INVERTIBILITY IN BICATEGORIES

NILES JOHNSON

Contents

Abstract. (yet to be written)

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1. Bicategories

We assume the reader is familiar with basic definitions for bicategories, however we do not assume the reader is an expert and will review some important examples and relevant additional structures. A precise and concise introduction can be found in [Lei98], while [Lac07] provides a more expanded guide.

We use arrows such as $f : M \to M'$ to denote that f is a 2-cell with source M and target M', and slashed arrows such as $M : A \nrightarrow B$ to denote that M is a 1-cell with source A and target B. We use $\circ$ or juxtaposition to denote vertical composition of 2-cells, and $\odot$ to denote horizontal composition of 1-cells and of 2-cells.

1.1. Examples.

Let R be a commutative ring. The collection of R-algebras is the collection of 0-cells for the bicategory $\mathscr{M}_R$. The 1-cells of $\mathscr{M}_R(A, B)$ are the B-A-bimodules, and the 2-cells are bimodule homomorphisms. The horizontal composition, $\odot$, is given by the tensor product: If $M \in \mathscr{M}_R(B, C)$ and $N \in \mathscr{M}_R(A, B)$, then

$$M \odot N = M \otimes_B N \in \mathscr{M}_R(A, C).$$

If k is a commutative DG-algebra, we have the bicategory DG_k, defined similarly: the 0-cells are DG k-algebras, 1-cells are DG bimodules, and 2-cells are maps of bimodules. We also have the derived bicategory $\mathscr{D}_k$, with the same 0-cells as DG_k, but the category of 1- and 2-cells between two 0-cells is the derived category of bimodules.

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If k denotes a commutative ring spectrum, we likewise have a bicategory of k-algebras, with the category of 1-cells and 2-cells between two 0-cells being the homotopy category of bimodule spectra and bimodule morphisms. This bicategory is denoted $\mathscr{S}_k$.

More generally, any cocomplete symmetric monoidal category with unit k gives rise to a bicategory whose 0-cells are monoids and 1-cells are bimodules. Colimits are required to construct the $\odot$, just as they are in the case of tensor over an R-algebra or smash over a k-algebra.

1.2. **Monoidal bicategories.** A monoidal bicategory can be defined as a tricategory with one object. In practical terms, this means that the bicategory is equipped with an additional monoidal product on 0-, 1, and 2-cells, satisfying reasonable associativity and unit constraints. In $\mathscr{M}_R$, the monoidal product is $\otimes_R$; in $\mathscr{S}_k$, it is $\wedge_k$. In these examples, the monoidal product is symmetric, and hence these are symmetric monoidal bicategories. More generally, if $\mathscr{C}_k$ is a cocomplete monoidal category with unit k, and $\mathscr{B}_k$ the bicategory of monoids and bimodules in $\mathscr{C}_k$, then $\mathscr{B}_k$ is a symmetric monoidal bicategory with monoidal product induced by that of $\mathscr{C}_k$.

1.3. **Closed structure.** A *closed structure* for a bicategory, $\mathscr{B}$, defines right adjoints for $\odot$. For a 1-cell M, the right adjoint to $-\odot M$ is called "source-hom", or "right-hom", and denoted $M \triangleright -$. The adjoint to $M \odot -$ is called "target-hom", or "left-hom", and denoted $- \triangleleft M$. The adjunctions are written as

$$\begin{aligned} \mathscr{B}(V \odot M, W) &\cong \mathscr{B}(V, M \triangleright W) \\ \mathscr{B}(M \odot T, U) &\cong \mathscr{B}(T, U \triangleleft M) \end{aligned}$$

The existence of left and right hom functors defines a *closed bicategory.* Formal definitions and a complete description of closed structures can be found in [MS06].

Notation 1.1. To clarify understanding, we will occasionally make use of the following more explicit notation, familiar from algebraic contexts: For $M : A \nrightarrow B$, $W : A \nrightarrow C$, and $U : D \nrightarrow A$,

$$_C[\mathrm{Hom}_A({}_BM_A, {}_CW_A)]_B = M \triangleright W$$

$$_A[\mathrm{Hom}_B({}_BM_A, {}_BU_D)]_D = U \triangleleft M$$

Neither form of the notation is ideal, but it is our hope that using them together will aid readability more than either could alone.

Note that our examples above have internal hom functors, and this defines a closed structure. In general, if $\mathscr{C}_k$ is a bicomplete closed symmetric monoidal category, then the bicategory $\mathscr{B}_k$ built from $\mathscr{C}_k$ is a closed symmetric monoidal bicategory.

1.4. **Pseudofunctors.** If $\mathscr{A}$ and $\mathscr{B}$ are bicategories, a pseudofunctor $\mathcal{P} : \mathscr{A} \to \mathscr{B}$ (also called a *morphism*) is the bicategorical version of a functor. It is a function on 0-cells, and for each pair of 0-cells a functor

$$\mathscr{A}(A, B) \xrightarrow{\mathcal{P}_{AB}} \mathscr{B}(\mathcal{P}A, \mathcal{P}B).$$

These functors are compatible with $\odot$-composition in that there are 2-cell isomorphisms

$$\mathcal{P}_{BC}X' \odot \mathcal{P}_{AB}X \xrightarrow{\cong} \mathcal{P}_{AC}(X' \odot X)$$

satisfying the natural associativity and unit compatibility conditions.

Our focus is on the represented pseudofunctors. These are a bicategorical version of represented functors for categories, and they take values in the bicategory Cat. In this bicategory, the 0-cells are categories, the 1-cells are functors, and the 2-cells are natural transformations of functors. (Since the composition of functors isstrictly associative, Cat is actually a 2-category, but this is not relevant here.) For any bicategory $\mathscr{B}$ with 0-cell A, we have the represented pseudofunctor $\mathscr{B}(A, -) : \mathscr{B} \to Cat$. For a 0-cell $E \in \mathscr{B}$, this pseudofunctor gives a category, $\mathscr{B}(A, E)$. For a 1-cell $M : E \nrightarrow E'$, we have the functor $M \odot - : \mathscr{B}(A, E) \to \mathscr{B}(A, E')$, and 2-cells $M \to M'$ give natural transformations of such functors. The compatibility isomorphisms which make $\mathscr{B}(A, -)$ a pseudofunctor are precisely the associativity isomorphisms $(M_2 \odot (M_1 \odot -)) \cong (M_2 \odot M_1) \odot -$.

