PROJECT DESCRIPTION

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1. Background

The PI proposes to work under the sponsorship of Hal Sadofsky at the University of Oregon. The development of good categories of spectra has enabled topologists to make wholesale translations of classical algebra into a 'brave new' algebra of spectra. This theory enables not only further calculational work but also an organization of topological objects and theory according to algebraic properties and concepts. Such algebraic organization has been important in other fields, and the PI's interest is in a categorical perspective on algebraic concepts in topology and related areas of mathematics.

The PI's thesis develops Morita theory in generalized, bicategorical contexts. Generalizations of Morita theory have been of consistent interest in a broad range of mathematical disciplines from algebraic, to categorical, to homotopy-theoretic. In derived and homotopical contexts, the development of Morita theory has required more delicacy than algebraic and categorical treatments. The work of Rickard in [Ric89] and [Ric91] generalizes Morita theory to derived categories of rings, and the work of Schwede-Shipley [SS03] and Dugger-Shipley [DS07b] gives partial extensions of Morita theory to differential graded rings and ring spectra. These results are known collectively as tilting theory. However, counterexamples in [DS07b] demonstrate a barrier to the expected progression of Morita theory.

The PI integrates a bicategorical perspective on Morita theory with the previous work in homotopical contexts to provide a synthesis of Morita theory in algebraic, topological, and algebro-geometric contexts. The exposition is aimed at an audience not necessarily familiar with either categorical or topological Morita theory, and it is the PI's express intention that bicategorical Morita theory support the broad network of researchers studying Morita theory in their various fields.

One feature of the bicategorical perspective is the explanation of classical and modern Morita theory in terms of the bicategorical Yoneda lemma. This provides a vantage from which to distinguish between those parts of the theory which are formal and those which depend on additional structure in particular contexts. The PI makes use of this perspective to highlight the relationship between all forms of Morita theory and questions of enrichment. The theory proceeds in parallel with the classical theory when one restricts scope to equivalences which preserve ambient closed structure; the need for this restriction is not apparent in the classical theory or its extension to derived categories of rings because the algebraic structure is rigid enough for the necessary enrichment to be automatic. When one generalizes to consider differential graded rings or ring spectra, this is no longer the case. Work of Dugger-Shipley [DS07a] develops specific model-theoretic conditions in the topological setting which guarantee the necessary enrichment is preserved, and the PI hopes to work at the University of Oregon, in part, to gain from Dugger's experience with topological developments of Morita theory.

As a continuation of the PI's work in Morita theory, the PI's thesis develops the following characterization of Azumaya objects in bicategorical contexts, formally generalizing the several equivalent conditions which characterize classical Azumaya algebras. The statement holds generally in a symmetric monoidal bicategory with unit R, but for concreteness R may be read as a commutative ring, differential graded algebra, or ring spectrum. In any of these contexts, we have a corresponding notion of R-algebra, and A denotes an R algebra regarded as a left-module over R and a right module over its enveloping algebra, A^e. The monoidal product, $\otimes$, may be read either as the tensor product in the category of modules over a ring (or differential graded algebra), or as the derived tensor or smash product in the derived (or homotopy) category of modules, depending on the reader's choice for R and A. The internal (derived) hom in the appropriate category is denoted simply Hom. Recall that in the derived category of modules over an algebra, the isomorphisms are the quasi-isomorphisms and their formal inverses.

Theorem. *The following statements for an R-algebra A are equivalent and define the notion of an Azumaya object over R.*

i. $(A, \mathrm{Hom}_{A^e}(A, A^e))$ *is an invertible pair.*
ii. $(\mathrm{Hom}_R(A, R), A)$ *is an invertible pair.*
iii. *a)* *The evaluation* $\mathrm{Hom}_{A^e}(A, A^e) \otimes_R A \to A^e$ *is an isomorphism.*
b) *The coevaluation* $A \otimes_{A^e} \mathrm{Hom}_{A^e}(A, A^e) \to \mathrm{Hom}_{A^e}(A, A)$ *is an isomorphism.*
c) *The unit map* $R \to \mathrm{Hom}_{A^e}(A, A)$ *is an isomorphism.*
iv. *a)* *The evaluation* $A \otimes_{A^e} \mathrm{Hom}_R(A, R) \to R$ *is an isomorphism.*
b) *The coevaluation* $\mathrm{Hom}_R(A, R) \otimes_R A \to \mathrm{Hom}_R(A, A)$ *is an isomorphism.*
c) *The unit map* $A^e \to \mathrm{Hom}_R(A, A)$ *is an isomorphism.*
v. *a)* *A is dualizable as an A^e-module.*
b) *The unit induces* $R \cong \mathrm{Hom}_{A^e}(A, A)$.
c) *A^e is A-local.*
vi. *a)* *A is dualizable as an R-module.*
b) *The unit induces* $A^e \cong \mathrm{Hom}_R(A, A)$.
c) *R is A-local.*

Conditions *i* and *ii* say that A^e is Morita equivalent to R. In the classical case of an algebra over a commutative ring, condition *iii* says that A is a central separable R-algebra, and *iv* says that A is faithfully projective over R. In the topological case, conditions *v* and *vi* are proven to be equivalent by Baker-Lazarev [BL04], however the formal connection to classical conditions is new.

