Generalized Witt Schemes

Justin Noel

Department of Mathematics
University of Chicago

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Goals

- Extend the correspondence between formal algebraic geometry to include the Witt scheme and its dual.
- Use this to realize the curves functor in topology.
- We then see that "p-typification" of curves gives rise to
  - Generalized Husemoller-Witt splittings
  - Quillen's idempotent operation splitting $MU_{(p)}$.
- $\mathrm{Spf}(E^0(J)) \cong \widehat{\mathbb{W}}^{\mathbb{Z}_p^\times}$, for p odd and E Landweber exact or Morava K-theory.

Outline

- Recall how formal groups arise in topology.
- Scheme theoretic perspective of formal groups and the Witt Vectors.
- Extending this correspondence to include $\widehat{\mathbb{W}}_{E_0} \cong \mathrm{Spf}(E^0(BU))$.
- Applications of this correspondence.

Generalized Chern Classes

- Assume E is an even-periodic cohomology theory, i.e. $\pi_* E$ is non-canonically isomorphic to $E_0[\lambda, \lambda^{-1}]$ with $|\lambda| = 2$.
- For each such E we have a theory of Chern classes.

$$E^0(BU(n)) \cong E_0[[c_1, \ldots, c_n]]$$

- The space $BU = \operatorname{colim} BU(n)$ inherits the operations $\oplus$ and $\otimes$ which make it the 0-component of a ring up to homotopy.
- This implies that $E^0(BU) \cong E_0[[c_1, c_2, \ldots]]$ is a coring object (without unit) in the category of E_0-algebras.
- Up to completion, $E^0(BU)$ is isomorphic to its dual $E_0(BU)$ as Hopf algebras.

Quillen's Theorem

- The tensor product operation makes $E^0(BU(1)) \cong E_0[[c_1]]$ into a bicommutative Hopf-algebra.
- After choosing a generator c_1, set

$$F(x, y) = \Delta_{\otimes}(c_1) \in E_0[[x, y]].$$

- Quillen's Theorem: The formal group associated to

$$E = MP \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU$$

is universal:

$$\mathrm{Ring}(MP_0, R) \cong \text{Set of formal group laws defined over R}$$

.

Extensions of Quillen's Theorem

- Adams showed that the map $MP_0 \to E_0$ classifying the above formal group law is realized by a map of ring spectra $MP \to E$.
- This example illustrates a correspondence between the algebra of formal groups and even-periodic ring spectra.
- This correspondence has been extended in many ways (the chromatic filtration, elliptic cohomology, Strickland's equivalence for Landweber exact formal groups, etc.)

Schemes and Formal Schemes

- Given our natural aversion to co-operations such as $\Delta_\oplus$ and $\Delta_\otimes$, we can rephrase the algebra in the opposite category.
- The category of affine schemes is the subcategory of representable functors in $\mathrm{Set}^{\mathrm{Rings}}$.
- We obtain the category of formal schemes by formally adjoining filtered colimits to affine schemes.
- Examples:
  - $\mathrm{Spec}(\mathbb{Z}[x]/(x^n)) = \mathrm{Rings}(\mathbb{Z}[x]/(x^n)), -) \cong \mathrm{Nil}_n$.
  - $\widehat{\mathbb{A}}^1 = \mathrm{Spf}(\mathbb{Z}[[x]]) \cong \mathrm{colim}\,\mathrm{Spec}(\mathbb{Z}[x]/(x^n)) \cong \mathrm{Nil}$.
- The Yoneda Lemma gives an equivalence between Affine schemes and $\mathrm{Ring}^{\mathrm{op}}$ and between formal schemes and pro–$\mathrm{Ring}^{\mathrm{op}}$.

Formal Groups as Formal Schemes

- We can now rephrase the above to say:
  - $\mathrm{Spf}(E^0(BU(1)))$ is a formal group (a group object in formal schemes).
  - $\mathrm{Spf}(E^0(BU))$ is a formal ring scheme (without unit). If we forget the multiplication we have an infinite-dimensional formal group.
  - $\mathrm{Spec}(MP^0(S^0)$ is the functor that takes a ring to the set of formal group laws defined over that ring.
- Many natural objects in the theory of formal groups have similar analogues in topology.

The Witt Scheme

- The (big) Witt Scheme $\mathbb{W}$ is an affine ring scheme whose underlying scheme is $\mathbb{A}^{\infty}$.
- The ring structure is very unusual.
- For example, $\mathbb{W}(\mathbb{F}_p) \cong \prod_{(n,p)=1} \mathbb{Z}_p$

The Dual Witt Scheme

- The Cartier dual of the Witt Scheme $\mathbb{W}$ is an infinite-dimensional formal group $\widehat{\mathbb{W}}$.
- $\mathbb{W}$ and $\widehat{\mathbb{W}}$ are nearly the same (the representing Hopf-algebras are isomorphic after completion).
- This formal group represents the *Curves* functor:

$$\mathcal{C}(\widehat{\mathbb{G}}) = \mathrm{FSch}_*(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) \cong \mathrm{FGp}(\widehat{\mathbb{W}}, \widehat{\mathbb{G}})$$

.

