circ \ldots \circ d_{j_{n-k-1}} = d_0 \circ d_1 \circ \ldots \circ d_{r-1} \circ d_{r+1} \circ d_{j_{r+1}} \circ \ldots \circ d_{j_{n-k-1}}$. By the observation above, the reduced form of $d_1 \circ l_2$ will therefore not have a term d_r, but the reduced form of $d_1 \circ l_1$ will have a term d_r. Therefore $d_1 \circ l_1 \neq d_1 \circ l_2$. By the inductive hypothesis, we therefore have two distinct $(k-1)$-faces of x, $d_1 \circ l_1 (x) \neq d_1 \circ l_2 (x)$. Therefore $l_1 (x) \neq l_2 (x)$ as desired.

Case 2: $i_r > r$.
Let s be the unique integer such that for $a < s$, $i_a = a$, and $i_s > s$. Note that in Case 2, we must have $0 \leq s \leq r$ Then apply d_0 to each map. $d_0 \circ l_1 = d_0 \circ d_0 \circ \ldots \circ d_{s-1} \circ d_{i_s} \circ \ldots \circ d_{i_{r-1}} \circ d_{i_r} \circ \ldots \circ d_{i_{n-k-1}} = d_0 \circ \ldots \circ d_s \circ d_{i_s} \circ \ldots \circ d_{i_{r-1}} \circ d_{i_r} \circ \ldots \circ d_{i_{n-k-1}}$. And, $d_0 \circ l_2 = d_0 \circ d_0 \circ \ldots \circ d_{s-1} \circ d_{i_s} \circ \ldots \circ d_{i_{r-1}} \circ d_{j_r} \circ \ldots \circ d_{j_{n-k-1}} = d_0 \circ \ldots \circ d_s \circ d_{i_s} \circ \ldots \circ d_{i_{r-1}} \circ d_{j_r} \circ \ldots \circ d_{j_{n-k-1}}$. These are both written in reduced form; since $i_r \neq j_r$, we have $d_0 \circ l_1 \neq d_0 \circ l_2$. As in Case 1, this shows $l_1 (x) \neq l_2 (x)$ as desired.
. $\square$.

3 Property β

Say F has property β if for any 0-simplices $v_0, \ldots v_n$, there is at most one n-simplex x with vertex set $\{v_0, \ldots v_n\}$.

Claim 3: F has property α if and only if sdF has property β.

Proof: First suppose F has α. Take any $x_0, \ldots x_n$, 0-simplices in sdF (that is, simplices in F). Suppose this is the vertex set of some n-simplex in sdF. Then this n-simplex must be in the form $X = x_{i_0} \overset{Fl_1}{\mapsto} x_{i_1} \overset{Fl_2}{\mapsto} x_{i_2} \ldots \overset{Fl_n}{\mapsto} x_{i_n}$, where $\{i_0, \ldots i_n\} = \{0, \ldots n\}$ and (since the x_i's are distinct) $\deg(x_{i_0}) > \deg(x_{i_1}) >$

$\ldots > \deg(x_{i_n})$.

By α, the l_i's are therefore unique; therefore such X is unique since the order of the x_i's is determined according to degree.

Now suppose sdF has β, and suppose for some non-degenerate $x \in F_n$, for l_1, l_2 some two injective maps $[k] \to [n]$, have $Fl_1(x) = Fl_2(x) =: y$. Then $x \xrightarrow{Fl_1} y$ and $x \xrightarrow{Fl_2} y$ are 1-simplices with the same vertex set. Therefore by β, they must be equal, and therefore $l_1 = l_2$. Therefore α holds for F.
$\square$.

4 Relationship to simplicial complexes

There is a standard full embedding of the category of simplicial complexes, $\mathcal{SC}plx$, into $\mathcal{S}$, by creating all the necessary degenerate simplices. Call this functor D. It is known that for $F \in \mathcal{S}$, $F \in D(\mathcal{SC}plx)$ if and only if the vertex sets non-degenerate simplices of F satisfy the criteria for a simplicial complex.

Claim 4: For $F \in \mathcal{S}$, $F \in D(\mathcal{SC}plx)$ iff F has α and β.