Notation 1.2 (Yoneda Embedding)**.** The pseudofunctor represented by A is called "Yoneda-A" and denoted $\mathscr{B}(A, -)$. The corepresented pseudofunctor is called "op-Yoneda-A" and denoted $\mathscr{B}(-, A)$.

this may be unnecessary if the notation is intuitively clear

1.5. **(Strong) transformations.** A transformation is the bicategorical analog of a natural transformation between functors. A transformation between two represented pseudofunctors, $\mathscr{B}(B,-)$ and $\mathscr{B}(A,-)$ is given by

(1) A family of functors $F_C : \mathscr{B}(B,C) \to \mathscr{B}(A,C)$. These are the *components* of F.
(2) For each 1-cell $C \xrightarrow{K} C'$, a natural transformation which, for 1-cells $X \in \mathscr{B}(B,C)$, has component 2-cells

$$K \odot F_C(X) \to F_{C'}(K \odot X)$$

natural in K and X, with standard associativity and unit compatibilities; namely that the following diagrams commute, with K and X as above, and $L \in \mathscr{B}(C', C'')$.

$$\begin{array}{ccc} L \odot K \odot F_C(X) & \longrightarrow & L \odot F_{C'}(K \odot X) \\ & \searrow & \downarrow \\ & & F_{C''}(L \odot K \odot X) \end{array} \qquad \begin{array}{ccc} C \odot F_C(X) & \longrightarrow & F_C(C \odot X) \\ \downarrow \cong & \swarrow \cong & \\ F_C(X) & & \end{array}$$

Notation 1.3. In the following, we will frequently drop the subscripts on the components of our transformations since they may always be determined from context and they tend to make the text less readable.

For developing Morita theory and related concepts, our interest will be in *strong* transformations; these are transformations for which the component 2-cells shown above are natural isomorphisms. The appropriate morphisms of transformations are called *modifications*, but we will not make any explicit reference to them beyond the following definition.

Definition 1.4. For two bicategories $\mathscr{A}$ and $\mathscr{B}$, $\Psi_s[\mathscr{A},\mathscr{B}]$ denotes the bicategory whose 0-cells are pseudofunctors $\mathscr{A} \to \mathscr{B}$, 1-cells are strong transformations, and 2-cells are modifications.

1.6. **The bicategorical Yoneda lemma and Morita II.**

Lemma 1.5 (Yoneda [Str80])**.** *For a pseudofunctor of bicategories $\mathcal{P} : \mathscr{A} \to Cat$, evaluation at the unit 1-cell for each 0-cell, A, of $\mathscr{A}$ provides the components for an equivalence of categories*

$$\Psi_s[\mathscr{A}, Cat](\mathscr{A}(A,-), \mathcal{P}) \xrightarrow{\simeq} \mathcal{P}A.$$

Corollary 1.6 (Morita II)**.**

$$\Psi_s[\mathscr{A}, Cat](\mathscr{A}(A,-), \mathscr{A}(B,-)) \xrightarrow{\simeq} \mathscr{A}(B,A)$$

That is, strong transformations $\mathscr{A}(A,-) \to \mathscr{A}(B,-)$ are given (precisely) by $\odot$-composition with a 1-cell $B \nrightarrow A$. In particular, strong transformations which induce equivalences $\mathscr{A}(A,C) \simeq \mathscr{A}(B,C)$ for all 0-cells C are given by invertible 1-cells $B \nrightarrow A$.

The essential point of the proof, as in the 1-categorical case, is the observation that for a strong transformation, S, and a 1-cell $Z : A \nrightarrow C$,

$$S_C(Z) \cong S_C(Z \odot A) \cong Z \odot S_A(A)$$

so that, for any C, the functor S_C is determined by $S_A(A)$, an object in the category $\mathscr{A}(B,A)$. Natural transformations of these functors are determined by morphisms in $\mathscr{A}(B,A)$.

2. Duality and invertibility in bicategories

Throughout this section we consider fixed 1-cells $X : B \nrightarrow A$ and $Y : A \nrightarrow B$ in a closed bicategory $\mathscr{B}$.

Definition 2.1 (Dual pair)**.** We say (X,Y) is a dual pair, or 'X is left-dual to Y' ('Y is right-dual to X'), or 'X is right-dualizable' ('Y is left-dualizable') to mean that we have 2-cells

$$\eta : A \to X \odot Y \quad \text{and} \quad \varepsilon : Y \odot X \to B$$

such that the following composites are the respective identity 2-cells.

$$X \cong A \odot X \xrightarrow{\eta \odot \mathrm{id}} X \odot Y \odot X \xrightarrow{\mathrm{id} \odot \varepsilon} X \odot B \cong X$$

$$Y \cong Y \odot A \xrightarrow{\mathrm{id}\odot\eta} Y \odot X \odot Y \xrightarrow{\varepsilon\odot\mathrm{id}} B \odot Y \cong Y$$

Definition 2.2 (Base and cobase for a dual pair)**.** When (X, Y) is a dual pair in a bicategory $\mathscr{B}$, we term the source of X (the target of Y) the *base* of the dual pair, and we term the source of Y (the target of X) the *cobase* of the dual pair. Thus, the evaluation map of the dual pair is a two-cell from $Y \odot X$ to the base 1-cell, and the coevaluation (unit) is a two-cell from the cobase 1-cell to $X \odot Y$.

Definition 2.3 (Invertible pair)**.** A dual pair (X, Y) is called invertible if the maps η and ε are isomorphisms. Equivalently, the adjoint pairs described above are adjoint equivalences.

Duality for monoidal categories has been studied at length, and duality in a bicategorical context has been introduced in [MS06, §16.4]. The definition of duality does not require $\mathscr{B}$ to be closed, but we will make use of the following basic facts about duality, some of which do require a closed structure on $\mathscr{B}$.

