NOTES ON PICARD GROUPOIDS

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We consider connective spectra X. If the only nonzero homotopy group of X is the zeroth, then X is an Eilenberg–Mac Lane spectrum $K(\pi_0(X), 0)$, and π_0 establishes an equivalence between the homotopy category of such spectra and the category of Abelian groups. The same holds for the category of spaces X with a unique non-zero homotopy group for any fixed $n \geq 2$. That is, we are in the stable range for such spaces at dimension 2. We want to understand the analogous statements for connective spectra X such that $\pi_i(X) = 0$ for all $i > q$ for some fixed q. It is an exercise to determine the relevant stable range for spaces with non-zero homotopy groups between n and $n + q$ which can be read off from known results. When $q = 1$, it is $n \geq 3$.

In these notes, I'll discuss the case $q = 1$, which is used non-trivially by Sasha Beilinson and Deepam Patel, to whom the basic result is due.

Definition 0.1. A Picard groupoid is a small symmetric monoidal groupoid

$$(\mathscr{P}, \otimes, I, c)$$

in which every object is invertible under $\otimes$.

Standard categorical yogi (e.g. [4, 4.2], [5, VI.3.2]) shows that $\mathscr{P}$ is equivalent as a symmetric monoidal category to a permutative category, in which $\otimes$ is strictly associative and unital, with unit object I. Moreover, any skeleton of a permutative category is again permutative. Passage to the skeleton of an equivalent permutative category preserves the property of being a groupoid and of having invertible objects, so it takes Picard groupoids to permutative and skeletal Picard groupoids. Since the term strict permutative groupoid is sometimes used for the strictly commutative case (which is uninteresting to us), we call these "elementary Picard groupoids". We have indicated the proof of the following result.

Proposition 0.2. *Any Picard groupoid is equivalent (as such) to an elementary Picard groupoid.*

We abbreviate π_0 for the group of objects of $\mathscr{P}$ under $\otimes$ and we let π_1 denote the group of automorphisms of the object I. As in any monoidal category (symmetric not being needed for this), π_1 is an Abelian group. Since $\mathscr{P}$ is skeletal, $A \otimes B$ and $B \otimes A$ for objects A and B are the same element of π_0, but we must remember and encode the commutativity isomorphism c.

We give a way to think about this.

Think of π_0 as a discrete category (only identity maps), and write A for both an object and its identity map. It still makes sense to think of π_0 as a group, but now the product and inverse are thought of as functors between discrete categories. This makes π_0 a discrete monoidal category, and we could view it as a strict permutative

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groupoid. Think of the Abelian group π_1 as a category with a single object $*$. Using $I \otimes I = I$, we see as usual that the tensor product and composite of automorphisms of I agree. Form the product category $\pi_1 \times \pi_0$. Give it the evident product $\otimes$:

$$(*, A) \otimes (*, B) = (*, A \otimes B)$$

on objects, and

$$(f, A) \otimes (g, B) = (fg, A \otimes B)$$

on morphisms. The unit object is $(*, I)$. This product is strictly associative and unital, and it makes sense for any pair of Abelian groups (π_0, π_1). We have just perversely denoted the group structure on π_0 as $\otimes$.

We can ask for a commutativity isomorphism. It assigns to each pair (A, B) a morphism

$$(c_I(A, B), A \otimes B)$$

where $c_I(A, B) \in \pi_1$. Thus c_I is a function

$$c_I : \pi_0 \times \pi_0 \longrightarrow \pi_1.$$

Since π_1 is an Abelian group, we automatically have the "naturality condition" $fc_I(A, B) = c_I(A, B)f$ for all $f \in \pi_1$. We must impose the symmetry condition

$$c_I(A, B)^{-1} = c_I(B, A)$$

and the coherence conditions

$$c_I(I, A) = id = c_I(A, I)$$

for all objects A and the triple permutation condition

$$c_I(A \otimes B, C) = c_I(A, C)c_I(B, C)$$

for all triples of objects A, B, C. In the presence of the first of these three conditions, if we change to additive notation for the product (and inverse) in both π_0 and π_1, we see that the second and third conditions amount to bilinearity, so that c_I is really nothing but a homomorphism

$$c_I : \pi_0 \otimes \pi_0 \longrightarrow \pi_1$$

of Abelian groups satisfying $c_I(A \otimes B) = -c_I(B \otimes A)$. Written in this alternative notation, the trivial choice $c_I(A, B) = \text{id}$ for all pairs A, B is given by $c_I(A \otimes B) = 0$. This gives a strict Picard groupoid, but there are other choices with the same underlying monoidal category. Any choice of a commutativity isomorphism c_I, meaning a function $c_I : \pi_0 \times \pi_0 \longrightarrow \pi_1$ satisfying the anticommutativity relation $c_I(A \otimes B) = -c_I(B \otimes A)$ gives an example

$$\mathscr{P}(\pi_0, \pi_1, c_I)$$

of an elementary Picard groupoid. Up to isomorphism, these are the only examples.