Structure of $\mathbb{W}_p$ and $\widehat{\mathbb{W}}_p$

- There are operations $V^m, F^n, \langle a \rangle$ on $\widehat{\mathbb{W}}$ for $m, n \in \mathbb{N}$ and $a \in \mathbb{Z}$.
- Over a p-local ring we can use these to construct a splitting:

$$\widehat{\mathbb{W}} \cong \prod_{(n,p)=1} \widehat{\mathbb{W}}_p$$

- This splitting is constructed in the *same* way as the splitting

$$MU_{(p)} \simeq \bigvee_{(n,p)=1} BP$$

- Niles Johnson and I have shown that the projection $MU_{(2)} \to BP$ is not H_∞ and hence not E_∞.

Realizing $\widehat{\mathbb{W}}_{E_0}$ as $\mathrm{Spf}(E^0(BU))$

- $\mathbb{W}_R$ is represented by $R[\theta'_1, \theta'_2, \dots]$ with

$$\Delta_{\oplus}(\theta'_n) = \sum_{i=0}^{n} \theta'_i \otimes \theta'_{n-i}$$

- This makes its (topological) dual $R[[\theta_1, \theta_2, \dots]]$ with analogous coproduct.
- So, clearly we have $\mathrm{Spec}(E_0(BU)) \cong \mathbb{W}_{E_0}$.

$Spf(E^0(BU))$ Represents Curves

- We have $\widehat{\mathbb{W}} = \operatorname{colim} \widehat{\mathbb{W}_n}$ where $\widehat{\mathbb{W}}_n$ is determined by

$$fSch_*(\mathrm{Spec}(\mathbb{Z}[x]/(x^{n+1}), \widehat{\mathbb{G}}) \cong fGpSch(\widehat{\mathbb{W}}_n, \widehat{\mathbb{G}}).$$

- Recall $\operatorname{colim} \mathbb{C}P^n \cong \mathbb{C}P^\infty$ and $\operatorname{colim} \Omega SU(n+1) \simeq BU$.
- We realize the topological analogue is

$$fSch_*(\mathrm{Spec}(E^0(\mathbb{C}P^n, \widehat{\mathbb{G}}) \cong fGpSch(\mathrm{Spf}(E^0(\Omega SU(n+1)), \widehat{\mathbb{G}}).$$

- Moreover, if $\widehat{\mathbb{G}} \cong \mathrm{Spf}(E^0(BU(1)))$ then we can trace through a series of adjunctions (after dualizing) to see that a curve (which induces an isomorphism $\widehat{\mathbb{A}}^1 \cong \widehat{\mathbb{G}}$) defines a map of *ring spectra* $MU \to E$.

Husemoller-Witt Splitting

- The splitting for $\widehat{\mathbb{W}}_{E_0}$ when E is p-local gives us

$$E^0(BU) \cong \prod_{(n,p)=1} B(n,p)$$

.

- We can see that this splitting induces the maps that give the splitting of $MU_{(p)}$.
- The first splitting is *purely algebraic* while the second is topological, but they both arise from the same construction.

New multiplicative structures on $\widehat{\mathbb{W}}_{E_0}$

- Each formal group law F, associated to E (given by an isomorphism with the formal affine line) determines a multiplicative (without unit) structure on the $\widehat{\mathbb{W}}_{E_0}$.
- This structure is determined by the formula for the total E-theory Chern class for a tensor product of two stable bundles.

$$c((\eta_1 - [n]) \otimes (\eta_2 - [m])) = \frac{c(\eta_1 \otimes \eta_2)}{c(\eta_1)^m c(\eta_2)^n}$$

- If $\eta_1 = \sum_{j=1}^{n} \eta_{1,j}$ and $\eta_2 = \sum_{k=1}^{m} \eta_{2,k}$ then

$$c(\eta_1 \otimes \eta_2) = \prod_{i,j} (1 + (c_1(\eta_{1,i}) +_F c_1(\eta_{2,j})))$$

Application to J

- Set E to be an even-periodic Landweber exact cohomology theory or even Morava K-theory and p-complete all spaces.
- The unstable Adams operations on BU act on this formal group scheme. We can identify these with $\mathbb{Z}_p^\times$.
- The fixed points of this group action on $\widehat{\mathbb{W}}_{E_0}$ give the formal group scheme $\operatorname{Spf} E^0(J)$.
- This follows from the work of RWY that give us the short exact sequence of Hopf algebras:

$$E^0(J) \leftarrow E^0(BU) \overset{\psi^k - 1}{\longleftarrow} E^0(BU)$$

Summary

- We've setup an algebro-geometric interpretation of the E cohomology of BU and J.
- We've used the algebraic geometry to *simultaneously* construct the algebraic splitting of $E^0(BU)$ and the topological splitting of $MU_{(p)}$.
- The topology constructs a new product structure on $\widehat{\mathbb{W}}_{E_0}$ that the algebraic geometers didn't see.