Proposition 2.4. *A 1-cell $X \in \mathscr{B}(A, B)$ is right-dualizable if and only if the coevaluation*

$$\nu : X \odot (X \rhd A) \to X \rhd X$$

is an isomorphism. Moreover, this is the case if and only if the map

$$\nu_Z : X \odot (X \rhd Z) \to X \rhd (X \odot Z)$$

is an isomorphism for all 1-cells Z with target A.

Proposition 2.5. *Let (X, Y) be a dual pair in $\mathscr{B}$, with $X : B \nrightarrow A$ and $Y : A \nrightarrow B$.*

(1) *For any 0-cell C, we have two adjoint pairs of functors, with left adjoints written on top:*

$$\mathscr{B}(A, C) \underset{-\odot Y}{\overset{-\odot X}{\rightleftarrows}} \mathscr{B}(B, C)$$

$$\mathscr{B}(C, A) \underset{X\odot -}{\overset{Y\odot -}{\rightleftarrows}} \mathscr{B}(C, B)$$

The structure maps for the dual pair give the triangle identities necessary to show that the displayed functors are adjoint pairs.

(2) *If $\mathscr{B}$ is closed, then Y is canonically isomorphic to $X \rhd B$, and for any 1-cell $Z : B \nrightarrow D$, the natural map $Z \odot (X \rhd B) \to X \rhd Z$ is an isomorphism.*

Lemma 2.6. *Let $X : A \nrightarrow B$ be a 1-cell in $\mathscr{B}(A, B)$. If X is right-dualizable and the unit $B \to X \rhd X$ is an isomorphism, then the evaluation $X \odot (B \lhd X) \to B$ is an isomorphism. Likewise, if X is left-dualizable and the unit $A \to X \lhd X$ is an isomorphism, then the evaluation $(X \rhd A) \odot X \to A$ is an isomorphism.*

Proof. We prove the first statement, leaving the second as an exercise in opposites. Let Y denote the canonical right dual of X. Since X is right-dualizable, Y is left-dualizable and X is isomorphic to the canonical left dual of Y: $X \cong A \lhd Y$. The isomorphism $B \xrightarrow{\cong} X \rhd X$ implies that the unit for the duality is an isomorphism: $B \xrightarrow{iso} X \odot Y$. Now we have the following commutative square:

$$\begin{array}{ccc} X \odot (X \lhd B) & \xrightarrow{\text{evaluation}} & B \\ \downarrow \cong & & \downarrow \cong \\ (A \lhd Y) \odot (X \lhd B) \xrightarrow{\cong} (B \lhd X) \lhd Y & \xrightarrow{\cong} & B \lhd (X \odot Y) \end{array}$$

where the two vertical isomorphisms are described above, the left-hand isomorphism is a consequence of dualizability for Y, and the right-hand isomorphism is an exercise in adjunction. ∎

2.1. **Examples.** The right-dualizable 1-cells in the bicategory $\mathscr{M}_R$ are the finitely-generated projective bimodules. More precisely, they are finitely-generated projective as right-modules over their source (the base of the duality). In $\mathscr{D}_k$, Lemma B.9 shows that the retracts of *finite cell* bimodules (Definition A.1) are right-dualizable and Lemma B.10 shows that the converse is also true. There is a similar characterization of the dualizable objects in $\mathscr{S}_k$.

Definition 2.7 (Picard Group)**.** Let A be a 0-cell of a bicategory $\mathscr{B}$. The *Picard group* of A, denoted $Pic(A)$, is the group of isomorphism classes of invertible 1-cells $A \nrightarrow A$.

2.2. **Azumaya objects and the Brauer group.** For the following, we let A be a fixed 0-cell of $\mathscr{D}$. We denote the enveloping 0-cell, $A \otimes_R A^{op}$ by A^e. Let $A_r \in \mathscr{D}(A^e, R)$ denote A regarded as a 1-cell $A^e \nrightarrow R$.

Proposition 2.8. *For A_r and A_r^* as above, the following are equivalent.*

i. $(A_r, A_r \triangleright A^e)$ *is an invertible pair of 1-cells.*
ii. $(R \triangleleft A_r, A_r)$ *is an invertible pair of 1-cells.*
iii. a) *The evaluation* $(A_r \triangleright A^e) \otimes_R A_r \to A^e$ *is an isomorphism.*
b) *The coevaluation* $A_r \otimes_{A^e} (A_r \triangleright A^e) \to A_r \triangleright A_r = {}_R[\mathrm{Hom}_{A^e}({}_R A_{r A^e}, {}_R A_{r A^e})]_R$ *is an isomorphism.*
c) *The unit map* $R \to A_r \triangleright A_r$ *is an isomorphism.*
iv. a) *The evaluation* $A_r \otimes_{A^e} R \triangleleft A_r \to R$ *is an isomorphism.*
b) *The coevaluation* $(R \triangleleft A_r) \otimes_R A_r \to A_r \triangleleft A_r = {}_{A^e}[\mathrm{Hom}_R({}_R A_{r A^e}, {}_R A_{r A^e})]_{A^e}$ *is an isomorphism.*
c) *The unit map* $A^e \to A_r \triangleleft A_r$ *is an isomorphism.*

Proof. In general, a pair of 1-cells (X, Y) is invertible if and only if X is right-dualizable, Y is isomorphic to the canonical right dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *iii*. Likewise, (Y, X) is invertible if and only if X is left-dualizable, Y is isomorphic to the canonical left dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *ii* and *iv*. Since (X, Y) is an invertible pair if and only if (Y, X) is so, this finishes the proof. □

Definition 2.9. If A satisfies the four equivalent conditions above, A is an *Azumaya object* of $\mathscr{D}$.

Remark 2.10. Let R be a commutative ring. If $\mathscr{D}$ is the bicategory of R-algebras and their bimodules, this recovers the classical definition of Azumaya algebras over R. We have the following translations in this case:

1. A is *seperable* over R if and only if *iii.b* holds.
2. The center of A is equal to R if and only if *iii.c* holds.
3. A is *faithfully projective* over R if and only if *iv.a* and *iv.b* hold.