Proposition 0.3. *Any elementary Picard groupoid $\mathscr{P}$ is isomorphic to an elementary Picard groupoid of the form $\mathscr{P}(\pi_0, \pi_1, c_I)$.*

Proof. We have already described how to construct π_0 and π_1 from $\mathscr{P}$. We begin by defining inverse isomorphisms of categories

$$F : \mathscr{P} \longrightarrow \pi_1 \times \pi_0$$

$$F^{-1} : \pi_1 \times \pi_0 \to \mathscr{P}.$$

Note that since we start with a skeletal groupoid, there are no maps $A \longrightarrow B$ for $A \neq B$. For a map $g\colon A \to A$, define

$$g_I = g \otimes A^{-1}\colon I = A \otimes A^{-1} \to A \otimes A^{-1} = I.$$

For a map $f\colon I \to I$, define

$$f_A = f \otimes A : A = I \otimes A \to I \otimes A = A.$$

We then define F by $F(A) = (*, A)$ on objects and $F(g) = (g_I, A)$ on morphisms $g\colon A \longrightarrow A$. We define F^{-1} by $F^{-1}(*, A) = A$ on objects and $F^{-1}(f, A) = f_A$ on morphisms $f\colon I \longrightarrow I$. This already gives an isomorphism of monoidal categories. What about c? We are given isomorphisms

$$c(A, B) : A \otimes B \longrightarrow B \otimes A$$

in $\mathscr{P}$. Applying F to this morphism gives a morphism

$$(c_I(A, B), A \otimes B)\colon A \otimes B \longrightarrow A \otimes B,$$

where $c_I(A, B) \in \pi_1$. Coherence in $\mathscr{P}$ says that these elements satisfy the algebraic properties of a commutativity isomorphism that we prescribed above. Conversely, we can reconstruct the coherence isomorphism c from a commutativity isomorphism $c_I : \pi_0 \times \pi_0 \longrightarrow \pi_1$. □

By the two results above, any Picard groupoid is equivalent to an elementary Picard groupoid $\mathscr{P}(\pi_1, \pi_0, c_I)$. Consider the classifying space functor B. As a space, and ignoring c_I, $B(\pi_1 \times \pi_0)$ is just the disjoint union over the elements $A \in \pi_0$ of copies of the group π_1, so that $B(\pi_1 \times \pi_0)$ is the disjoint union over A of copies of the classifying space $B\pi_1 = K(\pi_1, 1)$. Therefore the homotopy groups of $B(\pi_1 \times \pi_0)$ are π_0 (an Abelian group by definition) and π_1. We need to understand what c_I corresponds to on the classifying space level, and it is clear conceptually that it must correspond to the unique k-invariant of a two-stage spectrum or space (in a stable range).

Form the associated spectrum $E\mathscr{P}(\pi_1, \pi_0, c_I)$, as characterized axiomatically in [6]. The unique axiom on such a functor E from permutative categories $\mathscr{C}$ to spectra is that there is a natural group completion map $\eta\colon B\mathscr{C} \longrightarrow E_0\mathscr{A}$. In our case, $B\mathscr{P}(\pi_1, \pi_0, c_I)$ is grouplike (π_0 is a group), and this implies that η is a weak homotopy equivalence. Therefore π_0 and π_1 are the only non-vanishing homotopy groups of $E\mathscr{P}(\pi_1, \pi_0, c_I)$.

Conversely, consider spectra X with two non-vanishing homotopy groups π_0 and π_1. Such an X fits into an exact triangle (cofiber sequence)

$$K(\pi_1, 1) \longrightarrow X \longrightarrow K(\pi_0, 0) \longrightarrow K(\pi_1, 2)$$

of spectra; call the last map k: it is the unique k-invariant of this two-stage Postnikov system of spectra. Working with spaces in the stable range, we can think of this as a two-stage Postnikov system

$$K(\pi_1, n+1) \longrightarrow X \longrightarrow K(\pi_0, n) \longrightarrow K(\pi_1, n+2)$$

of spaces for $n \geq 3$.

We know how to describe such spaces purely algebraically, for example by work of Loday [3] and, especially relevant, [2]; see also [1, 7]. For $n = 1$, this goes all the way back to very old work of J.H.C. Whitehead, but we are interested only in the simpler case of large n. We need only compare definitions to see that one of the several known algebraic descriptions of such two-stage spaces is equivalent to the description given by a commutativity isomorphism c_I, as above. This amounts to describing algebraically how to calculate $H^{n+2}(K(\pi_0, n); \pi_1)$. In fact, the triple (π_0, π_1, C_I) is a very special case of a stable crossed module [2, 3.1] or stable quadratic module [1, IV.C.1], and one way to proceed is to check that every stable quadratic module is suitably equivalent to one of this form. This exercise is carried out, not very explicitly, in [2, 3.10].

An alternative, purely stable, verification that E specifies an equivalence from the category of Picard groupoids to the homotopy category of spectra whose only non-vanishing homotopy groups are π_0 and π_1 can be obtained by using the Segal machine to construct E and then reconstructing from such a two-stage spectrum a very special Γ-space that is equivalent to one that comes from a Picard groupoid. This approach is worked out in Deepam Patel's thesis [8].

What happens for larger q might make a good thesis topic. We know the connection between symmetric monoidal categories and spectra. But making π_0 an Abelian group necessarily kills all π_i with $i > 2$, so it doesn't seem that an answer in terms of symmetric monoidal categories with special properties makes sense. Rather, we should think in terms of higher categorical versions of grouplike symmetric monoidal categories, starting with the appropriate formulation of the notion of a Picard 2-groupoid.

Interesting!

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