Proof: Suppose the non-degenerate simplices of F form a simplicial complex. Then each non-degenerate simplex is uniquely determined by its vertices and therefore β holds.

Conversely suppose α and β hold. Let V be the set of 0-simplices of F and set $K := \{S \subset V \mid \exists n,\ \exists x \in F_n,\ \{v_0(x), \ldots v_n(x)\} = S\}$. By α, any n-simplex x has $\binom{n+1}{m+1}$ distinct m-simplices as its faces, i.e. obtainable via the $\binom{n+1}{m+1}$ different injective maps $[m] \to [n]$. So, for any $S \in K$, for $T \subset S$, have $T = \{v_0(y), \ldots v_m(y)\}$ for some $y = Fl(x)$, for some injective map $[m] \xrightarrow{l} [n]$. Therefore K is a set of finite subsets of V such that for $S \in K$, any subset of S is also in K.
$\square$.

5 Relationship to quasicategories, categories, and the nerve functor

A ‘quasicategory’ is a simplicial set F such that for any n, $0 < k < n$, and map $\Lambda_n^k \to F$ extends to Δ_n as in the diagram below:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & B \\
C \arrow[ur, dashed, "\widetilde{f}"] &
\end{tikzcd}$$

Here Λ_n^k is the $(n-1)$-faces of Δ_n minus the kth $(n-1)$-face, where Δ_n is the simplicial set whose q-simplices are maps $[q] \to [n]$ with faces and degeneracies via precomposition. Note that there are $\binom{n+1}{q+1}$ non-degenerate q-simplices for each q.

Claim 5: For any simplicial set F, sdF is a quasicategory.

Proof: Take some map $\Lambda_n^k \to sdF$, with $0 < k < n$. We will show that the map extends to some $\Delta_n \to sdF$.

For $n \leq 1$ the statement is trivial as there is no such k. For $n = 2$, the map $\Lambda_2^1 \to sdF$ specifies 0-simplices x_0, x_1, x_2 and 1-simplices $x_0 \overset{Fl_1}{\to} x_1$ and $x_1 \overset{Fl_2}{\to} x_2$. Composing $Fl_2 \circ Fl_1 = F(l_1 \circ l_2)$ gives the desired third 1-simplex, and the desired 2-simplex is $x_0 \overset{Fl_1}{\to} x_1 \overset{Fl_2}{\to} x_2$. Note that by construction this extension is unique.

For $n > 2$, this is trivial again as any n composable 1-simplices of sdF (composable in F) specify a unique n-simplex with the given 1-faces.
$\square$.

Let $\alpha\mathcal{S}$, $\alpha\beta\mathcal{S}$ be the full subcategories of $\mathcal{S}$ whose objects are simplicial sets with α, and simplicial sets with both α and β, respectively. Let $\mathcal{Q}$ be the full subcategory of $\mathcal{S}$ whose objects are quasicategories. Define $\alpha\mathcal{Q} := \mathcal{Q} \cap \alpha\mathcal{S}$ and $\alpha\beta\mathcal{Q} := \mathcal{Q} \cap \alpha\beta\mathcal{S}$.

Now let N be the nerve functor, $\mathcal{C}at \overset{N}{\to} \mathcal{S}$. For $\mathcal{C} \in \mathcal{C}at$, $N(\mathcal{C})_n = \{A_0 \overset{f_1}{\to}$

$A_1 \xrightarrow{f_2} A_2 \ldots A_{n-1} \xrightarrow{f_n} A_n\}$, where the A_i's and f_i's are objects and morphisms in $\mathcal{C}$. It is known that N is an embedding, i.e. it is injective on objects and full and faithful on morphism sets. Therefore we can regard $\mathcal{C}at$ as a full subcategory of $\mathcal{S}$. It is also known that a quasicategory F is the nerve of a category if and only if any map $\Lambda_n^k \to F$ extends uniquely to a map from Δ_n. Note that by definition, $N(\mathcal{C})$ is a quasicategory for any $\mathcal{C}$, and so we have $\mathcal{C}at \subset \mathcal{Q} \subset \mathcal{S}$.