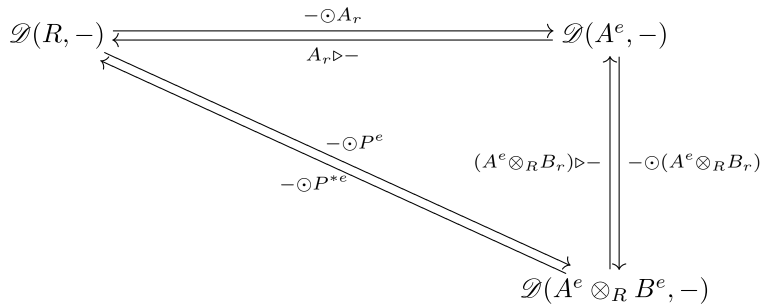
Note, moreover, that in this case *iii.b* and *iii.c* together imply *iii.a*. The proof makes use of ideal theory, and hence we do not expect such a result to hold in general, but we do not have a counterexample*.

*look for a counterexample

Proposition 2.11. *Let A be a 0-cell of a bicategory $\mathscr{D}$. Then A is Azumaya in $\mathscr{D}$ if and only if there is a 0-cell, B, such that B_r is left-dualizable and there is an invertible 1-cell $P : R \nrightarrow A \otimes_R B$.*

Proof. If A is Azumaya, then taking $B = A^{(op)}$ gives one direction. Given B and P as above, let $P^* = P \triangleright R$, and let $P^e = P \otimes_R (P^*)^{op}$, $P^{*e} = P^* \otimes P^{op}$. Then (P^e, P^{*e}) is an invertible pair of 1-cells between the 0-cells R^e and $(A \otimes_R B)^e \cong A^e \otimes_R B^e$. Moreover, since (P, P^*) is an invertible pair, $P^e = P \otimes_R (P^*)^{op} \cong (A \otimes_R B)_r \cong A_r \otimes_R B_r$. Now B_r is a 1-cell $B_r : B^e \nrightarrow R$, and so taking the tensor product with A^e gives $A^e \otimes B_r : A^e \otimes B^e \nrightarrow A^e$ and the unit isomorphism $A_r \odot A^e \cong A_r$ induces $P^e \cong A_r \otimes_R B_r \cong A_r \odot (A^e \otimes_R B_r)$. Thus we have the following diagram of adjunctions, where the left adjoints commute up to natural isomorphism.

By uniqueness of adjoints, the right-adjoints therefore also commute up to natural isomorphism.



Now the diagonal adjunction is an equivalence, and Lemma 2.6 shows that the evaluation map for $(B_r, B_r \triangleright B^e)$ is an isomorphism, hence the counit of the vertical adjunction is an isomorphism. The counit for the horizontal adjunction is therefore also an isomorphism and now a diagram chase shows that the units are also isomorphisms. Explicitly, we have:

$$\begin{aligned} A_r \triangleright (- \odot A_r) &\cong A_r \triangleright (- \odot A_r) \odot P^e \odot P^{*e} \\ &\cong A_r \triangleright (- \odot A_r) \odot A_r \odot (A^e \otimes_R B_r) \odot P^{*e} \\ &\cong - \odot A_r \odot (A^e \otimes_R B_r) \odot P^{*e} \\ &\cong - \odot P^e \odot P^{*e} \end{aligned}$$

Since (P^e, P^{*e}) is an invertible pair, this shows that the horizontal adjunction is an equivalence, and therefore A is an Azumaya object. □

Note in the proof above that B is also an Azumaya object; interchanging the roles of A and B we thus have the following.

Corollary 2.12. *For A as above, A is Azumaya if and only if A_r is left-dualizable and there is a 0-cell B with an invertible 1-cell $P : R \nrightarrow B \otimes_R A$.*

Definition 2.13. Let $\mathscr{B}$ be a symmetric monoidal bicategory with unit 0-cell R. The *Brauer group* of R, denoted $Br(R)$, is the group of 1-cell-equivalence-classes of 0-cells A for which there exists a 0-cell B such that $A \otimes_R B$ is equivalent to R.

Remark 2.14. The Azumaya objects are the objects $A \in Br(R)$ such that A_r is left-dualizable. Classically, 1-cell-equivalence is Morita equivalence, and if R is a commutative ring, every $A \in Br(R)$ satisfies the left-dualizability condition. (In this context, it means that A is finitely-generated and projective as an R-module.)

3. Triangulated bicategories

We recall first the definitions of localizing subcategory and generator for a triangulated category, and then give a definition (3.4) of triangulated bicategory suitable for our purposes. In particular, under this definition $\mathscr{D}_k$ is a triangulated bicategory.

Definition 3.1 (Localizing subcategory)**.**
If $\mathscr{T}$ is a triangulated category with infinite coproducts, a *localizing* subcategory, $\mathscr{S}$, is a full triangulated subcategory of $\mathscr{T}$ which is closed under coproducts from $\mathscr{T}$.

Remark 3.2. This is equivalent to the definition for arbitrary triangulated categories of [Hov99], (which requires that a localizing subcategory be thick) because a triangulated subcategory automatically satisfies the 2-out-of-3 property and because in any triangulated category with countable coproducts, idempotents have splittings. See [Nee01, 1.5.2, 1.6.8, and 3.2.7] for details.

Definition 3.3 (Triangulated generator)**.**
A set, $\mathscr{P}$, of objects in $\mathscr{T}$ (triangulated category with infinite coproducts, as above) is a set of *triangulated generators* (or simply *generators*) if the only localizing subcategory containing $\mathscr{P}$ is $\mathscr{T}$ itself.

Definition 3.4 (Triangulated bicategory [MS06, §16.7])**.**
A closed bicategory $\mathscr{B}$ will be called a *triangulated bicategory* if for each pair of 0-cells, A and B, $\mathscr{B}(A,B)$ is a triangulated category with infinite coproducts, and if the suspension, Σ, is a pseudofunctor (Subsection 1.4) on $\mathscr{B}$, and furthermore the local triangulations on $\mathscr{B}$ are compatible as described in the following two axioms.