This result allows the PI to give a generalized definition of Azumaya objects as elements of the categorical Brauer group – a definition familiar to category theorists – and yet retain a calculable description of these objects. A similar result on invertibility for modules instead of algebras gives a new proof of the tilting theory results for derived categories of differential graded algebras or ring spectra.

2. Proposed Work: Objectives, Methods, and Expected Significance

The proposed work has three parts: one main project, and two projects in preliminary stages, possibly appropriate as concurrent or future work.

The central focus of the proposed work is to make use of the classification for Azumaya objects to approach Brauer group calculations in topological or derived contexts. As the project develops, this work can expand to consider other practical questions of invertibility in homotopy-theoretic settings. A secondary project, related to this one, seeks to develop a spectrum-level analog of categorical constructions giving generalized Picard and Brauer groups. The objects of study in this case are 'three-stage spectra', having at most three non-trivial homotopy groups. These groups would be the unit, Picard, and Brauer groups of a commutative ring spectrum.

A third project grows out of joint work between Justin Noel and the PI. This project uses computer calculations to build a body of evidence for H_∞ structure on the Brown-Peterson spectrum, BP. This project has already yielded results of interest at the prime 2, and has the potential to motivate renewed interest in the subject.

2.1. Brauer Groups of Ring Spectra.

The proposed work will develop an approach to calculations of Brauer groups for spectra and other generalized contexts. Much of the classical work with Brauer groups depends crucially on ideal theory and the reduction from general commutative rings to fields. In areas which lack these tools, an alternative approach is needed. The brave new algebra of spectra, for example, lacks any known form of ideal theory and consequently Brauer group calculations in topological contexts are essentially nonexistent. The only known non-trivial example of an Azumaya ring spectrum is the mod-2 complex K-theory spectrum, $KU/2$ as an algebra over the 2-adic completion $\widehat{KU}_2$. This example was developed by Baker-Lazarev [BL04], which points out how topological Azumaya algebras provide topological Hochschild cohomology calculations.

Vitale [Vit02] develops a notion of exact sequence for categorical groups (cat-groups, also known as weak 2-groupoids or Picard-groupoids) and proves that a symmetric monoidal functor between symmetric monoidal categories induces a 5-term exact sequence of naturally constructed Picard and Brauer cat-groups. This, in turn, gives rise to an exact sequence which relates unit, Picard, and Brauer groups of the two categories.

Hopkins-Mahowald-Sadofsky [HMS94] and Hovey-Sadofsky [HS99] have studied Picard groups for the $K(n)$-local and $E(n)$-local sphere spectra, and the PI has considered Vitale's exact sequences with calculations from this work as input. To obtain Brauer group results, it is necessary to have a better understanding of Azumaya objects and the middle term in Vitale's sequence, whose objects may be thought of as "relative Azumaya objects". This was the genesis of the PI's interest in generalized Azumaya objects.

The PI will to use the characterization of Azumaya objects to carry out these calculations. For this, it will be necessary for the PI to understand how the categorical characterization of the PI's thesis works out in topological practice; that is, how to use this characterization to understand specific examples and computations of topological Azumaya algebras. Hal Sadofsky's work on Picard groups makes him uniquely suited to mentor this aspect of the PI's research.

Morita equivalences, Picard groups, Azumaya algebras, and Brauer groups are all ideas from classical algebra which find natural categorical generalizations in the notion invertibility for 1-cells in a bicategory. It is likely that the experience gained through work with Brauer groups will enable the PI to study invertibility more generally, and this is a possible future direction. Although the focus will be applications to ring spectra, there is also potential for this project to be applicable in algebro-geometric and other homotopical contexts. By developing the applications in categorical language, this proposed work supports communication among researchers throughout the mathematical disciplines interested in invertibility.