Claim 5.5: For $F \in \mathcal{S}$, sdF is the nerve of a category.

Proof: This is equivalent to showing that any map $\Lambda_n^k \to sdF$ extends uniquely to a map from Δ_n. But in the proof in Claim 5, the extensions given are unique.

□.

Now define $\alpha\mathcal{C}at := \mathcal{C}at \cap \alpha\mathcal{S}$ and $\alpha\beta\mathcal{C}at := \mathcal{C}at \cap \alpha\beta\mathcal{S}$.

Claim 6: For $\mathcal{C} \in \mathcal{C}at$, $\mathcal{C} \in \alpha\mathcal{C}at$ if and only if for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\mathcal{C}$, $A = B$ and $f = g = id$.

Proof: Suppose $\mathcal{C} \in \alpha\mathcal{C}at$ and have $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$. Then 2-simplex $A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$ in $N(\mathcal{C})$ does not have enough distinct 1-faces, and must therefore be degenerate. Therefore f or g must be an identity map and the claim is proved.

Conversely suppose that for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\mathcal{C}$, $A = B$ and $f = g = id$. Now take an n-simplex $A_0 \xrightarrow{f_1} A_1 \ldots A_n$ in $N(\mathcal{C})$. If this simplex is non-degenerate, then by the assumptions on $\mathcal{C}$, the A_i's are distinct. Then there are clearly $\binom{n+1}{k+1}$ distinct k-faces of this simplex, and so α holds.

□.

6 Relationship to posets

Claim 7: For $\mathcal{C} \in \mathcal{C}at$, $\mathcal{C} \in \alpha\beta\mathcal{C}at$ if and only if $\mathcal{C}$ is a poset.

Proof: Suppose $\mathcal{C}$ is a poset. Then for an n-simplex $A_0 \xrightarrow{f_1} A_1 \ldots A_n$ in $N(\mathcal{C})$, if it is non-degenerate (does not contain any identity maps) then the A_i's are distinct. Property α follows. Furthermore, for a set $\{A_0, \dots A_n\}$ of 0-simplices (objects in $\mathcal{C}$), an n-simplex with this vertex set gives a total ordering on these objects, agreeing with the partial order on $\mathcal{C}$. This total ordering, if it exists, must be unique. Therefore property β holds.

Conversely suppose $\mathcal{C}$ is a small category and $N(\mathcal{C})$ has α and β. Then for any x and y objects in $\mathcal{C}$, there is at most one 1-simplex in $N(\mathcal{C})$ with vertices x and y, and therefore there is at most one morphism $x \to y$ for any $x, y \in \mathcal{C}$, and morphisms $x \to y$ and $y \to x$ imply $x = y$. Therefore $\mathcal{C}$ is a poset.
$\square$.

Claim 8: The embedding $\mathcal{P} \cong \alpha\beta\mathcal{C}at \to \alpha\beta\mathcal{Q}$ is an isomorphism.
Proof: This is equivalent to showing that if $F \in \alpha\beta\mathcal{Q}$, then F is the nerve of a category (because this will show surjectivity, and already know that we have a full embedding, which gives an isomorphism).

Suppose we have a map $\Lambda_n^k \xrightarrow{f} F$ and two extensions, $\Delta_n \overset{g_1, g_2}{\to} F$. Each extension is uniquely determined by the image of the single non-degenerate n-simplex in Δ_n, call them x_1 and x_2. But the vertex sets of x_1 and x_2 are identical, because f determines the images of the 0-simplices in Δ_n (assuming here that $n \geq 2$ for non-triviality). Then by condition β, $x_1 = x_2$ and therefore the extension is unique.
$\square$.