(TC1) For a 1-cell $X : A \to\!\!\!\!\!\!\!\!\;+ B$, there is a natural isomorphism
$$\alpha : X \odot \Sigma A \to \Sigma X$$
such that the composite below is multiplication by -1.
$$\Sigma^2 A = \Sigma(\Sigma A) \xrightarrow{\alpha^{-1}} \Sigma A \odot \Sigma A \xrightarrow{\gamma} \Sigma A \odot \Sigma A \xrightarrow{\alpha} \Sigma(\Sigma A) = \Sigma^2 A$$

(TC2) For any 1-cell, W, the functors $W \odot -$, $- \odot W$, $W \rhd -$, and $- \rhd W$ are exact.

If $\mathscr{B}$ is a triangulated bicategory and P, Q are 1-cells in $\mathscr{B}(A,B)$, we emphasize that $\mathscr{B}$ is triangulated by writing the abelian group of 2-cells $P \to Q$ as $\mathscr{B}[P,Q]$ and by writing the graded abelian group obtained by taking shifts of Q as $\mathscr{B}[P,Q]_*$. To emphasize the source and target of P and Q, we may also write $\mathscr{B}(A,B)[P,Q]_*$.

Definition 3.5 ($\odot$-faithful 1-cells)**.**
In any locally additive bicategory, $\mathscr{B}$, a 1-cell $W : A \to\!\!\!\!\!\!\!\!\;+ B$ is called *left-faithful* if triviality for any 1-cell $Z : C \to\!\!\!\!\!\!\!\!\;+ A$ is detected by triviality of the composite $W \odot Z$. That is, $Z : C \to\!\!\!\!\!\!\!\!\;+ A$ is zero if and only if $W \odot Z = 0$. A collection of 1-cells, $\mathscr{E}$, in $\mathscr{B}(A,B)$ is called *jointly left-faithful* if the objects have this property jointly; that is, $Z = 0$ if and only if $W \odot Z = 0$ for all $W \in \mathscr{E}$. The term *left-faithful* is defined similarly, considering $- \odot W$ instead of $W \odot -$.

Remark 3.6. If $\mathscr{B}$ is a monoidal additive category with monoidal product $\odot$, the unit object is both left- and right-faithful. In arbitrary locally additive bicategories, if $A \neq B$ then $\mathscr{B}(A,B)$ may not have a single object with this property, but in relevant examples the collection of all 1-cells, $\mathrm{ob}\mathscr{B}(A,B)$, does have this property jointly. As a counter-point to this remark, we have the following lemma.

Lemma 3.7. *Let $\mathscr{B}$ be a triangulated bicategory, and let $P : A \to\!\!\!\!\!\!\!\!\;+ B$ be a generator for $\mathscr{B}(A,B)$. If the collection of all 1-cells, $\mathscr{B}(A,B)$, is jointly left-faithful (resp. right-faithful), then P is left-faithful (resp. right-faithful).*

Proof. Consider the left-faithful case; the right-faithful case is similar. Given any 1-cell $Z : C \to\!\!\!\!\!\!\!\!\;+ A$ with $P \odot Z = 0$, let $\mathscr{S}$ be the full subcategory of 1-cells, $W : A \to\!\!\!\!\!\!\!\!\;+ B$ for which $W \odot Z = 0$. This is a localizing subcategory of $\mathscr{B}(A,B)$, and by assumption $P \in \mathscr{S}$, so $\mathscr{S} = \mathscr{B}(A,B)$, and hence $Z = 0$. ☐

Remark 3.8. Since the functors $P \odot -$ are exact, the property of $P \odot -$ detecting trivial objects is equivalent to $P \odot -$ detecting isomorphisms (meaning that a 2-cell f is an isomorphism if and only if $P \odot f$ is so).

3.1. **Tilting theory.** For this subsection, we let $\mathscr{D}$ denote a closed, symmetric monoidal bicategory with unit 0-cell R. We do not require that $\mathscr{D}$ be triangulated, but simply that $\mathscr{D}$ have a 0-object in each 1-cell category.

Definition 3.9. Let $T : A \to\!\!\!\!\!\!\!\!\;+ B$ be a 1-cell in $\mathscr{D}(A,B)$.

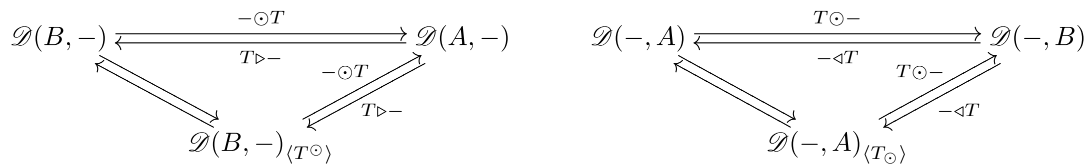
A 1-cell $M : C \to\!\!\!\!\!\!\!\!\;+ A$ is *left-T-acyclic* if $T \odot M = 0$. A 1-cell $N : C \to\!\!\!\!\!\!\!\!\;+ A$ is *left-T-local* if $\mathscr{D}(C,A)[M,N]_* = 0$ for all T-acyclic 1-cells $M \in \mathscr{D}(C,A)$. The full subcategory of left-T-local 1-cells in $\mathscr{D}(C,A)$ is denoted $\mathscr{D}(C,A)_{\langle T_\odot \rangle}$. (The notation $T_\odot$ is intended to remind the reader of push-forward via $\odot$-composition.)

A 1-cell $M' : B \to\!\!\!\!\!\!\!\!\;+ C$ is *right-T-acyclic* if $M \odot T = 0$. A 1-cell $N' : B \to\!\!\!\!\!\!\!\!\;+ C$ is *right-T-local* if $\mathscr{D}(B,C)[M',N']_* = 0$ for all right-T-acyclic 1-cells $M' \in \mathscr{D}(B,C)$. The full subcategory of right-T-local 1-cells in $\mathscr{D}(B,C)$ is denoted $\mathscr{D}(B,C)_{\langle T^\odot \rangle}$.(The notation $T^\odot$ is intended to remind the reader of pull-back via $\odot$-composition.)