2.2. **Three-Stage Spectra.** During the PI's study of Vitale's 5-term exact sequence, Peter May observed a relationship to current work in algebraic geometry. Beilinson [Bei06] relates ε-factors to cat-groups (there called Picard groupoids), and the thesis of Patel [Pat08] develops the relationship between Picard-groupoids and spectra having non-trivial homotopy only in π_0 and π_1 – making use of Segal's very special Γ-spaces. An alternative approach has been outlined by May, and is potentially applicable to the study of spectra with at most three non-trivial homotopy groups. The PI will investigate whether this can yield a topological version of Vitale's work.

In addition to the connection with algebraic geometry, such work would be relevant to higher category theory. There is considerable interest in the relationship between homotopy n-types and weak n-groupoids. Duskin [Dus] uses the inputs of unit, Picard, and Brauer groups for a commutative ring to develop a simplicial set with precisely these three non-trivial homotopy groups. An analogous construction of a spectrum would fit precisely into Vitale's work. Moreover, this may yield insight more generally on the problem of modeling, in categorical terms, spectra with finitely many non-zero homotopy groups.

2.3. **Calculational Evidence for H_∞ structure on BP.** Working at fixed prime, p, McClure [BMMS86, VIII.7.8] gives a completely explicit power series in $BP^*(B\mathbb{Z}/p)$ for each $n \geq 0$. This power series comes from the study of power operations on $MU_{(p)}$ and the Quillen idempotent; McClure shows that this series is identically zero for all n not of the form $p^k - 1$ if and only if BP inherits an H_∞ structure from $MU_{(p)}$ via the Quillen projection $r : MU_{(p)} \to BP$. Currently, this is the only imaginable source of ring structure for BP; Hu-Kriz-May [HKM] use Dyer-Lashof operations to prove that there can be no S-algebra map $BP \to MU_{(p)}$, and so the Quillen inclusion $s : BP \to MU_{(p)}$ cannot be a map of structured ring spectra.

In joint work, Noel and the PI developed a Mathematica package which computes approximations to this power series for various choices of p and n. The calculations clearly indicate that BP at $p = 2$ does not inherit an H_∞ structure from the map r. Emboldened by these calculations, Noel and the PI give an elementary proof of this fact simply by considering the constant term of the power series in McClure's formula. However, the calculations at odd primes so far have given zero for the necessary values of n.

Basic questions about the structure of BP have gone unanswered for more than 30 years. The evidence compiled by this work has the potential to motivate renewed interest and effort in this direction, and may give insight toward new conceptual approaches. Observing some calculational similarities, Noel and the PI have begun to investigate whether the perspective on power operations developed by Ando [And92] may be helpful. On the other hand, this work has the appeal of reducing a subtle and difficult topological question to a series of combinatorial problems. Although these are quite complex, and formally no less difficult, they put the question in a form that is accessible to a much more general audience. This enables Noel and the PI to generate interest among a public with minimal mathematical background.

3. Career Development

Understanding how to apply categorical results in a specific setting, making use of additional structure to improve the results in that setting, is a potentially subtle task. Sadofsky's experience with Picard groups for spectra will be especially valuable for this aspect of the PI's work with Brauer groups of spectra. The PI has worked hard – and enjoyed the work – toward bicategorical results, and the PI has learned a great deal of both calculational and theoretical algebraic topology. However, other than the work with BP the PI has limited experience carrying out independent calculations in specialized topological settings. Substantial experience with new calculations in algebraic topology will be one of the largest benefits of this project for the PI. Dugger, Sinha, and Sadofsky are experienced topologists at the University of Oregon, and the PI expects future interaction with each of them to be a valuable part of this development.

The PI has been an active member of the Midwest topology group, giving talks at the two most recent Graduate Student Topology Seminars and participating in the quarterly Midwest Topology Seminar. This has been both enjoyable and developmentally fruitful; the PI looks forward to participating in the community of algebraic topologists and geometers in the Northwest. Benefit from community involvement in the Midwest motivates the PI to stay involved with undergraduate students in mathematics as well. The PI has mentored five students in four projects (one joint) through the University of Chicago's Directed Reading Program. Mentors and mentees meet during the academic term to study a subject of mutual interest, and at the end of the term students give a brief presentation of their study. The PI enjoys undergraduate education in any capacity, but hopes specifically to develop a similar program at the University of Oregon. The PI has worked as a graduate student mentor for three summers during the University of Chicago Research Experience for Undergraduates program, and hopes to continue undergraduate mentorship informally if not formally at the University of Oregon.

In addition to the mathematical work, the PI has shown two paintings and a photograph at the 2007 University of Chicago Science in Art Exhibit [Art]. This juried exhibit uses art as a vehicle for creating connections between scientists and non-scientists, and the PI was pleased to participate. The PI's artistic interest will benefit from the community and surroundings at the University of Oregon, and the PI looks forward to engaging the broader public with science through visual art.

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