Now we have the following diagram of subcategories of $\mathcal{S}$. Here $\mathcal{C}at$, $\alpha\mathcal{C}at$, and $\alpha\beta\mathcal{C}at$ are represented by their images under the functor N, and are therefore subcategories of $\mathcal{S}$. Arrows in the diagram represent subdivision (and show the strongest possible claims that can be made).

$$\begin{tikzcd}
\mathcal{C}at \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cup"] & N(\mathcal{C}at) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] \arrow[d, bend right] & \mathcal{Q} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] \arrow[dl, bend right] & \mathcal{S} \arrow[d, phantom, "\cup"] \arrow[dll, bend right] & \\
\alpha\mathcal{C}at \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cup"] & N(\alpha\mathcal{C}at) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] \arrow[d, bend right] & \alpha\mathcal{Q} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] \arrow[dl, bend right] & \alpha\mathcal{S} \arrow[d, phantom, "\cup"] \arrow[dll, bend right] & \\
\alpha\beta\mathcal{C}at \arrow[r, phantom, "\cong"] \arrow[d, phantom, "="] & N(\alpha\beta\mathcal{C}at) \arrow[r, phantom, "="] & \alpha\beta\mathcal{Q} \arrow[r, phantom, "\subset"] & \alpha\beta\mathcal{S} \arrow[r, phantom, "="] \arrow[ll, bend left] & D(\mathcal{S}\mathcal{C}plx) \arrow[d, phantom, "\cong"] \\
\mathcal{P} & & & & \mathcal{S}\mathcal{C}plx
\end{tikzcd}$$

7 Subdivision of categories

Next, I will construct a subdivision functor $\mathcal{C}at \overset{sd}{\to} \mathcal{C}at$ that agrees with the above subdivision process; that is, the following diagram will commute:

$$\begin{tikzcd}
\mathcal{C}at \arrow[r, "N"] \arrow[d, "sd"] & \mathcal{S} \arrow[d, "sd"] \\
\mathcal{C}at \arrow[r, "N"] & \mathcal{S}
\end{tikzcd}$$

For $\mathcal{C} \in \mathcal{C}at$, construct $sd\mathcal{C}$ as follows. The objects of $sd\mathcal{C}$ will be the non-degenerate simplices of $N(\mathcal{C})$, that is, chains (of any length) of composable non-identity arrows, e.g. $\mathbf{A} = A_0 \overset{f_1}{\to} A_1 \ldots A_n$. For $\mathbf{B} = B_0 \overset{g_1}{\to} B_1 \ldots B_m$, set $sd\mathcal{C}(\mathbf{A}, \mathbf{B})$ to equal the set of 1-simplices x in $N(\mathcal{C})$ such that $d_0(x) = \mathbf{A}$, $d_1(x) = \mathbf{B}$.

In other words, a morphism from $\mathbf{A}$ to $\mathbf{B}$ is a map $[m] \overset{l}{\to} [n]$ such that when l is applied to $\mathbf{A}$, we obtain $\mathbf{B}$. Note that l must be injective for $\mathbf{B}$ to be non-degenerate; therefore we can restrict our attention to injective l, and the morphisms in $sd\mathcal{C}$ with domain $\mathbf{A}$ are the k-faces of $\mathbf{A}$ (ranging over all k between 0 and n).

It is clear by definition that the above diagram will commute. Furthermore, for the adjoint to N, $\mathcal{S} \overset{c}{\to} \mathcal{C}at$, it is known that $c \circ N \cong id_{\mathcal{C}at}$. Therefore we have $sd\mathcal{C} \cong c \circ N(sd\mathcal{C}) = c \circ sd \circ N(\mathcal{C})$, and so up to isomorphism, this definition is the same as definining the subdivision of $\mathcal{C}$ to be $c \circ sd \circ N(\mathcal{C})$.

We can add this to the diagram (here double arrows represent subdivision of categories):

$$\begin{tikzcd}
\mathcal{C}at \arrow[dd, Rrightarrow, bend right] & \cong & N(\mathcal{C}at) & \subset & \mathcal{Q} \arrow[ddll, bend right] & \subset & \mathcal{S} \arrow[ddllll, bend right] & \\
\cup & & \cup & & \cup & & \cup & \\
\alpha\mathcal{C}at \arrow[dd, Rrightarrow, bend right] & \cong & N(\alpha\mathcal{C}at) \arrow[dd, bend right] & \subset & \alpha\mathcal{Q} \arrow[ddll, bend right] & \subset & \alpha\mathcal{S} \arrow[ddllll, bend right] & \\
\cup & & \cup & & \cup & & \cup & \\
\alpha\beta\mathcal{C}at & \cong & N(\alpha\beta\mathcal{C}at) & = & \alpha\beta\mathcal{Q} & \subset & \alpha\beta\mathcal{S} \arrow[llll, bend left] & = D(\mathcal{SC}plx) \\
= & & & & & & & \cong \\
\mathcal{P} & & & & & & & \mathcal{SC}plx
\end{tikzcd}$$

8 The claims above are as strong as possible

I will show that stronger claims about these properties cannot be made, through ‘counterexamples’.