Notation 3.10. The pseudofunctor $\mathscr{D}(-,A)_{\langle T_\odot \rangle}$ is called "T-local op-Yoneda A". The pseudofunctor $\mathscr{D}(B,-)_{\langle T^\odot \rangle}$ is called "T-local Yondea B".

Baker and Lazarev describe the following in the context of spectra, but their methods generalize to our setting. The key observation is that for any 1-cell P whose source is A, $T \rhd P$ is right-T-local. Likewise, if P' is any 1-cell whose target is B, $P' \lhd T$ is left-T-local.

Proposition 3.11 (Baker-Lazarev factorization [BL04])**.** *Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}(A,B)$. The adjunctions induced by T factor through the T-local pseudofunctors; we have the following diagrams of adjoint transformations:*



Proposition 3.12 ([BL04])**.** *If a 1-cell $T \in \mathscr{D}(A,E)$ is right-dualizable and the unit map induces an isomorphism $E \cong T \triangleright T = {}_E[\mathrm{Hom}_A({}_ET_A, {}_ET_A)]_E$, then the induced adjoint pair is an equivalence $\mathscr{D}(E,-)_{\langle T\odot\rangle} \simeq \mathscr{D}(A,-)$. Likewise, if T is left-dualizable and the unit induces an isomorphism . . .* `finish statement` *. . .*

Proof. Let T^* denote the right-dual to T. Since T is right-dualizable, T^* is left-dualizable and the evaluation map $T \odot (E \triangleleft T) \to E$ is an isomorphism (Lemma 2.6). Moreover, $- \triangleleft T$ takes values in the T-local category and hence the fact that the unit of the adjunction is an isomorphism follows from the fact that the evaluation is so. ☐

Proposition 3.13. *If T satisfies the hypotheses of 3.12 and if in addition T is left-faithful (Definition 3.5), then all three of the adjoint pairs above are equivalences.*

Proposition 3.14. *Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}$. The following are equivalent:*

- *i.* *T is invertible.*
- *ii.* *a) T is right-dualizable.*
  - *b) The unit induces $B \cong T \triangleright T$.*
  - *c) A is left-T-local.*
- *iii.* *a) T is left-dualizable.*
  - *b) The unit induces $A \cong T \triangleleft T$.*
  - *c) B is right-T-local.*

Let A be a 0-cell of $\mathscr{D}$ and (as above) let A_r denote A regarded as a 1-cell $A^e \nrightarrow R$. Applying the above proposition to A_r, we have the following. For comparison with [BL04], note that in topological contexts $A_r \triangleright A_r$ is denoted $THH_R(A,A)$.

Corollary 3.15 ([BL04, 2.1,2.3])**.** *The following are equivalent:*

- *i.* *A is Azumaya in $\mathscr{D}$.*
- *ii.* *a) A_r is right-dualizable.*
  - *b) The unit induces $R \cong A_r \triangleright A_r$.*
  - *c) A^e is left-A_r-local.*
- *iii.* *a) A_r is left-dualizable.*
  - *b) The unit induces $A^e \cong A_r \triangleleft A_r$.*
  - *c) R is right-A_r-local.*

Corollary 3.16 (Rickard)**.** *Let S be a DG k-algebra, and let T be a DG S-module. If T has the following two properties, then $\mathscr{D}_k(S)$ and $\mathscr{D}_k(End_S(T))$ are equivalent as triangulated categories.*

- *(i) T is a right-dualizable S-module.*
- *(ii) T generates the triangulated category $\mathscr{D}_k(S)$.*

Proof. Let $\widetilde{T}$ denote T regarded as a bimodule over $E = {}_k[\mathrm{Hom}_S({}_kT_S, {}_kT_S)]_k$. Since T is (right-)dualizable, $\widetilde{T}$ is (right-)dualizable in $\mathscr{D}_k(S,E)$.

Since k is the unit of the symmetric monoidal bicategory $\mathscr{D}_k$, the 1-cells of $\mathscr{D}_k(S,k)$ are jointly left-faithful (Definition 3.5). Hence Lemma 3.7 shows that T is left-faithful. This means that $\widetilde{T}$ is also left-faithful, and thus the evaluation $(\widetilde{T} \triangleright S) \odot \widetilde{T} \to S$ is an isomorphism in $\mathscr{D}_k(S,S)$: The composite below is the identity and the first map, induced by the unit of the adjunction, is an isomorphism so the second must be also.

$$\widetilde{T} \xrightarrow{\cong} \widetilde{T} \odot (\widetilde{T} \triangleright S) \odot \widetilde{T} \xrightarrow{1\odot\mathrm{eval}} \widetilde{T}$$

□

Corollary 3.17. *Let k be a commutative ring spectrum, and let $\mathscr{D}_k$ denote the bicategory of k-algebras and homotopy categories of bimodules. Suppose A is a k-algebra, and let T be a fibrant and cofibrant A-module, with endomorphism k-algebra $E = F_A(T,T)$. If T has the following two properties, then $\mathscr{D}_k(A)$ and $\mathscr{D}_k(E)$ are equivalent categories.*

(*i*) *T is (right-)dualizable as an A-module.*
(*ii*) *T generates the triangulated category $\mathscr{D}_k(A)$.*

Notation 3.18. Given a map of k-algebras $\iota : B \to E$, we have two restriction-of-scalars functors: one for restriction of left modules, and another for restriction of right modules. For any k-algebra A, We let $\iota_L^* : \mathscr{S}_k(A,E) \to \mathscr{S}_k(A,B)$ denote restriction on the left (target), and $\iota_R^* : \mathscr{S}_k(E,A) \to \mathscr{S}_k(B,A)$ denote restriction on the right (source). Both functors create weak-equivalences and fibrations.

