NOTES ON VITALE PAPER

Categorial groupoids (cat-groups) are Baez-Lauda coherent 2-groups. For such a $\mathscr{G}$, $\pi_0(\mathscr{G})$ is the group of isomorphism classes of objects of G and $\pi_1(\mathscr{G})$ is the Abelian group $\mathscr{G}(I, I)$.

For a bicategory $\mathscr{B}$, $cl(\mathscr{B})$, the classifying category has objects those of $\mathscr{B}$, maps 2-isomorphism classes of 1-morphisms.

Fix $F\colon \mathscr{G} \longrightarrow \mathscr{H}$ (strong monoidal understood) of cat-groups.

(i) F is essentially surjective iff $\pi_0 F$ is surjective.
(ii) F is faithful iff $\pi_1 F$ is injective.
(iii) F is full iff $\pi_0 F$ is injective and $\pi_1 F$ is surjective.
(iv) F is an equivalence iff $\pi_0 F$ and $\pi_1 F$ are isomorphisms.

$0\colon \mathscr{G} \longrightarrow \mathscr{H}$: all arrows go to identity of I.

$k\colon Ker(F) \longrightarrow \mathscr{G}$: map of cat-groups with nat trans $\varepsilon\colon F \circ k \longrightarrow 0$, universal property (p. 388). Kernel exists (p.389).

If $\mathscr{G}$ and $\mathscr{H}$ are braided and F is braided, then $Ker(F)$ is braided, universal property for braided functors.

$\pi_1 k\colon \pi_1(Ker(F)) \longrightarrow Ker(\pi_1(F)$ is iso.

$\pi_0 k\colon \pi_1(Ker(F)) \longrightarrow Ker(\pi_1(F)$ is epi.

(i) F is faithful iff $\pi_1(Ker(F)) = 0$.
(ii) F is full iff $\pi_0(Ker(F)) = 0$.
(iii) F is full and faithful iff $Ker(F)$ is equivalent to 1.

$c\colon \mathscr{G} \longrightarrow Coker(F)$: map of cat-groups with nat trans $\pi\colon p \circ F \longrightarrow 0$, universal property (p. 391). Cokernel exists when $\mathscr{G}$ and $\mathscr{H}$ are symmetric and so is F (p.391). Braided needed to get monoidal structure, symmetric needed to show that the commutativity constraint is natural (p. 393).

If $\mathscr{G}$ and $\mathscr{H}$ are braided and F is braided, then $Ker(F)$ is braided, universal property for braided functors.

$\pi_0 c\colon Coker(\pi_0 F) \longrightarrow \pi_0(Coker(F))$ is iso.

$\pi_1 c\colon Coker(\pi_1 F) \longrightarrow \pi_1(Coker(F))$ is mono.

Assuming everything is symmetric:

(i) F is essentially surjective iff $\pi_0(Coker(F)) = 0$.
(ii) F is full iff $\pi_1(Coker(F)) = 0$.
(iii) F is full and essentially surjective iff $Coker(F)$ is equivalent to 1.

(i) F is full and essentially surjective iff $\pi_0(Ker(F)) = 0 = \pi_0(Coker(F)) = 0$.
(ii) F is full and faithful iff $\pi_1(Ker(F)) = 0 = \pi_1(Coker(F)) = 0$.
(iii) F is an equivalence iff $Ker(F)$ and $Coker(F)$ and equivalent to 1.
(iv) $\pi_0(Ker(F)) \cong \pi_1(Coker(F))$.

Exact sequence $\mathscr{G} \xrightarrow{F} \mathscr{H} \xrightarrow{G} \mathscr{K}$, with $\phi\colon G \circ F \longrightarrow 0$. Get $F'\colon G \longrightarrow Ker(G)$. Say (F, ϕ, G) is 2-exact if F' is full and essentially surjective. In symmetric case, this is true iff $Coker(F)$ is equivalent to 1.

Applying π_0 and π_1, get two three term exact seqeunces.

Let $\mathscr{C}$ be monoidal, and let $\mathscr{BC}$ be the bimonoidal category of monoids, bimodules, and maps of bimodules. For latter, assume that coequalizers are stable under tensor product (automatic in closed case). If $\mathscr{C}$ is braided, $cl(\mathscr{BC})$ is monoidal but not braided. If $\mathscr{C}$ is symmetric, then $cl(\mathscr{BC})$ is symmetric. Assuming braided where needed, get

(i) Picard cat-group $\mathscr{P}(\mathscr{C})$ of invertible objects and isomorphisms of $\mathscr{C}$.
(ii) Brauer cat-group $\mathscr{B}(\mathscr{C}) = \mathscr{P}(cl(\mathscr{BC}))$.

(i) $\pi_0(\mathscr{P}(\mathscr{C}))$ is the Picard group $Pic(\mathscr{C})$.
(ii) $\pi_1(\mathscr{P}(\mathscr{C}))$ is the unit Abelian group $U(\mathscr{C})$.
(iii) $\pi_0(\mathscr{B}(\mathscr{C}))$ is the Brauer group of $\mathscr{C}$.
(iv) $\pi_1(\mathscr{B}(\mathscr{C}))$ is again the Picard group $Pic(\mathscr{C})$.

$F\colon \mathscr{C} \longrightarrow \mathscr{D}$ coequalizer preserving (strong) monoidal functor, braided when needed, induces functors on these constructions. Assume henceforward that everything is symmetric, as is in fact needed.

Construct a bicategory $\mathscr{E}$ [$\mathbb{F}$ in Vitale, p.398], give $cl(\mathscr{E})$ a symmetric monoidal structure, and define define $\tilde{\mathscr{F}} = \mathscr{P}(cl(\mathscr{E}))$ [$\mathscr{F}$ in Vitale]. Construct a five term sequence of cat-groups

$$\mathscr{P}(\mathscr{C}) \longrightarrow \mathscr{P}(\mathscr{D}) \longrightarrow \tilde{\mathscr{F}} \longrightarrow \mathscr{B}(\mathscr{C}) \longrightarrow \mathscr{B}(\mathscr{D})$$

Not exact on π_0, so not 2-exact. Construct quotient $\mathscr{F}$ [$\bar{\mathscr{F}}$ in Vitale] to obtain a 2-exact five term sequence of symmetric cat-groups

$$\mathscr{P}(\mathscr{C}) \longrightarrow \mathscr{P}(\mathscr{D}) \longrightarrow \mathscr{F} \longrightarrow \mathscr{B}(\mathscr{C}) \longrightarrow \mathscr{B}(\mathscr{D})$$

Apply π_0 and π_1 to obtain two five term exact sequences. Splice to obtain a ten term exact sequence as follows, where F_2 and F_{-1} are the evident kernel and cokernel, $F_1 = \pi_1(\mathscr{F})$ and $F_0 = \pi_0(\mathscr{F})$.

$$0 \longrightarrow F_2 \longrightarrow U(\mathscr{C}) \longrightarrow U(\mathscr{D}) \longrightarrow F_1 \longrightarrow Pic(\mathscr{C})$$

$$\longrightarrow Pic(\mathscr{D}) \longrightarrow F_0 \longrightarrow Br(\mathscr{C}) \longrightarrow Br(\mathscr{D}) \longrightarrow F_{-1} \longrightarrow 0.$$