Check 1: For a simplicial set F, sdF does not necessarily have β.

Let the non-degenerate simplices of F be $x, y \in F_0$ and $a, b \in F_1$ with $d_0(a) = d_0(b) = x$ and $d_1(a) = d_1(b) = y$. Then in sdF, non-degenerate simplices $x \overset{a}{\mapsto} y$ and $x \overset{b}{\mapsto} y$ violate β.

Check 2: $\alpha\mathcal{S}$ is a proper subcategory of $\mathcal{S}$.

Let the non-degenerate simplices of F be $x \in F_0$ and $a \in F_1$ with $d_0(a) = d_1(a) = x$. This violates α.

Check 3: $\alpha\beta\mathcal{S}$ is a proper subcategory of $\alpha\mathcal{S}$.

The counterexample for Check 1 works here as well.

Check 4: $N(\alpha\mathcal{C}at)$ is a proper subcategory of $\alpha\mathcal{Q}$.

Let F have non-degenerate simplices $x, y, z \in F_0$, $a, b, c \in F_1$, and $m, n \in F_2$, with $d_0(a) = d_0(c) = x$, $d_0(b) = d_1(a) = y$, $d_1(b) = d_1(c) = z$, $d_0(m) = d_0(n) = a$, $d_1(m) = d_1(n) = b$, and $d_2(m) = d_2(n) = c$. This gives a counterexample as desired.

Check 5: $\alpha\mathcal{Q}$ is a proper subcategory of $\mathcal{Q}$.

The counterexample for Check 2 works here. (Note that because the only non-degenerate edge is a, and its vertices are both equal to x, and because there are no non-degenerate n-simplices for $n \geq 2$, any map $\Lambda^k_n \xrightarrow{f} F$ for any k, n is trivially extendable).

Check 6: $\alpha\beta\mathcal{Q}$ is a proper subcategory of $\alpha\mathcal{Q}$.

The counterexample for Check 1 works here. (Note that because there is no pair of non-degenerate edges with $d_1(edge_1) = d_0(edge_2)$, and no non-degenerate n-simplices for $n \geq 2$, any map $\Lambda_n^k \xrightarrow{f} F$ for any k, n is trivially extendable).

Check 7: $\alpha\mathcal{Q}$ is a proper subcategory of $\alpha\mathcal{S}$.

Let the non-degenerate simplices of F be $x, y, z \in F_0$, and $a, b \in F_1$, with $d_0(a) = x$, $d_0(b) = d_1(a) = y$, $d_1(b) = z$.

Check 8: $\alpha\beta\mathcal{Q}$ is a proper subcategory of $\alpha\beta\mathcal{S}$.

The counterexample for Check 8 works here.

Check 9: $\alpha\mathcal{C}at$ is a proper subcategory of $\mathcal{C}at$.

Most categories do not have α, e.g. the category of groups.

Check 10: $\alpha\beta\mathcal{C}at$ is a proper subcategory of $\alpha\mathcal{C}at$.

Let $\mathcal{C}$ be the category with object set $\{A, B\}$ and non-identity morphisms $\{A \xrightarrow{f} B,\ A \xrightarrow{g} B\}$.

To do:

1. prove that my definition of subdivision coincides with the tensor thing. DONE
2. give 'counterexamples' to show that claims are as strong as possible. DONE but need to make sure I'm not missing any.
3. include the functors between posets=a-spaces and simplicial complexes.
4. see if there's any relationship with Top, with other functors, etc.
5. prove claim 2 (about an equivalent condition for alpha). DONE