Proof of 3.17. Let $E = T \triangleright T = F_A(T,T)$. The unit map $k \to E$ is obtained as the composite of algebra maps $k \to B \to E$. Let $\widetilde{T}$ be a cofibrant replacement for T in $\mathscr{S}_k(A,E)$. Recall that T is cofibrant in $\mathscr{S}_k(A,B)$, and hence has the LLP with respect to acyclic fibrations. We construct $\widetilde{T}$ by the usual factorization of the map from the initial object, and the forgetful functor ι_L^* creates weak equivalences and fibrations, so the lifting property for T gives a weak equivalence $T \xrightarrow{\simeq} \iota_L^* \widetilde{T}$.

The canonical dual of T is $F_A(T,A) = T \triangleright A \in \mathscr{S}_k(B,A)$, and we let D denote a cofibrant replacement for $F_A(T,A)$ in $\mathscr{S}_k(B,A)$, so that we have a weak equivalence $D \xrightarrow{\simeq} F_A(T,A)$. The canonical dual of T has a right-action of the endomorphism k-algebra, E, and we let $\widetilde{D}$ be a cofibrant replacement for $F_A(T,A)$ in $\mathscr{S}_k(E,A)$, constructed again by the usual factorization. Since the forgetful functor ι_R^* creates weak equivalences and fibrations, we have an acyclic fibration $\iota_R^* \widetilde{D} \xrightarrow{\simeq} F_A(T,A)$ in $\mathscr{S}_k(B,A)$. Because D is cofibrant, the weak equivalence $D \xrightarrow{\simeq} F_A(T,A)$ lifts with respect to acyclic fibrations and hence we have a weak equivalence $D \xrightarrow{\simeq} \iota_R^* \widetilde{D}$.

Now we show that $(\widetilde{T}, \widetilde{D})$ is a dual pair in $\mathscr{D}_k$. The weak equivalences $\widetilde{T} \to T$ and $\widetilde{D} \to F_A(T,A)$ in $\mathscr{S}_k(A,E)$ and $\mathscr{S}_k(E,A)$, respectively, give maps

$$\widetilde{T} \odot \widetilde{D} \to T \odot F_A(T,A) \to E \text{ and } \widetilde{D} \odot \widetilde{T} \to F_A(T,A) \odot T \to A$$

in $\mathscr{S}_k(E,E)$ and $\mathscr{S}_k(A,A)$, respectively. Moreover, the first map is an isomorphism in $\mathscr{D}_k(E,E)$ because its image under $\iota_L^* \iota_R^*$ is a composite of two isomorphisms in $\mathscr{D}_k(B,B)$:

$$\iota_L^* \widetilde{T} \odot \iota_R^* \widetilde{D} \cong T \odot D \cong \iota_L^* \iota_R^* E.$$

The inverse to this map gives the unit for the dual pair, and the duality diagrams commute because the corresponding diagrams for T and $F_A(T,A)$ do. Hence the functors $- \odot \widetilde{T}$ and $- \odot \widetilde{D}$ induce an adjunction

$$\mathscr{D}_k(A,C) \underset{-\odot \widetilde{D}}{\overset{-\odot \widetilde{T}}{\rightleftarrows}} \mathscr{D}_k(E,C)$$

and the unit of this adjunction is an isomorphism.

As in the algebraic case, the 1-cells of $\mathscr{D}_k(A,k)$ are jointly left-faithful and hence the generator T is left-faithful. Since ι_L^* creates weak equivalences, $\widetilde{T}$ is also left-faithful and the result follows just as in the algebraic case. □

Appendix A. Model Structure for DG Algebras

Definition A.1 (Relative cell module)**.** A map of A-modules $C_0 \to C$ is called a relative cell A-module if C is the colimit of a sequence of maps $C_r \to C_{r+1}$, with each map obtained as a pushout

$$\begin{array}{ccc} \bigoplus_{q_i} S^{q_i} & \longrightarrow & C_r \\ \downarrow & & \downarrow \\ \bigoplus_{q_i} D^{q_i+1} & \longrightarrow & C_{r+1} \end{array}$$

The maps $S^{q_i} \to C_r$ above are called the *attaching maps* for C_r. If $0 \to C$ is a relative cell A-module, C is called a cell A-module. If there are only finitely many cells, then C is called a finite cell A-module.

Definition A.2 ([MS06, 4.5.1])**.** Let $\mathscr{I}$ be a set of maps in a category $\mathscr{C}$ with coproducts $\oplus$.

(a) A *relative $\mathscr{I}$-cell module* is a map $C_0 \to C$, with C obtained as a colimit of maps, $C_r \to C_{r+1}$, formed by pushouts

$$\begin{array}{ccc} \bigoplus_{q\in\mathscr{I}} X_q & \longrightarrow & C_r \\ \downarrow & & \downarrow \\ \bigoplus_{q\in\mathscr{I}} Y_q & \longrightarrow & C_{r+1} \end{array}$$

where each $X_q \to Y_q$ is a map in $\mathscr{I}$.

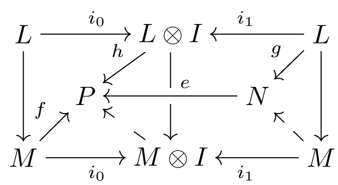
(b) The set $\mathscr{I}$ is *compact* if, for every map $X \to Y$ in $\mathscr{I}$, the source object, X, is small with respect to countable colimits. That is, for every relative $\mathscr{I}$-cell module $C_0 \to C$ as above, the natural map below is an isomorphism.

$$\operatorname{colim} \operatorname{Hom}_A(X, C_r) \xrightarrow{\cong} \operatorname{Hom}_A(X, \operatorname{colim} C_r)$$

(c) An *$\mathscr{I}$-cofibration* is a map which satisfies the LLP with respect to any map satisfying the RLP with respect to all maps in $\mathscr{I}$.

Definition A.3 (Cell submodule)**.** If $M = \operatorname{colim} M_r$ and $L = \operatorname{colim} L_r$ are cell A-modules for which each L_r is a submodule of M_r and, for each attaching map $S^q \to L_r$, the composite $S^q \to L_r \subset M_r$ is one of the attaching maps for M_r, then L is called a *cell submodule* of M.

Theorem A.4 (HELP [KM95, III.2.2])**.** *Let L be a cell submodule of a cell A-module, M, and let $e : N \to P$ be a quasi-isomorphism of A-modules. Then, given maps which make the solid arrow diagram below commute, there are dashed lifts which commute with the rest of the diagram.*



Note. For one who compares this lemma with [KM95], it may be helpful to point out that the grading is cohomological there, so they use s and $s-1$ where we use n and $n+1$.

Lemma A.5. *For any integer n, a map $e : N \to P$ of DG-modules over A satisfies HELP with respect to the inclusion $S^n \to D^{n+1}$ if and only if $e_* : H_*(N) \to H_*(P)$ is a monomorphism in degree n and an epimorphism in degree $n+1$.*

Appendix B. Base Change for DG Algebras

In this section, we describe general results regarding change of base DG k-algebra. Suppose that A and B are DG k-algebras for a commutative ring, k, and suppose $f : A \to B$ is a map of DG k-algebras. There are two natural pull-backs of B to the category of DG A-modules: let ${}_AB_B \in DG_k(B, A)$ denote B with the action of A on the left via f, and let ${}_BB_A \in DG_k(A, B)$ denote B with the action of A on the right via f. Then the A-A bimodule obtained from B with A acting on both sides by f is given as $({}_AB_B)\otimes_B({}_BB_A) = ({}_AB_B)\odot({}_BB_A) \in DG_k(A, A)$. The map f can be regarded as a 2-cell

$$A \xrightarrow{f} {}_AB_A = ({}_AB_B)\odot({}_BB_A).$$

The multiplication for B gives a 2-cell in $DG_k(B, B)$

$$({}_BB_A) \odot ({}_AB_B) = ({}_BB_A) \otimes_A ({}_AB_B) \to B$$

and the duality relations hold, making $({}_AB_B, {}_BB_A)$ a dual pair. Hence we have an adjoint pair of strong transformations

$$\text{extension of scalars: } f_! = - \odot {}_AB_B : DG_k(A, -) \to DG_k(B, -)$$

and

$$\text{restriction of scalars: } f^* = - \odot {}_B B_A \cong {}_A B_B \triangleright - : DG_k(B,-) \to DG_k(A,-)$$

The transformation f^* is right adjoint to $f_!$, but since f^* is itself a strong transformation, it also has its own right adjoint,

$$f_* = {}_B B_A \triangleright - : DG_k(A,-) \to DG_k(B,-).$$

B.1. **Local model structure.** Each 1-cell category $DG_k(A,B)$ has a model structure, described by applying the theory of Section A to the DG k-algebra $A \otimes_k B^{op}$. We refer to this as a local model structure for the bicategory DG_k, meaning simply a model structre on each 1-cell category. The generating cofibrations and acyclic cofibrations of $DG_k(A,B)$ are denoted by $\mathscr{I}(A,B)$ and $\mathscr{J}(A,B)$, respectively, and the results below describe the behavior of the base-change transformations above with respect to this local model structre.

Notation B.1. In contrast with Section A, here we let S^n, D^m, and I denote the corresponding chain complexes over k, and we let $\otimes$ denote $\otimes_k$. For any chain complex, M, over k, we let ${}_B M_A$ denote the DG (B,A)-bimodule $B \otimes_k M \otimes_k A$. So ${}_B M_A \in DG_k(A,B)$.

Proposition B.2 (Push-out Products)**.** *The local model structure on each 1-cell category $DG_k(A,B)$ is compatible with $\odot$-composition of 1-cells in the following sense: If i and j are generating cofibrations, then their pushout-product is a cofibration, and if one of i or j is a generating acyclic cofibration and the other is a generating cofibration, then their pushout-product is an acyclic cofibration.*

Proposition B.3. *If $f : A \to B$ is a map of DG k-algebras, then the adjoint pair $(f_!, f^*)$ is a Quillen adjoint pair for each C.*

$$DG_k(A,C) \underset{f^*}{\overset{f_!}{\rightleftarrows}} DG_k(B,C)$$

Remark B.4. A similar statement for (f^*, f_*) is not true unless ${}_A B_A$ is cofibrant as an A-module, since otherwise f^* does not preserve cofibrations in general.

Lemma B.5 (f^* creates weak equivalences)**.** *If $e : X \to Y$ is a map of 1-cells in $DG_k(B,C)$ for which $f^*e : f^*X \to f^*Y$ is a weak equivalence, then the original map $e : X \to Y$ is a weak equivalence.*

Proposition B.6. *If f above is a weak equivalence, then the Quillen pair $(f_!, f^*)$ is a Quillen equivalence for all C.*

Corollary B.7. *For $f : A \xrightarrow{\simeq} B$ as above, the dual pair $({}_A B_B, {}_B B_A)$ is invertible when considered as a pair of 1-cells in the derived categories $\mathscr{D}_k(B,A)$ and $\mathscr{D}_k(A,B)$, respectively.*

B.2. **Duality in DG_k and $\mathscr{D}_k$.**

Lemma B.8. *If M is right-dualizable in $DG_k(A,B)$, then M is a retract of a finite free (right-)DG-module over $A \otimes_k B^{op}$.*

Lemma B.9. *Let $M \in \mathscr{D}_k(A,B)$ and suppose M is a retract of a finite cell (B,A)-bimodule. Then $M : A \nrightarrow B$ is (right-)dualizable in $\mathscr{D}_k$ and therefore the coevaluation $M \odot (M \triangleright A) \to M \triangleright M$ is an isomorphism in $\mathscr{D}_k$.*

Lemma B.10. *Let $M : A \nrightarrow B$ be a 1-cell in $\mathscr{D}_k$, and suppose the coevaluation $M \odot (M \triangleright A) \to M \triangleright M$ is an isomorphism. Then M is (quasi-)isomorphic to a retract of a finite cell (B,A)-module.*

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University of Chicago, Department of Mathematics, 5734 South University Ave., Chicago, IL 60637
URL: `http://www.math.uchicago.edu/~niles`
E-mail address: `niles@math.uchicago.edu`